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ON FUNCTIONS REALIZING THE MAXIMA OF TWO FUNCTIONALS AT A TIME

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Let $S_R(M)$, $M>1$, denote the family of functions $F(z)=z+\Sigma_{n=2}^{\infty}A_n z^n$ holomorphic and univalent in the disc $\Delta=\{z: |z|<1\}$, satisfying the conditions: $|F(z)|<M$ for $z\in\Delta$, $A_n=\overline{A}_n$ for $n=2, 3, \ldots$ In the paper it has been proved that if there exists a function $w=F(z)$ for which in the family $S_R(M)$ the maxima of the coefficients A_N and A_{N+1} $(N=4, 5, \ldots)$ are attained simultaneously, then it satisfies in the disc Δ the equation

$$\frac{w}{(\varepsilon-w/M)(\overline{\varepsilon}-w/M)}=\frac{z}{(\varepsilon-z)(\overline{\varepsilon}-z)}, \quad |\varepsilon|=1.$$

There has also been given an analogous theorem concerning the coefficients A_K, A_N, $N=p+1$, $2\leq K\leq N-1$, where p is an arbitrary prime number.

1. Consider the family $S_R(M)$, $M>1$, of functions

$$F(z)=z+A_2z^2+\cdots+A_nz^n+\cdots \tag{1}$$

holomorphic and univalent in the disc $\Delta=\{z:|z|<1\}$, satisfying the conditions $|F(z)|<M$ for $z\in\Delta$, $A_N=\overline{A}_N$ for $n=2, 3, \ldots$ Let $S_R=\bigcup_{M>1}S_R(M)$.

One knows numerous results concerning the problem of maximization of

$$H_N(F)=A_N \tag{2}$$

the functional defined in the class $S_R(M)$. Some of them are obtained as straightforward corollaries from the known earlier estimations of functional (2) considered in the family $S(M)$ of univalent functions of form (1), bounded, with arbitrary coefficients. And so, a consequence of the results of G. Pick [11] and A. Schaeffer, D. Spencer [12] (see also [19; 8]) are the sharp estimations from above of the coefficients A_2 and A_3 in the class $S_R(M)$. In those papers, for each $M>1$, the extremal functions were given. From the results included in the papers by L. Siewierski ([16; 17; 18]) and from those obtained in some other way by M. Schiffer and O. Tammi [15] it follows directly that, for each $N=2, 3, \ldots,$ there exists an M_N such that, for all $M\in(1, M_N)$, the maximum of functional (2) in the class $S_R(M)$ is attained only for a function $w=P^{(N-1)}(z; M)$, $P^{(N-1)}(0; M)=0$, defined in the disc Δ by the equation

$$\frac{w}{[1-(w/M)^{N-1}]^{2/(N-1)}}=\frac{z}{(1-z^{N-1})^{2/(N-1)}}.$$

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This theorem constitutes a positive solution to the hypothesis of Z. Charzyński and O. Tammi, formulated earlier (for the families $S(M)$). For $N=5, 6, \ldots$, the maximal interval $(1, \tilde{M}_N)$ in which the functions $P^{(N-1)}$ are extremal with respect to functional (2) has not been known as yet. Of course, $\tilde{M}_2 = +\infty$, $\tilde{M}_3 = e$, $\tilde{M}_4 = 2$.

In the last few years, the estimations of the coefficient A_4 in the family $S_R(M)$, $M>1$, were obtained [20-22; 14; 9]. It was also shown [24; 23; 6; 7] that, for any $N=6, 8, 10, \ldots$, there exists an M_N^* such that, for each $M \in (M_N^*, +\infty)$, the only extremal function with respect to functional (2) in the classes $S_R(M)$ is the Pick function $P(z; M) \equiv P^{(1)}(z; M)$. Similarly as in the case of the Charzyński–Tammi hypothesis, the greatest interval $(\tilde{M}_N^*, +\infty)$ in which the Pick function is extremal is not known (of course, $\tilde{M}_2^* = 1$, $\tilde{M}_4^* = 11$).

The above-mentioned theorems solve merely in part the hard task of determining the maximum of functional (2) in the class $S_R(M)$ for any admissible N and M. It can easily be noticed that from the cited results of Siewierski it follows that, for M sufficiently close to 1, in the class $S_R(M)$ there are no functions extremal with respect to A_N and A_{N+1} at the same time. In turn, from the result of Jakubowski, Zielińska, Zyskowska mentioned above we infer that, for M sufficiently large, the only such function can be the Pick function.

