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ON THE L^2 DISCREPANCY OF SOME INFINITE SEQUENCES

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In the present paper we consider the problem of finding infinite sequences of points in s-dimensional unit cube $E^s=[0, 1]^s$, the L^2 discrepancy of which has the best possible order (tending to zero). We solve completely the problem in the case $s=1$ by constructing infinite sequences of points in $E=[0, 1]$, the L^2 discrepancy of which has the order $O(N^{-1}(\log N)^{1/2})$. In the case $s\geqq 2$ we reduce the problem of finding infinite sequences in E^s, whose L^2 discrepancy has the order $O(N^{-1}(\log N)^{s/2})$ to the problem of finding the so-called points of constant type in $\mathbf{R}^s$.

1. Introduction. For $s\geqq 1$ let $E^s=[0, 1]^s$ be the s-dimensional unit cube. Let $X=\{a_k\}_{k=1}^N$ be a finite sequence of points in E^s. For every $\gamma=(\gamma_1, \ldots \gamma_s)$ in E^s we put $D(X; \gamma)=N^{-1}A(X; \gamma)-\gamma_1 \ldots \gamma_s$, where $A(X; \gamma)$ is the number of terms of X lying in the box $[0, \gamma_1)\times \ldots \times[0, \gamma_s)$.

The number $T(X)=(\int_{E^s}|D(X; \gamma)|^2 d\gamma)^{1/2}$ is called the L^2 discrepancy of X.

Now let $\sigma=\{a_k\}_{k=1}^\infty$ be an infinite sequence of points in E^s. For every natural number N we form the finite sequence $\sigma_N\{a_k\}_{k=1}^N$, where $a_k(k=1, \ldots, N)$ are the first N terms of σ, and put $T_N(\sigma)=T(\sigma_N)$.

It is known (see [1]) that the sequence σ is uniformly distributed in E^s if and only if $\lim_{N\to\infty} T_N(\sigma)=0$.

Niederreiter (see [1]) proved that for any infinite sequence σ in E^s we have

$$T_N(\sigma)>c(s)N^{-1}(\log N)^{s/2} \tag{1}$$

for infinitely many natural numbers N, where $c(s)>0$ is an absolute constant depending only on s.

It is not known whether the estimate (1) has the best possible order.

Halton [2] and Sobol [3; 4] construct infinite sequences σ in E^s for which $T_N(\sigma)=O(N^{-1}(\log N)^s)$.

We shall dwell more fully upon the one-dimensional case.

Let α be any irrational number having a continued fraction with bounded partial quotients. From Ostrowski's studies [5, p. 95] follows that for the sequence

$$\sigma=(\{\alpha\}, \{2\alpha\}, \{3\alpha\}, \ldots) \tag{2}$$

in $E=[0, 1]$ holds the estimate

$$T_N(\sigma)=O(N^{-1}\log N). \tag{3}$$

(Here $\{\alpha\}$ denotes the fractional part of α.) Niederreiter [1] showed that the estimate (3) is the best possible for the sequence (2).

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Later Van der Corput [6, Theorem 4] constructed another sequence σ in E for the L^2 discrepancy of which the estimate (3) holds, too. Haber [7] showed that the estimate (3) is the best possible one for the sequence of Van der Corput as well.

In the present paper, in the case $s=1$, we construct infinite sequences $\widetilde{\sigma}$ in E for which holds estimate

$$T_N(\widetilde{\sigma})=O(N^{-1}(\log N)^{1/2}). \tag{4}$$

It follows from (1) that the estimate (4) cannot be improved for any infinite sequence in E. Thus we prove in particular that for $s=1$ the estimate (1) of Niederreiter does not admit an improvement.

In the case $s\geqq 2$ we reduce the problem of finding infinite sequences $\widetilde{\sigma}$ in E^s such that

$$T_N(\widetilde{\sigma})=O(N^{-1}(\log N)^{s/2}) \tag{5}$$

to the problem of finding the so-called points of constant type in $\mathbf{R}^s$.

We shall remark that the main results of the present paper (Theorem 3 and Theorem 4) are published without proof in [8].

2. Notations. In this paper we shall make use of the following notations:

We shall denote the set of all real numbers by $\mathbf{R}$, the set of all integers by $\mathbf{Z}$, and the set of all natural numbers by $\mathbf{N}$.

