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HYPERGRAPH CHARACTERIZATIONS OF k-TOLERANCES

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Relations called k-tolerances are considered as hypergraphs. The connection between k-Helly property and k-tolerances is given and the complement of a hypergraph is constructed.

An undirected graph $G=(V, E)$ without loops and multiple lines illustrates a binary tolerance relation, briefly a 2-tolerance, T_2 on the point set V. The classes of T_2 are the maximal cliques of G[3]. Also a hypergraph $H=(V, \mathscr{E})$ can be interpreted as a 2-tolerance relation, if H is conformal, i. e. the maximal sets in $\mathscr{E}$ are the maximal cliques of a graph H_2 derived from H. A hypergraph $H=(V, \mathscr{E})$ is conformal, if its dual H^* satisfies the Helly property, and thus the Helly property is associated with a 2-tolerance on V. These observations concerning 2-tolerances given by Zelinka in [3] can be generalized for k-tolerances introduced in [2] and the generalization work is the purpose of this paper. As a by-product some properties of hypergraphs are also given.

A k-ary relation T_k on a set V is a k-tolerance on V if $(a, \dots, a)\in T_k$ for every $a\in V$ (reflexivity), and if $(a_1, \dots, a_k)\in T_k$ implies that $(b_1, \dots, b_k)\in T_k$ for all k elements b from the set $\{a_1, \dots, a_k\}$ (generalized symmetry). The k-tolerances on V can be characterized by means of coverings (set-systems) of V called τ_k-coverings. A family $\mathscr{W}_k=\{V_{ki} \mid i\in I_k\}$, where I_k is an index set, of subsets V_{ki} of a set V is a τ_k-covering of V if the following conditions (1)—(3) hold

(1) $V=\cup\{V_{ki} \mid i\in I_k\}$ (i. e. $\mathscr{W}_k$ is a covering of V);

(2) $V_{ki}\not\subset V_{kj}$ when $i\neq j$ and $i, j\in I_k$;

(3) if a set $N\subset V$ is not contained in any set of $\mathscr{W}_k$ there exists then a k-sequence $a_1, \dots, a_k$ of elements from N (not necessarily distinct) such that $\{a_1, \dots, a_k\}$ is not contained in any set of $\mathscr{W}_k$.

The correspondence between k-tolerances T_k on V and τ_k-coverings $\mathscr{W}_k$ of V is the following: the classes of T_k constitute a τ_k-covering of V, and every τ_k-covering $\mathscr{W}_k$ of V determines a k-tolerance T_k on V having the sets of $\mathscr{W}_k$ as its classes [2, Th. 2]. Note that a τ_k-covering of V is also a τ_h-covering of V when $h\geq k$, but a τ_k-covering need not be a τ_h-covering for $h<k$; this is a consequence of the condition (3). In this paper we shall consider *k-tolerances on a finite set V only.*

A hypergraph is a set-system, $H=(V, \mathscr{E})$ where V is a finite set of points of H, the family $\mathscr{E}=\{E_1, \dots, E_n\}$ is a collection of disjoint nonempty subsets of V called the lines of H, and $V=\cup\{E_i \mid E_i\in\mathscr{E}\}$. The collection of all maximal sets in $\mathscr{E}$ is denoted by $\mathscr{E}_{\max}$. A subset $C\subset V$ in a hypergraph $H=(V, \mathscr{E})$ is called a clique of rank r, if either $|C|<r$ or $|C|\geq r$ and each subset of C with cardinality r is contained in at least one line of H [1, Chapt. 19:2]. A

hypergraph $H_k=(V, \mathscr{E})$ is the hypergraph of a k-tolerance T_k on V if $\mathscr{E}$ is the τ_k-covering of V corresponding to T_k (i. e. $\mathscr{E}$ is the family of all classes of T_k).

Theorem 1. *A subset $C\subset V$ is a class of a k-tolerance T_k on the set V if and only if C is a maximal clique of rank k in the hypergraph $H_k=(V, \mathscr{E})$ of T_k.*

Proof. Let C be a class of T_k on V such that $|C|\geq k$. Because the lines of H_k are the classes of T_k, C is a line in H_k and thus trivially each subset of C with cardinality k is contained in a line $(=C)$ of H_k. Hence C is a clique of rank k in H_k. If C is not maximal, then $C\subset C'$ properly and every set of k elements from C' is contained in some $E\in\mathscr{E}$. The proper inclusion $C\subset C'$ implies by (2) that C' is not contained in any E from $\mathscr{E}$, and thus C' is a set N of (3). But this is impossible because every k-element set from C' is contained in some $E\in\mathscr{E}$. Hence C is a maximal clique of rank k.