In this situation, it seems essential to ask about the structure of functions which simultaneously maximize in the class $S_R(M)$, $M>1$, two distinct coefficients of power series (1), whose indices satisfy certain conditions. We assume here that such functions do exist. It turns out that functions of this type map the disc Δ onto the disc $|w|<M$ from which at most two rectilinear slits have been removed. The problems considered in the present paper refer to the results obtained in the class $S=\bigcup_{M>1} S(M)$ by A. K. Bahtin [1], and the method approximates the proof of Lemma 31 in monograph [13]. The analogue of this lemma for functions of the class $S(M)$ is contained in [25].

2. Let $\mathscr{F}_N(M)$, $N=2, 3, \ldots$, $M>1$, stand for the set of all functions for which functional (2) attains its maximum in the family $S_R(M)$. Since functional (2) is continuous, whereas the class $S_R(M)$ compact, therefore $\mathscr{F}_N(M) \neq \emptyset$. Next, let us denote by $\mathscr{P}(M)$, $M>1$, the family of functions $w=P(z; M; \varepsilon)$, $P(0; M; \varepsilon)=0$, $|\varepsilon|=1$, satisfying in the disc Δ the equation

$$\frac{w}{(\varepsilon - w/M)(\bar{\varepsilon} - w/M)} = \frac{z}{(\varepsilon - z)(\bar{\varepsilon} - z)}. \tag{3}$$

Obviously, $\mathscr{P}(M) \subset S_R(M)$, $P(z; M; 1) \equiv P(z; M)$.

The following theorem takes place:

Theorem 1. *If, for any fixed $M>1$, $N=4, 5, \ldots$, a function F belongs to $\mathscr{F}_N(M) \cap \mathscr{F}_{N+1}(M)$, then F belongs to $\mathscr{P}(M)$.*

Proof. Let F of form (1) be an arbitrary fixed function belonging to the intersection $\mathscr{F}_N(M) \cap \mathscr{F}_{N+1}(M)$. Then, from the fundamental theorem of I. Dziubiński [3] it follows that the function

$$\zeta = f(z) = F(z)/M = \sum_{n=1}^{\infty} a_n z^n, \quad \text{where} \quad a_1 = 1/M,\ a_n = A_n/M,$$

$n=2, 3, \ldots,$ satisfies the system of differential-functional equations of the form

(4) $$\left(\frac{z\zeta'}{\zeta}\right)^2 \sum_{j=1}^{N} D_{N-j}\left(\zeta^{N-j}+\frac{1}{\zeta^{N-j}}\right)=\sum_{j=1}^{N} E_{N-j}\left(z^{N-j}+\frac{1}{z^{N-j}}\right),$$

(5) $$\left(\frac{z\zeta'}{\zeta}\right)^2 \sum_{j=1}^{N+1} \mathscr{D}_{N-j+1}\left(\zeta^{N-j+1}+\frac{1}{\zeta^{N-j+1}}\right)=\sum_{j=1}^{N+1} \mathscr{E}_{N-j+1}\left(z^{N-j+1}+\frac{1}{z^{N-j+j}}\right),$$

where

$$D_{N-j}=a_N^{(N-j+1)},\quad j=1, 2, \ldots, N-1;\quad 2D_0=-\mathscr{P}_N,$$
$$E_{N-j}=ja_j,\quad j=1, 2, \ldots, N-1;\quad 2E_0=(N-1)a_N-\mathscr{P}_N,$$
$$\mathscr{D}_{N-j+1}=a_{N+1}^{(N-j+2)},\ j=1, 2, \ldots, N;\quad 2\mathscr{D}_0=-\mathscr{P}_{N+1},$$
$$\mathscr{E}_{N-j+1}=ja_j,\quad j=1, 2, \ldots, N;\quad 2\mathscr{E}_0=Na_{N+1}-\mathscr{P}_{N+1},$$
$$\mathscr{P}_N=\min_{0\leq x\leq 2\pi}\sum_{j=2}^{N} D_{j-1}\cos(j-1)x;\ \mathscr{P}_{N+1}=\min_{0\leqq x\leqq 2\pi}\sum_{j=2}^{N+1}\mathscr{D}_{j-1}\cos(j-1)x,$$
$$[f(z)]^m=\sum_{n=m}^{\infty} a_n^{(m)}z^n,\quad m=2, 3, \ldots$$

