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# GENERALIZED-ANALYTIC SETS IN A GLEASON PART

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We give conditions, under which a metric neighbourhood of a Gleason part in the maximal ideal space of a uniform algebra is homeomorphic to a set, quite different from a classical analytic set.

Let $A$ be a uniform algebra on the compact Hausdorff space $X$, i. e. $A$ is a closed point-separating subalgebra of $C(X)$, containing the constants. Sp$A$ will denote the spectrum of $A$, i. e. the space of linear multiplicative functionals on $A$, which is naturally equivalent to the space of the maximal ideals of $A$. An interesting and important problem in the theory of commutative Banach algebras is to find conditions which assure the existence of special structures in sp$A$, according to which the functions of $A$ belong to certain classes. Wermer showed first, that if $A$ is a Dirichlet algebra on $X$, then each Gleason part (i. e. each equivalence class of the relation $\|\varphi-\theta\|<2$, $\varphi$, $\theta\in$ sp$A$) of sp$A$ is either a single point or an analytic disc, according to which all functions of $A$ are analytic; then Hoffman generalized Wermer's result to logmodular algebras; finally Lumer observed that if a linear multiplicative functional $\varphi$ has a unique representing measure on $X$, then the Gleason part $P_\varphi$ of sp$A$, containing $\varphi$ consists either of $\varphi$ alone or of an analytic disc through $\varphi$ (see [3]). Later Gamelin proved, that if a linear multiplicative functional $\varphi$ with finite-dimensional space of representing measures has a unique logarithmic measure on $X$, then $P_\varphi$ consists of $\varphi$ alone or admits a structure of a connected open Riemannian surface, according to which all functions of $A$ are analytic. The further investigations of the matter have been connected with its local nature. In [1] Browder proved the following generalization of a theorem of Gleason [2; 3]: *If $A$ is a commutative Banach algebra with unit, and $\varphi$ is a linear multiplicative functional on A, which continuous-point-derivation space* Ker $\varphi/[(\text{Ker}\,\varphi)^2]$ *is finite-dimensional, then some neighbourhood of $\varphi$ in the metric topology, which* sp$A$ *inherits from $A^*$, is homeomorphic to an analytic subset of some finite-dimensional polydisc, according to which the Gelfand transforms of elements of $A$ are analytic functions.* Note, that any element $f$ of $A$ in this case has a unique presentation of the form: $f=\Sigma_1^n\lambda_j w_j+\Sigma_1^\infty\mu_k g_k h_k$ for some $w_1,\dots,w_n\in A$, where $\lambda_j,\mu_k\in\mathbf{C}$, $\Sigma_1^\infty|\mu_k|<\infty$, $g_k, h_k\in A$, $\|g_k\|,\|h_k\|\leq 1$. The further picture of the matter seemed to be somehow vague until the following enlarging of the concept of analyticity has been attracted into consideration.

Let $\Gamma$ be a subgroup of the group R of all rational numbers, provided with discrete topology, $G=\hat{\Gamma}$ is the compact connected group of characters of the group $\Gamma$ and $\Delta_G$ is the "big disc", i. e. the cone over $G$: $\Delta_G=[0,1)\times G/\{0\}\times G$ with $\{*\}=\{0\}\times G/\{0\}\times G$ as a peak. For any $p\in\Gamma_+=\Gamma\cap[0,+\infty)$ the character

$\chi_p^p(g)=g(p)$ on $G$ is extendable on the whole $\bar{\Delta}_G$ in the following way: $\chi^p(\lambda, g)=\lambda^p\chi^p(g)$ for $p\neq 0$, $\lambda\neq 0$; $\chi^p(*)=0$ for any $p\neq 0$ and $\chi^0\equiv 1$ on $\bar{\Delta}_G$. In 1956 Arens and Singer [5] introduced the algebra $A_G$ of generalized-analytic functions on a compact group $G$ with ordered group of characters $\Gamma=\hat{G}$. The following is an equivalent definition: a continuous on $\bar{\Delta}_G$ function $f$ is called generalized-analytic, iff $f$ is uniformly approximable on $\bar{\Delta}_G$ by generalized polynomials, i. e. by finite linear combinations over $\mathbf{C}$ of functions $\chi^p$, $p\in\Gamma_+$ (see e. g. [3; 6]). If $\Delta_G(\varepsilon)$ denotes the big disc with radius $\varepsilon>0$, i. e. the set $[0, \varepsilon)\times G/\{0\}\times G$, then $A_G(\varepsilon)$ is the uniform algebra of continuous on $\bar{\Delta}_G(\varepsilon)$ functions, uniformly approximable by gen. polynomials on $\bar{\Delta}_G(\varepsilon)$. If $V$ is a subset of some big disc $\Delta_G(\varepsilon)$, we call $V$ a generalized-analytic set, if it coincides with the vanishing set of some family of gen.-analytic functions on $\bar{\Delta}_G(\varepsilon)$.

