ON THE EXTREME AND $L^2$ DISCREPANCIES
OF SYMMETRIC FINITE SEQUENCES

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In the present paper we obtain estimates for the extreme and $L^2$ discrepancies of any symmetric finite sequence of points in $s$-dimensional unit cube $E^s=[0,1]^s$. Our estimate for the $L^2$ discrepancy of symmetric sequences is an analogue of Erdős-Turán-Koksma's inequality. The estimate for the extreme discrepancy of symmetric sequences in the case $s=1$ coincides with LeVeque's inequality.

1. Introduction. Let $s \geq 1$ and let $E^s$ denotes the unit cube consisting of points $x=(x_1, \ldots, x_s)$ with $0 \leq x_j \leq 1$ ($j=1, \ldots, s$). Let $X=\{a_k\}_{k=1}^N$ be a finite sequence of points in $E^s$. For every $\gamma=(\gamma_1, \ldots, \gamma_s)$ in $E^s$ we write $A(X;\gamma)$ for the number of terms of $X$ lying in the box $0 \leq x_j < \gamma_j$ ($j=1, \ldots, s$) and put $D(X;\gamma)=N^{-1}A(X;\gamma)-\gamma_1 \cdots \gamma_s$. The numbers

$$D(X)=\sup_{\gamma \in E^s} |D(X;\gamma)| \quad \text{and} \quad T(X)=(\int_{E^s} |D(X;\gamma)|^2 d\gamma)^{1/2}$$

are called the extreme and $L^2$ discrepancies of $X$, respectively.

We shall make use of the following notations:

For every integer $m$ we write $m=\max \{1, |m|\}$.

For every lattice point $m=(m_1, \ldots, m_s)$ in $Z^s$ we define

$$||m||=\max_{1 \leq j \leq s} |m_j| \quad \text{and} \quad R(m)=m_1 \cdots m_s.$$

For $\alpha=(\alpha_1, \ldots, \alpha_s)$ and $\beta=(\beta_1, \ldots, \beta_s)$ in $R^s$, let $\langle \alpha, \beta \rangle$ denote the standard inner product, that is $\langle \alpha, \beta \rangle=\alpha_1 \beta_1 + \cdots + \alpha_s \beta_s$.

It is well known that for any finite sequence $X=\{a_k\}_{k=1}^N$ in $E^s$ and any natural number $n$ we have

$$D(X) \leq c(s) \left( \frac{1}{n} + \sum_{0 < ||m|| \leq n} (R(m))^{-2} \right) \left( \frac{1}{N} \sum_{k=1}^N e^{2\pi i \langle m, a_k \rangle} \right),$$

where $c(s)>0$ is an absolute constant depending only on the dimension $s$.

It is also well known that in the case $s=1$ the discrepancy $D(X)$ of any finite sequence $X=\{a_k\}_{k=1}^N$ in $E=[0,1]$ satisfies

$$D(X) \leq \left( \frac{6}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \right) \left( \frac{1}{N} \sum_{k=1}^N e^{2\pi i m a_k} \right)^{1/3}.$$

Inequalities (1) and (2) are called Erdős-Turán-Koksma's inequality (see [1], p. 116) and LeVeque's inequality (see [1], p. 111), respectively.

In the present paper we obtain an inequality (see Theorem 2) for the $L^3$ discrepancy of the so-called symmetric finite sequences in $E^s$ which is an analogue of Erdős-Turán-Koksma’s inequality. For the extreme discrepancy of the symmetric finite sequences in $E^s$ we obtain an inequality (see Theorem 3) which in the case $s=1$ coincides with LeVeque’s inequality.