Dividing (4) by (5), we get the equation

(6) $$\frac{D_{N-1}(\zeta^{N-1}+\zeta^{1-N})+D_{N-2}(\zeta^{N-2}+\zeta^{2-N})+\ldots+D_1(\zeta+\zeta^{-1})+D_0}{\mathscr{D}_N(\zeta^N+\zeta^{-N})+\mathscr{D}_{N-1}(\zeta^{N-1}+\zeta^{1-N})+\ldots+\mathscr{D}_1(\zeta+\zeta^{-1})+\mathscr{D}_0}=z\Re(z),$$

where

(7)
$$\Re(z)=\frac{E_{N-1}z^{2(N-1)}+E_{N-2}z^{2N-3}+\ldots+E_1z^N+E_0z^{N-1}+E_1z^{N-2}+\ldots+E_{N-2}z+E_{N-1}}{\mathscr{E}_Nz^{2N}+\mathscr{E}_{N-1}z^{2N-1}+\ldots+\mathscr{E}_1z^{N+1}+\mathscr{E}_0z^N+\mathscr{E}_1z^{N-1}+\ldots+\mathscr{E}_{N-1}z+\mathscr{E}_N}$$

with that $E_{N-1}=\mathscr{E}_N=M^{-1}$.

In Eq. (6) let us put

(8) $$\omega=\frac{\zeta}{1+a\zeta+\zeta^2},\quad a\in\langle -2, 2\rangle.$$

Eq. (6) will then take the form

(9) $$\omega\,\frac{T_0\omega^{N-1}+T_1\omega^{N-2}+\ldots+T_{N-2}\omega+T_{N-1}}{\mathscr{T}_0\omega^N+\mathscr{T}_1\omega^{N-1}+\ldots+\mathscr{T}_{N-1}\omega+\mathscr{T}_N}=z\Re(z),$$

where $T_{N-1}=D_{N-1}=M^{-N}$ and $\mathscr{T}_N=\mathscr{D}_N=M^{-(N+1)}$.

Moreover, the substitution of (8) made in (4), (5) reduces those equations to the form

(10) $$\frac{\omega^{1-N}}{[1+(a+2)\omega]\,[1-(a-2)\omega]}\left(\frac{z\omega'}{\omega}\right)^2\sum_{j=1}^{N} T_{j-1}\omega^{N-j}=\sum_{j=1}^{N} E_{N-j}(z^{N-j}+z^{j-N}),$$

$$(11)\quad \frac{\omega^{-N}}{[1+(a+2)\omega]\,[1-(a-2)\omega]}\left(\frac{z\omega'}{\omega}\right)^2 \sum_{j=1}^{N+1} \mathcal{T}_{j-1}\omega^{N-j+1} = \sum_{j=1}^{N+1} \mathcal{E}_{N-j+1}(z^{N-j+1}+z^{j-N-1}).$$

The function $\omega=\omega(z)$ defined by Eq. (9) is algebraic and, besides, from (6), (9), (10) we obtain $\omega(0)=0$. In a sufficiently small neighbourhood of the point $\omega=0$ and the point $z=0$ let us expand, respectively, the left- and the right-hand sides of Eq. (9) in power series. We have

$$\omega(M+\lambda_1\omega+\cdots)=z(1+\mu_1 z+\cdots).$$

Hence, by means of the indefinite coefficients method, one can define uniquely the expansion

$$(12)\quad \omega=M^{-1}z+\nu_2 z^2+\cdots$$

in a sufficiently small neighbourhood of the point $z=0$.

If we represent Eq. (9) in the form

$$(13)\quad G(z,\ \omega)=0,$$

then $G(0,\ 0)=0$ and $G'_\omega(0,\ 0)=M^{-1}\neq 0$. Consequently, according to the implicit function theorem, Eq. (13) uniquely determines in a sufficiently small neighbourhood of the point $z=0$ a single-valued and holomorphic function $\omega=\omega(z)$ taking values in a sufficiently small neighbourhood of the point $\omega=0$.

The form of expansion (12) of the function $\omega(z)$ points out that, in a sufficiently small neighbourhood of $z=0$, all branches of this algebraic function are identical. In consequence, all branches of the function $\omega(z)$ are equal for any $z\in\mathbf{C}$. Whereas if an algebraic function is single-valued on $\mathbf{C}$, then it is a rational function. So,

$$(14)\quad \omega=M^{-1}z\,\frac{P_1(z)}{P_2(z)},\quad z\in\mathbf{C},$$

where P_1 and P_2 are assumed to be relatively prime polynomials and $P_1(0)=P_2(0)=1$.