In [7] it is shown, that if $A$ is an antisymmetric uniform algebra, generated by the elements of a multiplicative semigroup, isomorphic to the positive rational numbers, where the generators are constant on the Šilov boundary $\partial A$ of $A$, then $\operatorname{sp} A$ is homeomorphic to some big disc, according to which the functions of $A$ are gen.-analytic and $A$ is isometrically isomorphic to the algebra $A_G(1)$. The Gleason parts of the algebra $A_G(\varepsilon)$ are not interesting from our point of view, because $A_G(\varepsilon)$ is a Dirichlet algebra and according to Wermer's theorem all its parts are either separate points (as the point $*$) or analytic discs. Of a greater interest is the Gleason part $P_*$ of the algebra $A^\nu(\varepsilon)=\{f\in A_G(\varepsilon) \mid \int_G f(\lambda, g)\overline{\chi^p(g)}dg=0$ for any $p$: $0<p<\nu\}$, containing the point $*$, which coincides with the whole big disc $\Delta_G(\varepsilon)$. Indeed, an application of a generalized version of the classical Schwarz's Lemma (see [8]) shows that any point of the open big disc $\Delta_G(\varepsilon)$ belongs to $P_*$. Of course the point $*$ admits far from only one representing measure on $\varepsilon\times G$ in this case. In [6] it is proved, that if there exists a multiplicative semigroup of elements $\{u_{p(j)}\}_{j=1}^\infty$ from $\operatorname{Ker}\varphi$, where $\varphi\in\operatorname{sp} A$, $\| u_{p(j)}\|\leq 1$, isomorphic to the semigroup $Q^1$ of rational numbers, bigger or equal to 1, such that any element of $A$ is presentable in the form: $f=\varphi(f)+\Sigma_{j=1}^\infty f_j u_{p(j)}$ with some $f_j\in A$, $\Sigma_1^\infty \|f_j\|<\infty$, then near $\varphi$ the spectrum $\operatorname{sp} A$ of $A$, provided with the Gelfand (i. e. weak*-) topology is homeomorphic to a gen.-analytic set, according to which all functions of $A$ are gen.-analytic. Because any weak*-neighbourhood contains a metric neighbourhood, $\varphi$ is non-isolated in the metric topology of $\operatorname{sp} A$ and also a metric neighbourhood of $\varphi$ is homeomorphic to a gen.-analytic set of a big disc. Note, that in this case any element $f$ of $A$ admits a representation of the following type: $f=\varphi(f)+\Sigma_{j=1}^\infty\varphi(f_j)\,.\,u_{p(j)}+\Sigma_{j,k=1}^\infty f_{jk}u_{p(j)}u_{p(k)}$, where $f_{jk}\in A$, $\Sigma_{j,k}\| f_{jk}\|<\infty$, the last summand being a special element of the space $[(\operatorname{Ker}\varphi)^2]$.

In the present paper we show, that if each element $f$ of $A$ admits the following representation:

$$f=\varphi(f)+\sum_{\substack{j=1\\ p(j)\in Q^\nu}}^{\infty}\lambda_j u_{p(j)}+g,$$

where

$$g=\sum_1^{\infty} \mu_k g_k h_k, \quad \lambda_j, \mu_k \in \mathbf{C}, \quad \Sigma_1^{\infty} |\lambda_j| < \infty, \quad \Sigma_1^{\infty} |\mu_k| < \infty, \quad \varphi \in \operatorname{sp} A,$$

$$g_k, \ h_k \in \operatorname{Ker} \varphi, \quad \| g_k \|, \quad \| h_k \| \leq 1,$$

then either $\varphi$ is an isolated point of $\operatorname{sp} A$ with respect to the metric topology, which $\operatorname{sp} A$ inherits from $A^*$ (namely with respect to the metric $\rho(\varphi, \theta) = \| \varphi - \theta \|$) and $P_\varphi = \{\varphi\}$ or a metric neighbourhood of $\varphi$ is homeomorphic to a gen.-analytic set of a big disc, according to which the functions of $A$ belong to the algebra $A^{\nu}(\varepsilon)$ and hence are gen.-analytic. Note, that the space $\operatorname{Ker} \varphi / [(\operatorname{Ker} \varphi)^2]$ of continuous point derivations at $\varphi$ is infinite-dimensional in this case.