For a real number x, we use $[x]$ to denote the integral part of x, $\{x\}$ to denote the fractional part of x, and $\|x\|$ to denote the distance of x from the nearest integer. Thus $x=[x]+\{x\}$, $\|x\|=\min(\{x\}, 1-\{x\})$.

For an integer m, we write $\overline{m}=\max(1, |m|)$.

For a lattice point $m=(m_1, \ldots, m_s)$ in $\mathbf{Z}^s$, we define $|m|=|m_1|+\ldots+|m_s|$ and $R(m)=\overline{m}_1 \ldots \overline{m}_s$.

For $\alpha=(\alpha_1, \ldots, \alpha_s)$ and $\beta=(\beta_1, \ldots, \beta_s)$ in $\mathbf{R}^s$, let $\langle\alpha, \beta\rangle$ denote the standard inner product, that is $\langle\alpha, \beta\rangle=\alpha_1\beta_1+\ldots+\alpha_s\beta_s$.

For a finite set M, let $|M|$ denote the number of elements of M.

For every function f Riemann-integrable in E^s, we define $\|f\|_{L_2}=(\int_{E^s}|f(x)|^2dx)^{1/2}$.

3. Infinite symmetric sequences. First of all we shall remind some of the definitions given in [9].

Let $X=\{a_k\}_{k=1}^N$ be a finite sequence in E^s, and let $x=(x_1, \ldots, x_s)$ be a point in E^s. We say that the point x has a multiplicity $p(0\leq p\leq s)$ with respect to X, if exactly p terms of X coincide with x. The sequence X is called a symmetric one, if for any point $x=(x_1, \ldots, x_s)$ in E^s all points of the type

$$(\tau_1+(-1)^{\tau_1}x_1, \ldots, \tau_s+(-1)^{\tau_s}x_s) \tag{6}$$

have one and the same multiplicity with respect to X, when $\tau_1, \ldots, \tau_s$ take independently the values 0 and 1.

Let $\widetilde{X}=\{b_k\}_{k=1}^M$ be a symmetric sequence in E^s. We say that the sequence $\widetilde{X}$ is produced by $X=\{a_k\}_{k=1}^N$ if $M=2^sN$ and if the following holds: if a point $x=(x_1, \ldots, x_s)$ is a term of the sequence X, then each point of type (6) is a term of the sequence $\widetilde{X}$, where $\tau_1, \ldots, \tau_s$ take the values 0 and 1.

Now, we shall define the notion symmetric infinite sequence. Let $\tilde{\sigma}=\{b_k\}_{k=1}^{\infty}$ be a infinite sequence in E^s. We shall call the sequence $\tilde{\sigma}$ a symmetric one if for any natural number n the finite sequence

$$(7) \qquad b_{(n-1)p+1}, \quad b_{(n-1)p+2}, \ldots, \quad b_{np},$$

where $p=2^s$, is symmetric.

Every point x in E^s can be regarded as one-term sequence, so that every point x in E^s produces as least one symmetric sequence in E^s, consisting of $p=2^s$ points. Conversely, every symmetric sequence in E^s consisting of $p=2^s$ terms is produced by any of its term.

We shall say that the symmetric infinite sequence $\tilde{\sigma}=\{b_k\}_{k=1}^{\infty}$ is produced by the infinite sequence $\sigma=\{a_k\}_{k=1}^{\infty}$, if for any natural number n the finite sequence (7) is produced by the point a_n.

Obviously, every symmetric infinite sequence in E^s is produced exactly by one infinite sequence in E^s, and every infinite sequence in E^s produces at least one symmetric infinite sequence.