Assume conversely that C is a maximal clique of rank k in a hypergraph H_k of k-tolerance T_k on V. Because every k-element set from C is present in some $E\in\mathscr{E}$, all k elements from C are in the relation T_k and thus C is present in a class E of T_k. But E is a maximal clique of rank k in H_k as shown above, and then $C\subset E$ and the maximality of C implies that $C=E$. Thus C is a class of T_k. This completes the proof.

Let $H=(V, E)$ be a hypergraph with $V=\{v_1, \ldots, v_n\}$ and $E=\{E_1, \ldots, E_m\}$. In the dual hypergraph $H^*=(E, V)$ of H the point-set E is the set $\{e_1, \ldots, e_m\}$ (corresponding to $E_1, \ldots, E_m$ in H) and the line set V is the family $\{V_1, \ldots, V_n\}$ (corresponding to $v_1, \ldots, v_n$), where $V_j=\{e_i \mid i\leq m$ and $v_j\in E_i$ in $H\}$. A family $\{M_i \mid i\in I\}$ has the Helly property, if $J\subset I$ and $M_i\cap M_j\neq\emptyset$ for all $i, j\in J$ imply $\cap\{M_j \mid j\in J\}\neq\emptyset$ [1, Chapt. 17 : 3]. We shall say that a family $\{M_i \mid i\in I\}$ has a) k-Helly property if $J\subset I$ and $M_{j1}\cap M_{j2}\cap\ldots\cap M_{jk}\neq\emptyset$ for all $j_1, \ldots, jk\in J$ imply $\cap\{M_j \mid j\in J\}\neq\emptyset$. Thus the Helly property reported above is a 2-Helly property, and as it is well-known, the convex subsets of an Euclidean n-space have the $n+1$-Helly property. Now we can prove

Theorem 2. *In a hypergraph $H=(V, \mathscr{E})$ the family $\mathscr{E}_{\max}$ is a τ_k-covering of V if and only if in the dual $H^*_{\max}=(E_{\max}, \mathscr{V})$ of $H_{\max}=(V, \mathscr{E}_{\max})$ the family $\mathscr{V}$ satisfies the k-Helly property.*

Proof. Let $\mathscr{E}_{\max}=\{E_i \mid i\in I\}$ be a τ_k-covering of V, $\mathscr{V}=\{V_l \mid l\in L\}$ and $J\subset L$ such that $V_{j1}\cap\ldots\cap V_{jk}\neq\emptyset$ holds for all $j1, \ldots, jk\in J$. If now $\cap\{V_j \mid j\in J\}=\emptyset$, there is no element $e_i\in\cap\{V_j \mid j\in J\}$, which implies that the set $N=\{v_j \mid j\in J\}$ is not contained in any set from $\mathscr{E}_{\max}$ in H. Because $\mathscr{E}_{\max}$ is a τ_k-coverning of V, N contains by (3) a k-sequence $v_{j1}, \ldots, v_{jk}$ not contained in any $E_i\in\mathscr{E}_{\max}$, whence the corresponding inter-section in $H^*_{\max}$ is $V_{j1}\cap\ldots\cap V_{jk}=\emptyset$ for $j1, \ldots, jk\in J$. This is a contradiction, and hence $\cap\{V_j \mid j\in J\}\neq\emptyset$ and the family $\mathscr{V}$ has the k-Helly property.

Conversely, let the family $\mathscr{V}$ of $H^*_{\max}$ have the k-Helly property. The conditions (1) and (2) hold for $\mathscr{E}_{\max}$ because H is a hypergraph and $\mathscr{E}_{\max}$ contains only maximal sets. Let $N\subset V$, $N\not\subset E_i$ for any $E_i\in\mathscr{E}_{\max}$ and let every k-sequence of N be contained in some $E_i\in E_{\max}$. This implies that $V_{j1}\cap\ldots\cap V_{jk}\neq\emptyset$ for all $v_{j1}, \ldots, v_{jk}\in N$. Because of the k-Helly property of $\mathscr{V}$, then $\cap\{v_j \mid v_j\in N\}\neq\emptyset$, and thus there is a set $E_i\in\mathscr{E}_{\max}$ corresponding to $e_i\in\cap\{v_j \mid v_j\in N\}$ such that $N\subset E_i$. This is a contradiction and so N contains a k-sequence not contained in any $E_i\in\mathscr{E}_{\max}$ whence also (3) holds for $\mathscr{E}_{\max}$ and it is a τ_k-covering of V. This completes the proof.