Let $A$ be a commutative Banach algebra with unit. The maximal ideal space of $A$ provided with Gelfand (weak*-) topology we denote by $\operatorname{sp} A$. Let $z = (z_1, z_2, \dots)$ and $I$ be the set of these sequences $\alpha = (\alpha_1, \alpha_2, \dots)$ from non-negative integers $\alpha_j$, the most finite of which are different from zero. As usual $|\alpha| = \Sigma_1^{\infty} \alpha_j$ and $z^\alpha = z_1^{\alpha_1}, z_2^{\alpha_2} \dots$ for $z \in \mathbf{C}^\infty$, $\alpha \in I$. Analogously, if $\{q_j\}_1^\infty$ is a sequence of elements of $A$ and $\alpha \in I$, by $a^\alpha$ we denote the finite product $a_1^{\alpha_1} \cdot a_2^{\alpha_2} \dots$ A polynomial of infinite-dimensional argument we call any linear combination of functions $z^\alpha$, $\alpha \in I$.

Let $w(\varepsilon) = \{z \in \mathbf{C}^\infty \mid \| z \|_{l_1} = \Sigma_1^\infty |z_j| < \varepsilon\}$ and $0 < \rho \leqq 1/2$. Following the Brouder's technique for the finite-dimensional case [1], we consider the function

$$P(z) = \rho(1 - [1 - \frac{1}{\rho^2} (\Sigma_1^\infty z_j)]^{1/2})$$

on $w(\rho^2)$, where $(\cdot)^{1/2}$ is the principal value of the square root. According to the binomial theorem,

$$P(z) = -\rho \sum_{k=1}^{\infty} \binom{1/2}{k} (-\frac{1}{\rho^2})^k (\sum_{j=1}^{\infty} z_j)^k. \tag{1}$$

Applying the Cauchi rule for absolutely convergent series, (1) can be re-written as:

$$P(z) = \sum_{\alpha \in I} C_\alpha z^\alpha, \tag{2}$$

where $z \in w(\rho^2)$ and $C_\alpha = C_{(\alpha_1, \alpha_2, \dots)}$, $C_0 = 0$. Because $\left| \binom{1/2}{k} \right| \leqq 1/2$ for any $k$ we obtain the estimation:

$$| P(z) | \leqq \frac{\rho}{2} \sum_{k=1}^{\infty} (\frac{1}{\rho^2} \| z \|_{l_1})^k = \frac{\rho}{2} \frac{\frac{1}{\rho^2} \| z \|_{l_1}}{1 - \frac{1}{\rho^2} \| z \|_{l_1}} = \frac{\rho}{2} \frac{\| z \|_{l_1}}{\rho^2 - \| z \|_{l_1}}. \tag{3}$$

Let $P_r(z)$ denote the expression $P_r(z) = -\rho \, \Sigma_{k=r+1}^{\infty} \binom{1/2}{k} (-\frac{1}{\rho^2})^k (\Sigma_1^\infty z_j)^k$. Then on $w(\rho^2)$ we have analogously:

$$| P_r(z) | \leqq \frac{\rho}{2} \sum_{k=r+1}^{\infty} (\frac{1}{\rho^2} \| z \|_{l_1})^k = \frac{\rho}{2} \frac{(\frac{1}{\rho^2} \| z \|_{l_1})_{r+1}}{1 - \frac{1}{\rho^2} \| z \|_{l_1}} = \frac{\| z \|_{l_1}^{r+1}}{2\rho^{2r-1}(\rho^2 - \| z \|_{l_1})}. \tag{4}$$