2. **Symmetric finite sequences.** Let $X = \{a_k\}_{k=1}^N$ be a finite sequences in $E^s$. We shall say that a given point $x = (x_1, \ldots, x_s)$ in $E^s$ has a multiplicity $p(0 \leq p \leq s)$ with respect to the sequence $X$ if exactly $p$ terms of $X$ coincide with the point $x$. We call the sequence $X$ a symmetric one if for any point $x = (x_1, \ldots, x_s)$ in $E^s$ all points of the type

$$
(\tau_1 + (-1)^{\tau_1}x_1, \ldots, \tau_s + (-1)^{\tau_s}x_s)
$$

have one and the same multiplicity with respect to $X$, when $\tau_1, \ldots, \tau_s$ take independently the values 0 and 1.

It is obvious that if a sequence $X$ in $E^s$ is symmetric then any other sequence $Y$ in $E^s$ originating from $X$ by means of a transposition of its terms, is also symmetric.

Let $X$ be a finite sequence consisting of $N$ points in $E^s$ and let $\hat{X}$ be a symmetric sequence, consisting of $M = 2^sN$ points in $E^s$. We say that the symmetric sequence $\hat{X}$ is produced by the sequence $X$ if for every point $x = (x_1, \ldots, x_s)$ in $E^s$ the following is valid: if a point $x$ is a term of the sequence $X$, then each point of type (3) is a term of the sequence $\hat{X}$, where $\tau_1, \ldots, \tau_s$ take the values 0 and 1.

Apparently, each sequence $X$ in $E^s$ produces at least one symmetric sequence $\hat{X}$ in $E^s$. The inverse, of course, is true, too: if $\hat{X}$ is a symmetric sequence, consisting of $M$ points in $E^s$ and if $M = 0 \pmod{2^s}$ then there exists at least one sequence $X$ in $E^s$ which produces the sequence $\hat{X}$.

3. **Estimates for the $L^3$ discrepancy of symmetric finite sequences.**

Theorem 1. Let $\hat{X}$ be a symmetric finite sequence, consisting of $M = 2^sN$ terms in $E^s$ and let $X = \{a_k\}_{k=1}^N$ be any finite sequence in $E^s$ producing $\hat{X}$. Then

$$
T(\hat{X}) \leq c(s) \sum_{|m| > 0} |R(m)|^{-2} \frac{1}{N} \sum_{k=1}^N e^{2\pi i (m, a_k)} |y|^s,
$$

where

$$
c(s) = (3/4\pi^2)(1 - 2^{-s} + 3^{-s}).
$$

Proof. Let $X = \{a_k\}_{k=1}^N$ be a given sequence in $E^s$, and let $\hat{X} = \{b_k\}_{k=1}^N$, where $M = 2^sN$, be a symmetric sequence in $E^s$ which is produced by $X$. We put

$$
a_k = (\xi_1(k), \ldots, \xi_s(k)), \quad k = 1, \ldots, N.
$$

For $\gamma = (\gamma_1, \ldots, \gamma_s)$ in $E^s$, let $\varphi_\gamma(x) = \varphi_\gamma(x_1, \ldots, x_s)$ be the characteristic function of the box $0 \leq x_j < \gamma_j$ ($j = 1, \ldots, s$). Then

$$
A(\hat{X}; \gamma) = \sum_{k=1}^N \varphi_\gamma(b_k) = \sum_{k=1}^N \sum_{\tau_1 = 0}^{1} \varphi_\gamma(\tau_1 + (-1)^{\tau_1} \xi_1(k), \ldots, \tau_s + (-1)^{\tau_s} \xi_s(k)).
$$
Therefore

\[ D(\vec{X}; \gamma) = \frac{1}{2N} \sum_{k=1}^{N} \sum_{\tau_1, \ldots, \tau_s=0}^{1} \phi_s(a_1(k), \ldots, a_s(k)) - \gamma_1 \ldots \gamma_s, \]

where, for brevity, we have put \( a_j(k) = \tau_j + (-1)^j \tau_j(k) \), \( j = 1, \ldots, s \).