A further connection between a hypergraph and its dual is a direct corollary of Theorem 2 and hence its proof is omitted:

Corollary. *Let E be a τ_h-covering of set V, $H=(V, \mathcal{E})$ the corresponding hypergraph, $H^*=(\mathcal{E}, \mathcal{V})$ its dual and $\mathcal{V}_{\max}$ a τ_d-covering of $\mathcal{E}$, where $h=\min\{k \mid \mathcal{E}$ is a τ_k-covering of $V\}$ and $d=\min\{k \mid \mathcal{V}_{\max}$ is a τ_k-covering of $E\}$. Then $h=d$ if and only if $\mathcal{E}$ has the h-Helly property.*

Next we present a result on representative graphs. Let $H=(E, \mathcal{V})$ be a hypergraph with $\mathcal{V}=\{V_1, \ldots, V_n\}$. The representative graph of H is an undirected graph with points $v_1, \ldots, v_n$ corresponding to the lines of H, and v_i is adjacent to v_j whenever $V_i \cap V_j \neq \emptyset$. In [1, Proposition 17.1] it is proved that a graph G with a point-set V and a family $\mathcal{E}=\{E_1, \ldots, E_m\}$ of subsets of V, where (a) every E_i is a clique of G and (b) every point and line of G is covered by at least one E_i, is the representative graph of the dual H^* of the hypergraph $H=(V, \mathcal{E})$. Conversely, if G is the representative graph of the hypergraph $H=(E, \{V_1, \ldots, V_n\})$, then the sets in the dual $H^*=(V, \{E_1, \ldots, E_m\})$ have the properties in (a) and (b) above. Now we can generalize the theorem on representative graphs of maximal cliques of a graph (i. e. of τ_2-coverings of a set [2, Thm. 12]).

Theorem 3. *A graph is the representative graph of sets in a τ_k-covering of a set V if and only if there is a family $\{E_j \mid j \in J\}$ of cliques of G such that*

(i) *each line of G is covered by an E_j;*

(ii) *$\{E_j \mid j \in J\}$ satisfies the k-Helly property.*

Proof. Let G be a representative graph of the sets E_j in a τ_k-covering $\mathcal{E}$ of a set V. Then the pair $(V^{a} \mathcal{E})$ determines a hypergraph H, in the dual $H^*=(E, \mathcal{V})$ of which the family $\mathcal{V}=\{V_1, \ldots, V_n\}$ has the k-Helly property (ii). According to [1, Proposition 17.1] reported above, the sets in $\mathcal{V}$ are cliques of G and satisfy (i).

Conversely, let $\{E_j \mid j \in J\}$ be a family of cliques in G satisfying (i) and (ii). Let further $E_i'=\{v_i\}$, $i=1, \ldots, n$. Then $\mathcal{E}=\{E_j \mid j \in J\} \cup \{E_1', \ldots, E_n'\}$ satisfies both (i) and (ii). According to [1, Proposition 17.1], G is now the representative graph of the dual $H^*=(E, \mathcal{V})$ of $H=(V, \mathcal{E})$. Because $\mathcal{E}$ has the k-Helly property, the maximal sets of $\mathcal{V}_{\max}$ in $\mathcal{V}$ constitutute a τ_k-covering of E. But clearly $V_i=\{e_i'\} \cup \{e_j \mid v_i \in E_j$ in $H\}$ of $\mathcal{V}$ is maximal in $\mathcal{V}$, whence $\mathcal{V}=\mathcal{V}_{\max}$, and thus G represents the sets of the τ_k-covering $\mathcal{V}$ of E. This completes the proof.

Next we consider partial hypergraphs. A partial hypergraph $D=(P, \mathcal{N})$ of a hypergraph $H=(V, \mathcal{E})$ is generated by a subfamily $\mathcal{N} \subset \mathcal{E}$ and $P=\bigcup\{E_i \mid E_i \in \mathcal{N}\}$. A subhypergraph $F=(B, \mathcal{K})$ of H generated by $B \subset V$ has as the line set $\mathcal{K}=\{E_i \cap B \mid E_i \in \mathcal{E}$ and $E_i \cap B \neq \emptyset\}$.