Lemma 1. *Let σ be an infinite sequence in E^s, and let n and N be integers with $1\leqq n\leqq N$. Then $NT_N(\sigma)\leqq nT_n(\sigma)+N-n$.*

Proof. Let $\sigma=\{a_k\}_{k=1}^{\infty}$. For every natural number N we form the finite sequence $\sigma_N=\{a_k\}_{k=1}^{N}$, where a_k $(k=1,\ldots,N)$ are the first N terms of σ. For every $\gamma=(\gamma_1,\ldots,\gamma_s)$ in E^s, we put $A_N(\gamma)=A(\sigma_N;\gamma)$, $D_N(\gamma)=D(\sigma_N;\gamma)$ and $V(\gamma)=\gamma_1\ldots\gamma_s$. Then

$$(8) \qquad T_N(\sigma)=\|D_N(\gamma)\|_{L^2}.$$

Let now n and N be given integers with $1\leqq n\leqq N$. Obviously $A_N(\gamma)=A_n(\gamma)+p(\gamma)$, where the function $p(\gamma)$ satisfies the inequality

$$(9) \qquad 0\leqq p(\gamma)\leqq N-n.$$

It follows from here, that

$$(10) \qquad ND_N(\gamma)=A_N(\gamma)-NV(\gamma)=A_n(\gamma)-nV(\gamma)+q(\gamma)=nD_n(\gamma)+q(\gamma),$$

where

$$(11) \qquad q(\gamma)=p(\gamma)-(N-n)V(\gamma).$$

Taking into consideration that $0\leqq V(\gamma)\leqq 1$, from (9) and (11) we conclude that

$$(12) \qquad |q(\gamma)|\leqq N-n$$

for every point γ in E^s.

Now from (8), (10) and (12) we get

$$NT_N(\sigma)=\|nD_n(\gamma)+q(\gamma)\|_{L^2}\leqq n\|D_n(\gamma)\|_{L^2}+\|q(\gamma)\|_{L^2}=nT_n(\sigma)+\|q(\gamma)\|_{L^2}$$
$$\leqq nT_n(\sigma)+N-n.$$

Lemma 1 is proved.

Theorem 1. *Let $\sigma=\{a_k\}_{k=1}^{\infty}$ be an infinite sequence in E^s and let $\tilde{\sigma}=\{b_k\}_{k=1}^{\infty}$ be any symmetric infinite sequence in E^s produced by σ. Then for every integer $N\geqq 2^s$*

$$T_N(\widetilde{\sigma})\leqq\Big(c(s)\sum_{|m|>0}(R(m))^{-2}\Big|\frac{1}{q}\sum_{k=1}^{q}e^{2\pi i\langle m,\,a_k\rangle}\Big|^2\Big)^{1/2}+\frac{2^s-1}{N}, \tag{13}$$

where the outer sum is over $m\in\mathbf{Z}^s$ *with* $|m|>0$, $q=[N/2^s]$, *and*

$$c(s)=(3/4\ \pi^2)(1-2^{-s+1}+3^{-s}). \tag{14}$$

Proof. We put $n=q2^s$. It is obvious that the finite sequence $\widetilde{\sigma}_n=\{b_k\}_{k=1}^n$ which consists of the first n terms of the sequence $\widetilde{\sigma}$ is produced by the finite sequence $\sigma_q=\{a_k\}_{k=1}^q$ which consists of the first q terms of the sequence σ. Therefore (see [9, Theorem 1])

$$T_n(\widetilde{\sigma})=T(\widetilde{\sigma}_n)\leqq\Big(c(s)\sum_{|m|>0}(R(m))^{-2}\Big|\frac{1}{q}\sum_{k=1}^{q}e^{2\pi i\langle m,\,a_k\rangle}\Big|^2\Big)^{1/2}, \tag{15}$$

where the constant $c(s)$ is defined by (14). Since $q=[N/2^s]$ then $q2^s\leqq N<(q+1)2^s$
and therefore $1\leqq n\leqq N$ and $N-n\leqq 2^s-1$.

From Lemma 1 we have $NT_N(\widetilde{\sigma})\leqq nT_n(\widetilde{\sigma})+N-n\leqq NT_n(\widetilde{\sigma})+2^s-1$ or $T_N(\widetilde{\sigma})\leqq T_n(\widetilde{\sigma})+(2^s-1)/N$. From here and from (15) we obtain (13). Theorem 1 is proved.