The function \( D(\gamma_1, \ldots, \gamma_s) = D(\vec{X}; \gamma) \) is a piecewise continuous function in \( E^s \). Let

\[ \sum_{m_1, \ldots, m_s=-\infty}^{\infty} c(m_1, \ldots, m_s) e^{2\pi i (m_1 \gamma_1 + \ldots + m_s \gamma_s)} \]

be the Fourier series of \( D(\gamma_1, \ldots, \gamma_s) \). It is well known that the Fourier coefficients are given by

\[ c(m_1, \ldots, m_s) = \int_{E^s} D(\gamma_1, \ldots, \gamma_s) e^{-2\pi i (m_1 \gamma_1 + \ldots + m_s \gamma_s)} d\gamma_1 \ldots d\gamma_s. \]

From (7) and (8) it follows that

\[ c(m_1, \ldots, m_s) = \frac{1}{N^{2s}} \sum_{k=1}^{N} \sum_{\tau_1, \ldots, \tau_s=0}^{1} \prod_{j=1}^{s} \int_{a_j(k)}^{1} e^{-2\pi i m \tau_j} d\gamma_j \]

\[ - \prod_{j=1}^{s} \int_{0}^{1} \gamma_j e^{-2\pi i m \tau_j} d\gamma_j \]

\[ = \frac{1}{N^{2s}} \sum_{k=1}^{N} \prod_{j=1}^{s} \sum_{\tau_j=1}^{1} A(m_j, a_j(k)) - \prod_{j=1}^{s} B(m_j), \]

where we use the following notations

\[ A(m, a) = \int_{a}^{1} e^{-2\pi i m \gamma} d\gamma \quad B(m) = \int_{0}^{1} \gamma e^{-2\pi i m \gamma} d\gamma. \]

For every real number \( a \) and every integer \( m \) we have

\[ A(m, a) = \begin{cases} (1/2 \pi i m) (e^{-2\pi i m a} - 1), & \text{if } m \neq 0; \\ 1-a, & \text{if } m = 0; \end{cases} \]

and

\[ B(m) = \begin{cases} -1/2 \pi i m, & \text{if } m \neq 0; \\ 1/2, & \text{if } m = 0. \end{cases} \]

We verify immediately that

\[ \sum_{\tau_j=0}^{1} A(0, a_j(k)) = 1. \]

By (9), (11) and (12) we obtain

\[ c(0, \ldots, 0) = \frac{1}{N^{2s}} \sum_{k=1}^{N} 1 - \frac{1}{2^s} = 0. \]
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Then by Parseval's identity we have

$$ (T(\tilde{X}))^2 = \int_{E^s} |D(\gamma_1, \ldots, \gamma_s)\|^2 d\gamma_1 \ldots d\gamma_s $$

$$ = \sum_{m_1, \ldots, m_s = -\infty}^{\infty} |c(m_1, \ldots, m_s)|^2 = \sum_{m_1, \ldots, m_s = -\infty}^{\Sigma^*} |c(m_1, \ldots, m_s)|^2, $$

where $\Sigma^*$ means that $(m_1, \ldots, m_s) \neq (0, \ldots, 0)$.

Let $p$ be a natural number with $1 \leq p \leq s$ and let $\{j_1, \ldots, j_p\}$ be an arbitrary subset of the set $\{1, 2, \ldots, s\}$. We shall introduce the notation

$$ \sigma(j_1, \ldots, j_p) = \sum_{m_1, \ldots, m_s = -\infty}^{\Sigma^*} |c(m_1, \ldots, m_s)|^2, $$

where the sum $\Sigma^*$ is over $(m_1, \ldots, m_s)$ in $\mathbb{Z}^s$ with $m_j \neq 0$ ($1 \leq j \leq s$) if and only if $j$ coincides with any of the numbers $j_1, \ldots, j_p$.

It is easy to see that formula (13) can be written in the form

$$ (T(\tilde{X}))^2 = \sum_{p=1}^{s} \sum_{j_1, \ldots, j_p} \sigma(j_1, \ldots, j_p), $$

where the inner sum is over $\{j_1, \ldots, j_p\} \subset \{1, 2, \ldots, s\}$.