Theorem 4. *Let $H=(V, E)$ be a hypergraph, where $\mathcal{E}$ satisfies the d-Helly property, $\mathcal{E}_{\max}$ be a τ_h-covering of V such that $h=\min\{k \mid \mathcal{E}_{\max}$ is a τ_k-covering of $V\}$, and $2 \leq d < h$. Then H contains a partial subhypergraph $D=(M, \mathcal{N})$, where $\mathcal{N}$ has the h-Helly property but not the d-Helly property.*

Proof. When $\mathcal{E}_{\max}$ is a τ_h-covering of V, there is a set M containing h points, $M \not\subseteq E_i$ for any $E_i \in \mathcal{E}_{\max}$ and arbitrary h-1 points from M belong to some $E_i \in \mathcal{E}_{\max}$. Let us consider the partial subhypergraph $D=(M, \mathcal{N})$, where $\mathcal{N}$ contains all maximal sets of type $M \cap E_i$, $E_i \in \mathcal{E}$, without duplicates. Then $\mathcal{N}$ contains h sets $E_1', \ldots, E_h'$ such that each of them contains exactly h-1 disjoint points from M. Now, because E_j' contains h-1 disjoint points of M and M contains h disjoint points, the intersection of all h-1 sets E'_j from $\mathcal{N}$ is nonempty and $\cap\{E'_j \mid E'_j \in \mathcal{N}\} = \emptyset$, whence $\mathcal{N}$ satisfies at most h-Helly property. Because there are h sets in $\mathcal{N}$, it satisfies the h-Helly property, and the theorem follows.

A set $S \subset V$ in a hypergraph $H = (V, \mathcal{E})$ is strongly stable if $|S \cap E_i| \leq 1$ for every $E_i \in \mathcal{E}$. The maximum number of poins in a strongly stable set of H is denoted by $\alpha(H)$ and this number is called the strong stability number of H. The covering number $\rho(H)$ is the least number of lines of H that cover all points in H. The following theorem presents a connection between $\alpha(H^*)$ and $\rho(H^*)$ for a hypergraph H.

Theorem 5. *If a hypergraph $H = (V, E)$ contains a partial hypergraph $H' = (V', E')$ with a τ_h-covering $\mathcal{E}'$ of V', where $h \geq 3$ and $h = \min\{k \mid \mathcal{E}'$ is a τ_k-covering of $V'\}$, then H contains a partial subhypergraph $D = (W, \mathcal{F})$ such that $\alpha(D^*) = 1$ and $\rho(D^*) = 2$.*

Proof. Let $H' = (V', E^*)$ be a partial hypergraph of the theorem. Because $\mathcal{E}'$ is not a τ_{h-1}-covering of V', there is a subset $N \subset V'$, N is not contained in any set from $\mathcal{E}'$, such that any h-1-element subset of N extends to a member of $\mathcal{E}'$. Hence any h-element subset W of N, $W \not\subset \mathcal{E}'$, with $\mathcal{F} = \{F \mid h-1 = |F|$ and $F \subset W\}$ constitutes the partial subhypergraph D, where $F_i \cap F_j \neq \emptyset$. Thus $\rho(D^*) \leq 2$. Clearly $\rho(D^*) \geq 2$, and so we obtain $\rho(D^*) = 2$. Obviously $\alpha(D^*) = 1$, and the theorem follows.

Let $\nu(H)$ denote the maximum cardinality of a matching of a hypergraph H and $\xi(H)$ the transversal number of H. The hypergraph H is balanced if and only if $\nu(D) = \xi(D)$ for every partial subhypergraph D of H [1, Thm. 20:5]. On the other hand, $\nu(D) = \xi(D) \Leftrightarrow \alpha(D^*) = \rho(D^*)$. Now, Theorem 5 above shows that a balanced hypergraph H can contain only such partial hypergraphs $H' = (V', E')$, where $E'_{\max}$ is a τ_2-covering of V', whence H is conformal.