Theorem 2. *Let* $\alpha=(\alpha_1,\ldots,\alpha_s)$ *be a point in* $\mathbf{R}^s$ *with irrational coordinates and let* $\sigma=\{a_k\}_{k=1}^\infty$ *be the infinite sequence in* E^s *defined as follows* $a_k=(\{\alpha_1 k\},\ldots,\{\alpha_s k\})$, $k=1,2,\ldots$ *Suppose that* $\widetilde{\sigma}$ *is any symmetric infinite sequence in* E^s *produced by* σ. *Then for every integer* $N\geqq 2^s$

$$T_N(\widetilde{\sigma})\leqq\frac{2^s-1}{N}+\Big(c(s)\sum_{|m|>0}(R(m))^{-2}\min(1,\ (2q\,\|\langle m,\alpha\rangle\|)^{-2})\Big)^{1/2}. \tag{16}$$

Proof. It is well known that for every non-integer x the following estimate holds $|\Sigma_{k=1}^q e^{2\pi ikx}|\leqq\min(q,\ \frac{1}{2\|x\|})$. Whence we obtain

$$\frac{1}{q}\Big|\sum_{k=1}^{q}e^{2\pi i\langle m,\,a_k\rangle}\Big|=\frac{1}{q}\Big|\sum_{k=1}^{q}e^{2\pi i\langle m,\,\alpha\rangle k}\Big|\leqq\min(1,\ (2q\,\|\langle m,\alpha\rangle\|)^{-1}).$$

From here and from Theorem 1 we obtain (16). Thus Theorem 2 is proved.

4. Points of constant type and lemmas. Let $\alpha=(\alpha_1,\ldots,\alpha_s)$ be any point in $\mathbf{R}^s$. We shall say that the point α is a point of constant type if there exists such positive constant $c=c(\alpha)$, that the inequality

$$R(m)\,\|\langle m,\alpha\rangle\|\geqq c \tag{17}$$

is satisfied for every lattice point $m=(m_1,\ldots,m_s)$ in $\mathbf{Z}^s$ with $m\neq(0,\ldots,0)$.

We have given another definition in [8] for a point of constant type, but it follows from [10] that the two definitions are equivalent.

It is well known (see for example [11, p. 33]) that the point α in $\mathbf{R}$ will be a point of constant type if and only if α is an irrational number having a continued fraction with bounded partial quotients. So, there exist in $\mathbf{R}$ infinitely many points of constant type. However, it is not known whether there exist points of constant type in $\mathbf{R}^s$ if $s\geqq 2$. In all probabitity there exist such points in $\mathbf{R}^s$ for any dimension $s\geqq 1$.

Let $r=(r_1, \ldots, r_s)$ be a point in $\mathbf{N}^s$ and let $\alpha=(\alpha_1, \ldots, \alpha_s)$ be a point of constant type in $\mathbf{R}^s$. We introduce the following notations

$$M(r)=\{m\in \mathbf{Z}^s : 2^{r_j-1}\leqq \bar{m}_j <2^{r_j}, \quad j=1, \ldots, s\} \tag{18}$$

and

$$d=d(r, \alpha)=c/2^{|r|}, \tag{19}$$

where $c=c(\alpha)$ is the constant of inequality (17). We remind that $|r|=r_1+\ldots+r_s$.

Let k be a natural number. We put

$$M_k(r, \alpha)=\{m\in M(r): kd\leqq \|\langle m, \alpha\rangle\|<(k+1)d\}. \tag{20}$$

We shall remark that for a given k, r and α the set $M_k(r, \alpha)$ can turn out to be an empty set. Besides, it is obvious that for arbitrary k, r and α the point $\vec{0}=(0, \ldots, 0)$ does not belong to the set $M_k(r, \alpha)$.

Lemma 2. *Let α be a point of constant type in $\mathbf{R}^s$. Then*

$$\mathbf{Z}^s=\bigcup_{r\in \mathbf{N}^s} M(r) \tag{21}$$

and

$$M(r)\setminus\{\vec{0}\}=\bigcup_{k=1}^{\infty} M_k(r, \alpha). \tag{22}$$

Proof. Let $m=(m_1, \ldots, m_s)$ be a point in $\mathbf{Z}^s$. Since $\bar{m}_j\geqq 1$ $(j=1, \ldots, s)$ then there will exist such numbers $r_j\in \mathbf{N}$ $(j=1, \ldots, s)$ that

$$2^{r_j-1}\leqq \bar{m}_j< 2^{r_j}. \tag{23}$$

We put $r=(r_1, \ldots, r_s)$. From (18) and (23) it follows that $m\in M(r)$. Therefore $\mathbf{Z}^s\subset \bigcup_{r\in \mathbf{N}^s}M(r)$. Since the invers inclusion is obvious, then (21) is proved.