Now let $p$ be again a given natural number with $1 \leq p \leq s$ and let $\{j_1, \ldots, j_p\}$ be a given subset of $\{1, 2, \ldots, s\}$. We shall obtain an estimate for the sum $\sigma(j_1, \ldots, j_p)$ From lower exposition, it will become clear that, without any loss of generality, we can assume that $\{j_1, \ldots, j_p\}$ coincides with $\{1, 2, \ldots, p\}$.

Let us suppose that $\{j_1, \ldots, j_p\} = \{1, 2, \ldots, p\}$. Then

$$ \sigma(j_1, \ldots, j_p) = \sum_{m_1, \ldots, m_p = -\infty}^{\infty} |c(m_1, \ldots, m_p, 0, \ldots, 0)|^2. $$

Let $m_1 \neq 0, \ldots, m_p \neq 0$ ($1 \leq p \leq s$). Then from (9), (11) and (12) we obtain

$$ c(m_1, \ldots, m_p, 0, \ldots, 0) $$

$$ = \frac{1}{N^{2s}} \cdot \frac{1}{(2\pi)^p} \cdot \sum_{m_1, \ldots, m_p}^{\mathcal{N}} \prod_{j=1}^{p} \sum_{k=1}^{1} \frac{1}{\tau_j} \left( e^{-2\pi i m_j \xi_j(k)} - 1 \right) - \frac{(-1)^p}{(2\pi)^p} \frac{1}{2^{s-p}} \frac{1}{m_1 \ldots m_p}. $$

We make the following transformations

$$ \prod_{j=1}^{p} \sum_{\tau_j=0}^{1} \left( e^{-2\pi i m_j \xi_j(k)} - 1 \right) = \prod_{j=1}^{p} \left( e^{2\pi i m_j \xi_j(k)} + e^{-2\pi i m_j \xi_j(k)} - 2 \right) $$

$$ = \sum_{\varepsilon_1, \ldots, \varepsilon_p} \beta(\varepsilon_1, \ldots, \varepsilon_p) e^{2\pi i (\varepsilon_1 c_1 \xi_1(k) + \ldots + \varepsilon_p m_p \xi_p(k))}, $$

where the sum is over $\varepsilon_1, \ldots, \varepsilon_p$ which take the values $0, -1$ and $1$ (their number is $3^p$).

It is easily seen, that

$$ \beta(0, \ldots, 0) = (-1)^p 2^p $$
and

$$\beta(\varepsilon_1, \ldots, \varepsilon_p) \leq 2^{p-1} \text{ for } (\varepsilon_1, \ldots, \varepsilon_p) = (0, \ldots, 0).$$

Now, using (18), from (16) and (17) we conclude that

$$\frac{1}{m_1 \ldots m_p} \sum' \beta(\varepsilon_1, \ldots, \varepsilon_p) S(\varepsilon_1 m_1, \ldots, \varepsilon_p m_p, 0, \ldots, 0),$$

where $\sum'$ means that $(\varepsilon_1, \ldots, \varepsilon_p) \neq (0, \ldots, 0)$ and where for a lattice point $m = (m_1, \ldots, m_p)$ in $\mathbb{Z}^p$ we use the notation

$$S(m_1, \ldots, m_p) = \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i (m_1 \tilde{a}_1(k) + \ldots + m_p \tilde{a}_p(k))}.$$  

From (20) and (19) we obtain the estimate

$$|c(m_1, \ldots, m_p, 0, \ldots, 0)| \leq \frac{1}{2^{p-1} \pi^p} \sum' \frac{|S(\varepsilon_1 m_1, \ldots, \varepsilon_p m_p, 0, \ldots, 0)|}{m_1 \ldots m_p}.$$  