Now we shall deduce a recurrent dependence, connecting various coefficients $C_\alpha$ from (2). According to (2), the function $P(z)$ develops into an absolutely convergent series $\Sigma_{|\alpha|\geq 1} C_\alpha z^\alpha$ on $w(\rho^2)$. Then

$$(P(z))^2=(\sum_{|\beta|\geq 1} C_\beta z^\beta)(\sum_{|\gamma|\geq 1} C_\gamma z^\gamma)=\sum_{|\alpha|\geq 2}(\sum_{\beta+\gamma=\alpha} C_\beta C_\gamma) z^\alpha.$$

On the other hand, by definition of $P(z)$ we have that

$$(P(z))^2=\rho^2(1-2[1-\frac{1}{\rho^2}(\Sigma z_j)]^{1/2}+1-\frac{1}{\rho^2}(\Sigma z_j))=\rho^2(\frac{2}{\rho}P(z)-\frac{1}{\rho^2}\Sigma z_j)$$

$$=2\rho P(z)-\Sigma z_j=\sum_{|\alpha|\geq 1} 2\rho C_\alpha z^\alpha-\sum_{|\alpha|=1} z^\alpha=\sum_{|\alpha|=1}(2\rho C_\alpha-1)z^\alpha+\sum_{|\alpha|\geq 2} 2\rho C_\alpha z^\alpha.$$

Consequently

$$\sum_{|\alpha|\geq 2}(\sum_{\beta+\gamma=\alpha} C_\beta C_\gamma) z^\alpha=\sum_{|\alpha|=1}(2\rho C_\alpha-1)z^\alpha+\sum_{|\alpha|\geq 2} 2\rho C_\alpha z^\alpha,$$

from where $\Sigma_{|\alpha|=1}(2\rho C_\alpha-1)z^\alpha+\Sigma_{|\alpha|\geq 2}(2\rho C_\alpha-\Sigma_{\beta+\gamma=\alpha} C_\beta C_\gamma)z^\alpha=0$. Let $z=(z_1, z_2, \ldots)=(\rho_1 e^{i\theta_1}, \rho_2 e^{i\theta_2}, \ldots)$, $\theta=(\theta_1, \theta_2, \ldots)$ and $(\alpha, \theta)=\Sigma\alpha_j\theta_j$ for any $\alpha\in I$. Then

(5) $$\sum_{|\alpha|=1}(2\rho C_\alpha-1)\rho^{|\alpha|}e^{i(\alpha,\theta)}+\sum_{|\alpha|\geq 2}[2\rho C^\alpha-\sum_{\beta+\gamma=\alpha} C_\beta C_\gamma]\rho^{|\alpha|}e^{i(\alpha,\theta)}=0.$$

Let $\Phi$ denote the functional on the set of finite products of continuous functions $f_k(z_j)$ of one variable on the circles $|z_j|=1$, $j=1, 2, \ldots$, defined as the product of corresponding Lebesgue integrals on circles $|z_j|=1$, $j=1, 2, \ldots$ For any $\alpha\in I$, $\alpha\neq 0$ holds: $\Phi(e^{i(\alpha,\theta)})=0$. If we fix a $\alpha_0\in I$, multiply both sides in (5) by $e^{-i(\alpha_0,\theta)}$ and apply $\Phi$ to the expressions obtained, we get that the coefficient before $e^{i(\alpha_0,\theta)}$ is zero. Thus since all the coefficients in (5) are zero, we have:

(6)
$$C_\alpha=1/2\rho\geq 1 \quad \text{if } |\alpha|=1;$$
$$C_\alpha=1/2\rho\sum_{\beta+\gamma=\alpha} C_\beta C_\gamma \quad \text{for } |\alpha|\geq 2.$$

In the following by $B_E$ we denote the closed unit ball of the linear subspace $E\subset A$. Let $\theta\in \operatorname{sp} A$ and $M$ be the kernel of $\theta$. Suppose that there exists a sequence $\mathfrak{U}=\{u_j\}_1^\infty$, $\|u_j\|=1$, with elements of $M$, $u_i-u_j\notin[M^2]$ for $i\neq j$, such that every element $f\in M$ is presentable in a unique way in the form

(7) $$f=\Sigma_{j=1}^\infty\lambda_j u_j+g,$$

where $\lambda_j\in \mathbf{C}$, $j=1, 2, \ldots$, $\Sigma|\lambda_j|<\infty$ and $g\in[M^2]$. In the following Lemma $N$ denotes the linear subspace

$$\{f\in M \mid f=\sum_{k=1}^\infty \mu_k g_k h_k;\ g_k, h_k\in B_M,\ \|\{\mu_k\}\|_{l_1}<\infty\}.$$