Now let $m\in M(r)$ and let $m\neq\vec{0}$. Since $\alpha=(\alpha_1, \ldots, \alpha_s)$ is a point of constant type in $\mathbf{R}^s$, then from (17), (18) and (19) it follows that $\|\langle m, \alpha\rangle\|\geqq d$. Consequently there will exist such a number $k\in \mathbf{N}$ that $kd\leqq \|\langle m, \alpha\rangle\|<(k+1)d$, but according to (20), these inequalities mean that $m\in M_k(r, \alpha)$. Thus we proved that

$$M(r)\setminus\{\vec{0}\}\subset \bigcup_{k=1}^{\infty} M_k(r, \alpha).$$

It follows from here that the equality (22) is true, since the inverse inclusion is again obvious. Lemma 2 is proved.

Lemma 3. *For every $x, y\in \mathbf{R}$ we have*

$$|\,\|x\|-\|y\|\,|=\min(\|x+y\|, \quad \|x-y\|). \tag{24}$$

Proof. Since the function $\|x\|$ is even and periodic with period 1, then without loss of generality we can assume that x and y belong to the interval $[0, 1/2]$. Suppose, for example, that $0\leqq x\leqq y\leqq 1/2$. Then the equality (24) can be written in the form

$$y-x=\min(\|x+y\|, \quad y-x). \tag{25}$$

If $0\leq x+y\leq 1/2$, then $\|x+y\|=x+y\geq y-x$. Therefore (25) is satisfied.

If $1/2\leq x+y\leq 1$, then $\|x+y\|=1-(x+y)\geq y-x$ and therefore (25) is again satisfied. Lemma 3 is proved.

Lemma 4. *Let $\alpha=(\alpha_1,\ldots,\alpha_s)$ be a point of constant type in R^s. Then $|M_k(r,\alpha)|\leq 2^{s+1}$.*

Proof. Let $h=d/2^s$. Obviously the set $M_k(r,\alpha)$ can be presented as an union of the following (2^s in number) mutually non-intersecting sets:

$$M_k^{(j)}(r,\alpha)=\{m\in M_k(r,\alpha): kd+jh\leq\|\langle m,\alpha\rangle\|<kd+(j+1)h\}, \tag{26}$$

where $j=1, 2,\ldots, 2^s$. Consequently, for proving the lemma it is sufficient to prove that $|M_k^{(j)}(r,\alpha)|\leq 2$ for every $j=1, 2,\ldots, 2^s$. We shall prove, that if $M_k^{(j)}(r,\alpha)\neq\emptyset$ then $|M_k^{(j)}(r,\alpha)|=2$.

Let $m\in M_k^{(j)}(r,\alpha)$. Since the function $\|x\|$ is even then the point $-m=(-m_1,\ldots,-m_s)$ will also belong to $M_k^{(j)}(r,\alpha)$ and therefore

$$\{m,-m\}\subset M_k^{(j)}(r,\alpha). \tag{27}$$

We shall prove that we have here, in fact, an equality. Let us assume that $m'=(m_1',\ldots,m_s')\in M_k^{(j)}(r,\alpha)$ and $m'\neq\pm m$.

Since m and m' belong to $M_k^{(j)}(r,\alpha)$ then they belong to $M(r)$, too. Then

$$R(m\pm m')=\overline{m_1\pm m_1'}\ldots\overline{m_s\pm m_s'}\leq(\overline{m}_1+\overline{m}_1')\ldots(\overline{m}_s+\overline{m}_s')<2^s2^{|r|}. \tag{28}$$

From the assumption it follows that $m\pm m'\neq\vec{0}$. Now, from (17), (28) and (19) we obtain

$$\|\langle m,\alpha\rangle\pm\langle m',\alpha\rangle\|=\|\langle m\pm m',\alpha\rangle\|\geq c/R(m\pm m')>c/2^s2^{|r|}=d/2^s=h.$$

From here and from Lemma 3 it follows that

$$|\,\|\langle m,\alpha\rangle\|-\|\langle m',\alpha\rangle\|\,|>h. \tag{29}$$

But since $m\in M_k^{(j)}(r,\alpha)$ and $m'\in M_k^{(j)}(r,\alpha)$ then from (26) we have $kd+jh\leq\|\langle m,\alpha\rangle\|<kd+(j+1)h$, as well as $kd+jh\leq\|\langle m',\alpha\rangle\|<kd+(j+1)h$. Therefore $|\,\|\langle m,\alpha\rangle\|-\|\langle m',\alpha\rangle\|\,|<h$ which contradicts the inequality (29). The contradiction we obtain proves that in (27) we have an equality, that is $|M_k^{(j)}(r,\alpha)|=2$. Thus Lemma 4 is proved.