From the latter estimate and the well-known inequality $(\sum_{j=1}^{n} u_j)^2 \leq n \sum_{j=1}^{n} u_j^2$ we find

$$|c(m_1, \ldots, m_p, 0, \ldots, 0)|^2 \leq \frac{3^{p-1}}{4^{p-1} \pi^p} \sum' \frac{|S(\varepsilon_1 m_1, \ldots, \varepsilon_p m_p, 0, \ldots, 0)|^2}{(m_1 \ldots m_p)^2}.$$

From that inequality and from (15) we get

$$\sigma(j_1, \ldots, j_p) \leq \frac{3^{p-1}}{4^{p-1} \pi^p} \sum' \frac{|S(\varepsilon_1 m_1, \ldots, \varepsilon_p m_p, 0, \ldots, 0)|^2}{(m_1 \ldots m_p)^2}.$$  

Let us consider the inner sum of that inequality with fixed $\varepsilon_1, \ldots, \varepsilon_p$ and denote it by $\Omega(\varepsilon_1, \ldots, \varepsilon_p)$. Let $q(1 \leq q \leq p)$ be the number of those $\varepsilon_j$ ($j = 1, \ldots, s$) which are not equal to zero. With no loss of generality we can assume that $\varepsilon_1 = 0, \ldots, \varepsilon_q = 0$. Then

$$\Omega(\varepsilon_1, \ldots, \varepsilon_p) = \sum_{m_1, \ldots, m_p = -\infty}^{\infty} \frac{|S(\varepsilon_1 m_1, \ldots, \varepsilon_p m_p, 0, \ldots, 0)|^2}{(m_1 \ldots m_p)^2}.$$  

$$= \frac{\pi^2}{3} \sum_{m_1, \ldots, m_p = -\infty}^{\infty} \frac{|S(\varepsilon_1 m_1, \ldots, \varepsilon_p m_p, 0, \ldots, 0)|^2}{(m_1 \ldots m_p)^2}.$$  

$$\leq \frac{\pi^2}{3} \sum_{m_1, \ldots, m_p = -\infty}^{\infty} \frac{|S(\varepsilon_1 m_1, \ldots, \varepsilon_p m_p, 0, \ldots, 0)|^2}{(m_1 \ldots m_p)^2}.$$  

$$\leq \frac{\pi^2}{3} \sum_{m_1, \ldots, m_p = -\infty}^{\infty} \frac{|S(\varepsilon_1 m_1, \ldots, \varepsilon_p m_p, 0, \ldots, 0)|^2}{(m_1 \ldots m_p)^2}.$$
From here and from (22) we obtain

\[
\sigma(j_1, ..., j_p) \leq \frac{3(3^p - 1)}{4 + 1 + 3^p} \sum_{m_1, ..., m_s = -\infty}^{\infty} \frac{S(m_1, ..., m_s)^2}{(m_1, ..., m_s)^2} \sum_{\epsilon_1, ..., \epsilon_p} 1
\]

\[
= \frac{3(3^p - 1)}{4 + 1 + 3^p} \sum_{m_1, ..., m_s = -\infty}^{\infty} \frac{S(m_1, ..., m_s)^2}{(m_1, ..., m_s)^2} .
\]

We obtained the estimate (23) under the condition that \{j_1, ..., j_p\} coincides with \{1, 2, ..., p\}. However, it is evident from the proof that it holds for an arbitrary subset of the set \{1, 2, ..., s\}.

From (14) and (23), using

\[
\sum_{\rho=1}^{s} \frac{(3^\rho - 1)^2}{3^\rho} = \sum_{\rho=1}^{s} \frac{(3^\rho - 1)^2}{3^\rho} \binom{s}{\rho} = 4^s - 2^{s+1} + (4,3)^s
\]

we obtain the estimate

\[
(T(\widetilde{X}))^2 \leq \frac{3}{4^s + 1} \sum_{m_1, ..., m_s = -\infty}^{\infty} \frac{S(m_1, ..., m_s)^2}{(m_1, ..., m_s)^2} \sum_{\rho=1}^{s} \frac{(3^\rho - 1)^2}{3^\rho} \sum_{j_1, ..., j_p} 1
\]

\[
= C(s) \sum_{m_1, ..., m_s = -\infty}^{\infty} \frac{S(m_1, ..., m_s)^2}{(m_1, ..., m_s)^2} ,
\]

where the constant \(C(s)\) is defined by Eq. (5). Obviously the estimate (24) coincides with (4). Thus Theorem 1 is proved.