We shall now examine the form of the polynomials P_1 and P_2 in (14). Note first that if $\omega\to 0$, then from (4), (8) it follows (cf. [13]) that $z\to 0$; consequently, the polynomial P_1 has no zeros, which means that $P_1(z)\equiv 1$ for $z\in\mathbf{C}$. In turn, P_2 cannot be constant because the function $\omega=z/M$ does not satisfy (10). Hence it appears that

$$(15)\quad \omega=M^{-1}z\,\frac{1}{1+b_1z+\ldots+b_kz^k},\quad b_k\neq 0,\ k\geq 1.$$

Suppose that $k=1$. If we substitute the function

$$(16)\quad \omega=\frac{M^{-1}z}{1+b_1z},\quad b_1\neq 0,$$

in Eq. (10), then we shall easily find that, for $z\to\infty$, the left-hand side of (10) tends to zero, while its right-hand side tends to ∞. Hence, function (16) does not satisfy system (10)-(11); consequently, $k>1$.

Let $k=1+r$, where $r\geq 1$. Then, from (15) we have

$$\omega=\frac{M^{-1}}{b_{1+r}z^r}\,[1+0(z^{-1})],\quad |z|>\rho, \tag{17}$$

where ρ is sufficiently large.

Let us substitute (17) in Eq. (10), (11). Since

$$\sum_{j=1}^{N} E_{N-j}(z^{N-j}+z^{j-N})=M^{-1}z^{N-1}[1+O(z^{-1})],$$

$$\omega^{1-N}\sum_{j=1}^{N} T_{j-1}\omega^{N-j}=M^{-1}b_{1+r}^{N-1}z^{(N-1)r}[1+O(z^{-1})],$$

$$\frac{z\omega'}{\omega}=-r[1+O(z^{-1})],$$

$$\frac{1}{[1+(a+2)\omega]\,[1-(a-2)\omega]}=1+O(z^{-2r}),$$

therefore from Eq. (10), in a neighbourhood of the point ∞, we obtain

$$r^2M^{-1}b_{1+r}^{N-1}z^{(N-1)r}\,[1+O(z^{-1})]=M^{-1}z^{N-1}[1+O(z^{-1})]. \tag{18}$$

Analogously, from (11) we have

$$r^2M^{-1}b_{1+r}^{N}z^{Nr}\,[1+O(z^{-1})]=M^{-1}z^{N}[1+O(z^{-1})]. \tag{19}$$

So, from (18), (19) we obtain the system of equations

$$\begin{cases} (r-1)\,(N-1)=0\\ (r-1)\,N=0\\ r^2b_{1+r}^{N-1}=1\\ r^2b_{1+r}^{N}=1\end{cases} \tag{20}$$

whence $r=1$, $b_2=1$.

Consequently, function (15) has the form

$$\omega=\frac{M^{-1}z}{z^2+b_1z+1}\,.$$

Hence, by taking account of (8), the function $\zeta=f(z)$ satisfies the equation

$$\frac{\zeta}{\zeta^2+a\zeta+1}=\frac{M^{-1}z}{z^2+b_1z+1}\,. \tag{21}$$

Since f is holomorphic in Δ, therefore $z^2+b_1z+1=(\varepsilon-z)\,(\bar{\varepsilon}-z)$, where $|\varepsilon|=1$ while on the other hand, Eq. (21) determines the function $f\in S_R(M)$ for any $M>1$ only if $b_1=a$. Hence (21) takes the form

$$\frac{\zeta}{(\varepsilon-\zeta)(\bar{\varepsilon}-\zeta)}=\frac{M^{-1}z}{(\varepsilon-z)(\bar{\varepsilon}-z)}\,,\quad |\varepsilon|=1,$$

In consequence, the function $w=F(z)=Mf(z)$ satisfies Eq. (3), and thus belongs to $\mathscr{P}(M)$.

3. It turns out that in Theorem 1 the assumptions concerning the indices of the coefficients being maximized may be changed. We have

Theorem 2. *If, for any fixed $M>1$, $N=p+1$, $2\leq K\leq N-1$, where p is a prime number, a function F belongs to $\mathscr{F}_N(M)\cap\mathscr{F}_K(M)$, then F belongs to $\mathscr{P}(M)$.*

This theorem is proved similarly as Theorem 1. In particular, the assumptions about the numbers N and K are used while proving the form of the expansion of a suitable function ω in series (12). The assumption that p is a prime number is also essential in proving that the system analogous to system (20) has a solution of the form $r=1$, $b_2=1$.

4. To finish with, it is worth noting that, in the limit case (when $M\to+\infty$), the family $\mathscr{P}(+\infty)$ reduces to the function $P(z;\ +\infty;\ \varepsilon)=z/(\varepsilon-z)(\bar{\varepsilon}-z)$, $|\varepsilon|=1$. In particular, for $\varepsilon=1$, we obtain the Koebe function (see the result in the class S_R [2]). What is more, the functions $P(z;\ M;\ \varepsilon)$ are extremal with respect to functional (2) in some class of typically-real functions, investigated by Z. Lewandowski and S. Wajler [10].

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