As shown by Zelinka [3], every graph $G = (V, E)$ corresponds to a hypergraph $H = (V, \mathcal{E})$, where $\mathcal{E} = \mathcal{E}_{\max}$ is a τ_2-covering of V consisting of all maximal cliques of G, and vice versa. In particular, the lines of G show all pairs of disjoint points which are in the 2-tolerance relation determined by the τ_2-covering of maximal cliques of G. The complement G_c of G is the graph $G_c = (V, E_c)$, where $(a, b) \in E_c \Leftrightarrow (a, b) \notin E$ and $a \neq b$. If $E = \{E_i \mid i \in I\}$ is the family of maximal cliques of G, then the family E_c of maximal elements in $\{S \mid S \subset V$ and $|S \cap E| \leq 1$ for any $E \in \mathcal{E}\}$ is the family of maximal cliques of G_c. Thus the complement of a hypergraph $H = (V, \mathcal{E})$, where $\mathcal{E} = \mathcal{E}_{\max}$ is a τ_2-covering of V, is $H_c = (V, \mathcal{E}_c)$ with $\mathcal{E}_c$ given above. Analogously, every hypergraph $H = (V, \mathcal{E})$ with $\mathcal{E} = \mathcal{E}_{\max}$ corresponds to a "graph", the maximal cliques of which are the sets in $\mathcal{E}$. Unfortunately, the "lines" of this "graph" have not a simple pictorial illustration, when $\mathcal{E}$ is a τ_k-covering of V with $k \geq 3$; all points $a_1, \ldots, a_k \in E_i \in \mathcal{E}$, where at least two points a_s and a_t are disjoint, constitute a "line" of the "graph". In any way, the analogy offers a way for constructing the complement H_c for a $H = (V, \mathcal{E})$, where $\mathcal{E} = \mathcal{E}_{\max}$ is τ_h-covering of V with $h = \min\{k \mid \mathcal{E}$ is a τ_k-covering of $V\}$. We put $H_c = (V, \mathcal{E}_c)$, where $\mathcal{E}_c$ is the family of maximal elements in $\{S \mid S \subset V$ and $|S \cap E| \leq h$-1 for any $E \in \mathcal{E}\}$. Clearly $\cup\{S \mid \in \mathcal{E}_c\} = V$.

Let us consider as an example the hypergraph $H = (V, \mathcal{E})$ with $V = \{a, b, c\}$ and $\mathcal{E} = \{E_1, E_2, E_3\}$, where $E_1 = \{a, b\}$, $E_2 = \{b, c\}$ and $E_3 = \{a, c\}$. As easily seen, $\mathcal{E}$ is a τ_3-covering of V. According to the definition above, the family $\mathcal{E}_c$ of H_c contains only one line $E_{c1} = \{a, b, c\}$ and thus $\mathcal{E}_c$ is a τ_2-covering of V. The example shows that $\mathcal{E}$ and $\mathcal{E}_c$ need not be τ_h-coverings of V with the same value of h. According to the construction, $E_c = E_{c\max}$, and thus E_c is a τ_h-covering of V for some value of h.

A strong q-colouring of a hypergraph H is q-colouring of the points of H such that no two points in the same line have the same colour. The strong chromatic number $\gamma(H)$ of H is the smallest integer for which there is a strong q-colouring. Now we can prove the Nordhaus-Gaddum theorem for hypergraphs.

Theorem 6. *Let $H=(V, \mathscr{E})$ be a hypergraph with $\mathscr{E}=\mathscr{E}_{\max}$, γ its strong chromatic number, γ_c the strong chromatic number of H_c, and $|V|=p$. Then $2\sqrt{p}\leq\gamma+\gamma_c\leq 2p$ and $p\leq\gamma\gamma_c\leq p^2$.*

Proof. Let H be q-chromatic and $V_1, \ldots, V_q$ the colour classes of H, where $|V_i|=p_i$. Then $\Sigma p_i = p$ and $\max p_i \geq p/q$. Every V_i is contained in a line of H_c, whence $\gamma_c \geq \max p_i \geq p/q$. Thus $\gamma\gamma_c \geq p$. According to the relation between geometric and arithmetic means, $2\sqrt{p}\leq\gamma+\gamma_c$. Clearly $\gamma+\gamma_c\geq 2p$ and the example about H and its complement H_c above shows that $\gamma(H)=3=\gamma(H_c)$, whence the equality can also hold in $\gamma+\gamma_c\leq 2p$. Also the validity of $\gamma\gamma_c\leq p^2$ is obvious.

Note that the limitation to hypergraphs with $\mathscr{E}=\mathscr{E}_{\max}$ is not essential, because the strong colouring of H is determined by the lines in $\mathscr{E}_{\max}$.

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