It should be mentioned that Theorem 1 (with a different constant) has been proved first in [2], but with a certain inaccuracy in our proof. We express our gratitude to Professor V. Popov (Sofia) for drawing our attention to it.

From Theorem 1 for \(s=1\) we get the following assertion.

**Corollary 1.** Let \(X\) be a symmetric finite sequence consisting of \(M=2^sN\) terms in \(E=[0, 1]\) and let \(X=(a_k)_{k=1}^N\) be any finite sequence in \(E\) producing \(X\). Then

\[
T(\widetilde{X}) \leq (\frac{1}{2\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i a_k m^2})^{1/2} .
\]

We shall note that this estimate is precise. Indeed, if \(a_1 = a_2 = ... = a_N = 0\) then \(X\) will consist of \(N\) zeros and \(N\) ones. Now from (25) we obtain

\[
T(\widetilde{X}) \leq (\frac{1}{2\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2})^{1/2} = 1/\sqrt{12} ,
\]

and from the definition of \(L^2\) discrepancy we find that

\[
T(\widetilde{X}) = (\frac{1}{0} (\frac{1}{2} - \gamma)^2 d\gamma)^{1/2} = 1/\sqrt{12} .
\]

Therefore in this case (25) is an equality.
The following theorem is an analogue of Erdős-Turán-Koksma's inequality (1).

**Theorem 2.** Let \( X = \{ a_k \}_{k=1}^N \) be any symmetric finite sequence in \( E^s \). Then

\[
T(X) \leq \left( C(s) \sum_{|m| > 0} (R(m))^{-2} \left| \frac{1}{N} \sum_{k=1}^N e^{2\pi i (m \cdot a_k)} \right|^2 \right)^{1/2},
\]

where the constant \( C(s) \) is defined by (5).

**Proof.** Let \( X = \{ a_k \}_{k=1}^N \) be a given symmetric sequence in \( E^s \) and let \( \tilde{X} = \{ b_k \}_{k=1}^N \), where \( M = 2^s N \), be another symmetric sequence in \( E^s \) which is produced by \( X \). Then for every point \( \gamma \) in \( E^s \) we have \( A(\tilde{X}; \gamma) = 2^s A(X; \gamma) \). Therefore \( D(\tilde{X}; \gamma) = M^{-1} A(\tilde{X}; \gamma) = M^{-1} A(X; \gamma) = D(X; \gamma) \), and so

\[
T(X) = T(\tilde{X}).
\]

It follows from Theorem 1 that for the symmetric sequence \( \tilde{X} \) will be valid the estimate (4). Now from (4) and (27) we get (26). Theorem 2 is proved.

We proved that Theorem 2 follows from Theorem 1. It is not difficult to prove the inverse. Indeed, let \( X = \{ a_k \}_{k=1}^N \) be an arbitrary sequence in \( E^s \) and let \( \tilde{X} = \{ b_k \}_{k=1}^M \) be a symmetric sequence which is produced by \( X \).

From (26) we shall have

\[
(T(\tilde{X}))^2 \leq C(s) \sum_{|m| > 0} (R(m))^{-2} \left| \frac{1}{M} \sum_{k=1}^M e^{2\pi i (m \cdot b_k)} \right|^2.
\]

Using the notation (6) and (21) we can write

\[
\frac{1}{M} \sum_{k=1}^M e^{2\pi i (m \cdot b_k)} = \frac{1}{2s} \sum_{k=1}^N \sum_{\tau_1, \ldots, \tau_s = 0} e^{2\pi i (\tau_1 m_1 + \ldots + \tau_s m_s)}(k)
\]

\[
\frac{1}{2s} \sum_{\tau_1, \ldots, \tau_s = 0} S((-1)^{\tau_1} m_1, \ldots, (-1)^{\tau_s} m_s).
\]