Lemma 5. *For every real number $a\geq 1$ the following inequality holds*

$$\sum_{k>a} 1/k^2<2/a.$$

Proof. Obviously $a<[a]+1\leq 2[a]$. Hence

$$\sum_{k>a}\frac{1}{k^2}=\sum_{k>[a]}\frac{1}{k^2}\leq\int_{[a]}^{\infty}\frac{dx}{x^2}=\frac{1}{[a]}<\frac{2}{a}.$$

Lemma 5 is proved.

Lemma 6. *Let ν be a non-negative integer. Then for every real number $a\geq 1$ the following inequality holds*

$$\sum_{k>a} \frac{k^\nu}{2^k} < c(\nu)\frac{a^\nu}{2^a},$$

where $c(\nu)>0$ *is a constant depending only on* ν.

Proof. It is easily proved that for every natural number n holds the inequality

$$\int_n^\infty \frac{x^\nu}{2^x}\, dx \leq c_1(\nu)\frac{n^\nu}{2^n},$$

where $c_1(\nu)$ is a positive constant depending only on ν. This inequality can be proved by induction on ν using the equality

$$\int_n^\infty \frac{x^{\nu+1}}{2^x}\, dx = \frac{\nu+1}{\log 2}\int_n^\infty \frac{x^\nu}{2^x}\, dx + \frac{n^{\nu+1}}{2^n \log 2}.$$

It is easily verified that the function $f(x)=x^\nu/2^x$ for $x\geq \nu/\log 2$ is a decreasing function. Consequently, for every natural number $n>\nu/\log 2$ we have

$$r_n = \sum_{k>n}\frac{k^\nu}{2^k} \leq \int_n^\infty \frac{x^\nu}{2^x}\, dx \leq c_1(\nu)\frac{n^\nu}{2^n}.$$

By an appropriate increase of the constant $c_1(\nu)$ we find the constant $c_2(\nu)$ such that the inequality

$$r_n = \sum_{k>n}\frac{k^\nu}{2^k} \leq c_2(\nu)\frac{n^\nu}{2^n}$$

is to be satisfied for every $n\in \mathbf{N}$. Hence we obtain

$$\sum_{k>a}\frac{k^\nu}{2^k} = \sum_{k>[a]}\frac{k^\nu}{2^k} \leq c_2(\nu)\frac{[a]^\nu}{2^{[a]}} < c_2(\nu)\frac{a^\nu}{2^{a-1}} = c(\nu)\frac{a^\nu}{2^a},$$

where $c(\nu)=2c_2(\nu)$. Lemma 6 is proved.

5. Precise estimates for the L_2 discrepancy of some symmetric infinite sequences.

Theorem 3. *Let* α *be a point of constant type in* $\mathbf{R}^s$ *and let* $\sigma=\{a_k\}_{k=1}^\infty$ *be the infinite sequence in* E^s *defined as follows* $a_k=(\{\alpha_1 k\},\dots,\{\alpha_s k\})$, $k=1,2,\dots$ *Suppose that* $\tilde\sigma$ *is any symmetric infinite sequence in* E^s *produced by* σ. *Then the* L^2 *discrepancy* $T_N(\tilde\sigma)$ *of the infinite sequence* $\tilde\sigma$ *satisfies the estimate* (5).

Proof. Let $N\geq 2^s$ and

$$S_N = \sum_{|m|>0}(R(m))^{-2}\min(1,\ (N\|\langle m,\alpha\rangle\|)^{-2}).$$

Taking into consideration that $c(s)<1$ and $q=[N/2^s]$ (and consequently $2q>N/2^s$) from Theorem 2 we obtain

$$T_N(\tilde\sigma)\leq 2^s(S_N)^{1/2}+(2^s-1)/N. \tag{30}$$

So we have to estimate the sum S_N.