From (28) and (29) it follows that

\[
(T(\tilde{X}))^2 \leq C(s) \frac{1}{2s} \sum_{\tau_1, \ldots, \tau_s = 0} |S((-1)^{\tau_1} m_1, \ldots, (-1)^{\tau_s} m_s)|^2
\]

\[
= C(s) \frac{1}{2s} \sum_{m_1, \ldots, m_s = -\infty} \frac{|S(m_1, \ldots, m_s)|^2}{(m_1 \ldots m_s)^2}
\]

\[
= C(s) \frac{1}{2s} \sum_{m_1, \ldots, m_s = -\infty} \frac{|S(m_1, \ldots, m_s)|^2}{m_1 \ldots m_s}
\]

\[
= C(s) \sum_{|m| > 0} (R(m))^{-2} \left| \frac{1}{N} \sum_{k=1}^N e^{2\pi i (m \cdot a_k)} \right|^2.
\]

We see that Theorem 1 is equivalent to Theorem 2.
In the case $s=1$, it follows from Theorem 2 that for any symmetric sequence $X=\{a_k\}_{k=1}^N$ in $E$ we have

\[
T(X) \leq \left( \frac{1}{2\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \right)^{1/2} \left( \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i m a_k} \right)^{1/2},
\]

but from Koksma's equality (see [1], p. 110):

\[
(T(X))^2 = \left( \frac{1}{N} \sum_{k=1}^{N} (a_k - \frac{1}{2}) \right)^2 + \frac{1}{2\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \left( \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i m a_k} \right)^2,
\]

which holds for any sequence $X=\{a_k\}_{k=1}^N$ in $E$ it follows that

\[
T(X) \geq \left( \frac{1}{2\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \right)^{1/2} \left( \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i m a_k} \right)^{1/2}.
\]

That inequality shows that in fact we have in (30) an equality.

**Corollary 2.** Let $X=\{a_k\}_{k=1}^N$ be any symmetric finite sequence in $E$. Then

\[
T(X) = \left( \frac{1}{2\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \right)^{1/2} \left( \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i m a_k} \right)^{1/2}.
\]

Of course, corollary 2 can also be derived only from Koksma's equality.

**4. An estimate for the extreme discrepancy of symmetric finite sequences.** In the following theorem we shall show that for symmetric sequences LeVeque's inequality (2) can be generalized for any dimension $s \geq 1$.

**Theorem 3.** Let $X=\{a_k\}_{k=1}^N$ be any symmetric finite sequence in $E^s$. Then

\[
D(X) \leq C_1(s) \sum_{|m| > 0} (R(m))^{-2} \left( \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i \langle m, a_k \rangle} \right)^{(s+2)},
\]

where $C_1(s) > 0$ is an absolute constant depending only on $s$ and $C_1(1) = 3/\pi^2$.

**Proof.** It is known (see [3], Theorem 4.2 and Corollary 1.2) that for any sequence $X=\{a_k\}_{k=1}^N$ in $E^s$ the following inequality holds

\[
(D(X))^{(s+2)/2} \leq C_2(s)T(X),
\]

where $C_2(s)$ is an absolute constant depending only on $s$ and $C_2(1) = \sqrt{12}$. Therefore

\[
D(X) \leq (C_2(s)T(X))^{2/(s+2)}.
\]

Now, if $X$ is a symmetric sequence from (32) and from Theorem 2 we get the estimate (31) with the constant $C_1(s) = (C_2(s))^2C(s)^{(s+2)}$, where the constant $C(s)$ is defined by (5). It is easy to verify that $C_1(1) = 3/\pi^2$. Theorem 3 is proved.

Finally, we shall remark that for $s=1$ the estimate (31) coincides with LeVeque's inequality (2).

**REFERENCES**

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