Using Lemma 2, as well as equalities (18) and (20) by which the sets $M(r)$ and $M_k(r, \alpha)$ are difined, we make the following transformations

(31)

$$S_N = \sum_{\substack{m \in \mathbf{Z}^s \\ |m|>0}} (R(m))^{-2} \min(1, (N\|\langle m, \alpha\rangle\|)^{-2}) = \sum_{r \in \mathbf{N}^s} \sum_{\substack{m \in M(r) \\ |m|>0}} (R(m))^{-2} \min(1, (N\|\langle m, \alpha\rangle\|)^{-2})$$

$$\leqq 2^{2s} \sum_{r \in \mathbf{N}^s} 2^{-2|r|} \sum_{\substack{m \in M(r) \\ |m|>0}} \min(1, (N\|\langle m, \alpha\rangle\|)^{-2})$$

$$= 2^{2s} \sum_{r \in \mathbf{N}^s} 2^{-2|r|} \sum_{k=1}^{\infty} \sum_{m \in M_k(r, \alpha)} \min(1, (N\|\langle m, \alpha\rangle\|)^{-2})$$

$$\leqq 2^{2s} \sum_{r \in \mathbf{N}^s} 2^{-2|r|} \sum_{k=1}^{\infty} \min(1, (Ndk)^{-2}) \sum_{m \in M_k(r, \alpha)} 1.$$

From Lemma 4 it follows that $\sum_{m \in M_k(r, \alpha)} 1 \leqq 2^{s+1}$. Taking that estimate into consideration, as well as formula (19) by which the quantity $d = d(r, \alpha)$ is defined, we obtain from (31)

$$S_N \leqq 2^{3s+1} \sum_{r \in \mathbf{N}^s} \frac{1}{2^{2|r|}} \sum_{k=1}^{\infty} \min\left(1, \frac{2^{2|r|}}{c^2 N^2 k^2}\right) \tag{32}$$

$$\leqq c(s, \alpha) \sum_{r \in \mathbf{N}^s} \frac{1}{2^{2|r|}} \sum_{k=1}^{\infty} \min\left(1, \frac{2^{2|r|}}{N^2 k^2}\right),$$

where we have put $c(s, \alpha) = 2^{3s+1} \max(1, 1/c^2)$. Let us remind that $c = c(\alpha)$.

Further on we split the outer sum in (32) into two sums: over all $r = (r_1, \ldots, r_s) \in \mathbf{N}^s$ with $2^{|r|} \leqq N$ and over all $r = (r_1, \ldots, r_s) \in \mathbf{N}^s$ with $2^{|r|} > N$. Then we shall obtain from (32)

$$S_N \leqq c(s, \alpha) \sum_{2^{|r|} \leqq N} \frac{1}{2^{2|r|}} \sum_{k=1}^{\infty} \min\left(1, \frac{2^{2|r|}}{N^2 k^2}\right) \tag{33}$$

$$+ c(s, \alpha) \sum_{2^{|r|} > N} \frac{1}{2^{2|r|}} \sum_{k=1}^{\infty} \min\left(1, \frac{2^{2|r|}}{N^2 k^2}\right)$$

$$= \frac{c(s, \alpha)}{N^2} \sum_{2^{|r|} \leqq N} \sum_{k=1}^{\infty} \frac{1}{k^2} + c(s, \alpha) \sum_{2^{|r|} > N} \frac{1}{2^{2|r|}} \sum_{k=1}^{\infty} \min\left(1, \frac{2^{2|r|}}{N^2 k^2}\right).$$

Let $2^{|r|} > N$. Then from Lemma 5 we have

$$\sum_{k=1}^{\infty} \min\left(1, \frac{2^{2|r|}}{N^2 k^2}\right) = \sum_{k \leqq \frac{2^{|r|}}{N}} 1 + \sum_{k > \frac{2^{|r|}}{N}} \frac{2^{2|r|}}{N^2 k^2} \tag{34}$$

$$\leqq \frac{2^{|r|}}{N} + \frac{2^{2|r|}}{N^2} \sum_{k > \frac{2^{|r|}}{N}} \frac{1}{k^2} \leqq 3 \, \frac{2^{|r|}}{N}.$$

From (33) and (34) we obtain

$$S_N \leqq \frac{\pi^2}{6}\frac{c(s,\alpha)}{N^2}\sum_{2^{|r|}\leqq N} 1 + \frac{3c(s,\alpha)}{N}\sum_{2^{|r|}>N}\frac{1}{2^{|r|}}. \tag{35}$$

For the first sum in (35) we have

$$\sum_{2^{|r|}\leqq N} 1 \leqq \Big(\sum_{2^{r_j}\leqq N} 1\Big)^s \leqq (\log_2 N)^s = c_1(s)(\log N)^s, \tag{36}$$

where $c_1(s) = (\log 2)^{-s}$.

For the second sum in (35) we make the following transformations and use Lemma 6

$$\sum_{2^{|r|}>N}\frac{1}{2^{|r|}} = \sum_{2^k>N}\sum_{|r|=k}\frac{1}{2^{|r|}} = \sum_{k>\log_2 N}\frac{1}{2^k}\sum_{|r|=k} 1 \leqq \sum_{k>\log_2 N}\frac{k^{s-1}}{2^k} \tag{37}$$

$$\leqq c_2(s)N^{-1}(\log_2 N)^{s-1} = c_3(s)N^{-1}(\log N)^{s-1},$$

where $c_3(s)$ is a positive constant depending only on s.

From (35), (36) and (37) we obtain $S_N = O(N^{-2}(\log N)^s)$. From this estimate and from (30) we finally obtain the estimate (5). Thus, Theorem 3 is proved.

Let α be a noninteger. It is obvious then that the one-dimensional infinite sequences

$$\tilde\sigma = (\{\alpha\}, \{-\alpha\}, \{2\alpha\}, \{-2\alpha\}, \ldots) \tag{38}$$

and

$$\tilde\sigma = (\{-\alpha\}, \{\alpha\}, \{-2\alpha\}, \{2\alpha\}, \ldots) \tag{39}$$

are symmetric. They are produced by the sequence (2). Then from Theorem 3 in the one-dimensional case we get the following assertion:

Theorem 4. *Let α be an irrational number having a continued fraction with bounded partial quotients. Suppose that the infinite sequence $\tilde\sigma$ in $E = [0, 1]$ is defined by* (38) *or* (39). *Then the L^2 discrepancy $T_N(\tilde\sigma)$ of the infinite sequence $\tilde\sigma$ satisfies the estimate* (4).

REFERENCES

1. H. Niederreiter. Application of diophantine approximations to numerical integration. Diophantine Approximation and its Applications (Proc. Conf., Washington, 1972). New York, 1973. 129-199.
2. J. H. Halton. On the efficiency of certain quasi-random sequences of points in evaluating multi-dimensional integrals. *Numer. Math.*, **2**, 1960, 84-90.
3. И. М. Соболь. О распределении точек в кубе и сетках интегрирования. *Успехи матем. наук*, **21**, 1966, 271-272.
4. И. М. Собаль. О распределении точек в кубе и приближенном вычислении интегралов *Ж. вычисл. матем. и матем. физики*, 7, 1967, 784-802.
5. A. Ostrowski. Bemerkungen zur Theorie der Diophantischen Approximationen I. *Abt. Math. Sem. Hamburg*, **1**, 1922, 77-98.

6. J. G. van der Corput. Verteilungsfunktionen II. *Proc. Kon. Acad. Wetensch. Amsterdam,* **38**, 1935, 1058-1066.
7. S. Haber. On a sequence of points of interest for numerical quadrature. *J. Research Nat. Bur. Standards, Sect. B,* **70**, 1966, 127-136.
8. П. Д. Пройнов. Точки постоянного типа и оценка сверху квадратичного отклонения для одного класса бесконечных последовательностей. *Доклады БАН,* **35**, *1982*, 753-755.
9. P. D. Proinov. On the extreme and L^2 discrepancies of symmetric finite sequences. *Serdica,* 10, 1984, 376-383.
10. J. W. S. Cassels, H. P. F. Swinnerton-Dyer. On the product of three homogeneous linear forms and indefinite ternary quadratic forms. *Phil. Trans. Royal Soc. A,* **248**, 1955, 73-96.
11. С. Ленг. Введение в теорию диофантовых приближений. М., 1970.

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