Lemma 1. *There exists a constant $\rho$: $0<\rho\leq 1/2$, such that if $r$ is a positive integer, any element $f\in B_M$ takes the form $f=\Sigma_{|\alpha|\leq r}\xi_\alpha(\lambda u)^\alpha+F$ for some constants $\xi_\alpha$ and $\lambda_j$, $j=1, 2, \ldots$ with $\Sigma|\lambda_j|\leq 1/2\rho$, and for some element $F\in N$ with $|\varphi(F)|\leq\rho^{r+1}$ for all $\varphi\in \operatorname{sp} A$, $\|\varphi-\theta\|\leq\varepsilon=\rho^5$.*

Proof. We shall show a little more, namely that the constants $\xi_\alpha$ can be chosen so that $|\xi_\alpha|\leq C_\alpha$ for $|\alpha|\leq r$. Let

$$S=\{f\in A\,|\,f=\Sigma_1^\infty\lambda_j u_j+\Sigma\zeta_k g_k h_k,\ g_k,h_k\in B_M,\ \lambda_j,\zeta_k\in\mathbf{C},\ \Sigma|\lambda_j|+\Sigma|\zeta_k|\leq 1\}.$$

It is clear that $S\subset M$. Because $M=\bigcup_{m=1}^\infty mS=\bigcup_{m=1}^\infty[mS]$, according to the Baire category theorem there exists a $m>0$, such that $\operatorname{int}[mS]\neq\emptyset$. Consequently $\operatorname{int}[S]\neq\emptyset$. Because $S$ is a convex and symmetric set, then $[S]$ contains some neighbourhood of the origin, say $\{f\in M\,|\,\|f\|\leq 4\rho\}$. Hence, any fixed element $f$ from $B_M$ belongs to $[\frac{1}{4\rho}S]$. Denote by $f_1$ such an element of $\frac{1}{4\rho}S$, for which $\|f-f_1\|\leq 1/2$. Then $f-f_1\in[\frac{1}{8\rho}S]$ and we can find such $f_2\in S/8\rho$, that $\|f-f_1-f_2\|\leq 1/4$. By induction we obtain that for every $m>0$ there exist such $f_k\in\frac{1}{2^{k+1}\rho}S$, that $\|f-\Sigma_1^m f_k\|\leq\frac{1}{2^m}\cdot$ It is clear that $f=\Sigma_{k=1}^\infty f_k\in\frac{1}{4\rho}S+\frac{1}{8\rho}S+\frac{1}{16\rho}S+\cdots\subset\frac{1}{2\rho}S$, i. e. $B_M\subset\frac{1}{2\rho}S$. Consequently for any $f\in B_M$ there exist $\lambda_j$ and $\mu_k\in\mathbf{C}$, with $\Sigma|\lambda_j|+\Sigma|\mu_k|\leq 1/2\rho$ and such $g_k,h_k\in B_M$, that

$$f=\Sigma_1^\infty\lambda_j u_j+\Sigma_1^\infty\mu_k g_k h_k. \tag{8}$$

If $\|f\|=1$, then $1=\|f\|\leq\Sigma|\lambda_j|+\Sigma|\mu_k|\leq 1/2\rho$, from where $\rho\leq 1/2$. If we take $\xi_\alpha=1$ for $|\alpha|=1$, then $|\xi_\alpha|=1\leq 1/2\rho=C_\alpha$, $|\alpha|=1$. Now for $F=\Sigma_1^\infty\mu_k g_k h_k$ we have

$$|\varphi(F)|\leq\Sigma_1^\infty|\mu_k|\,|\varphi(g_k)|\,|\varphi(h_k)|\leq\frac{1}{2\rho}\|\varphi-\theta\|^2\leq\frac{\varepsilon^2}{2\rho}\leq\frac{\rho^{10}}{2\rho}\leq\rho^2$$

for all $\varphi\in\{\varphi\in\operatorname{sp}A\,|\,\|\varphi-\theta\|\leq\varepsilon\}$. The case $r=1$ is proved. As an immediate corollary from it we see that $N=[M^2]$. Let now $r\geq 2$ and suppose that the assertion of the Lemma is true for the case $r-1$. Then the elements $q_k$ and $h_k$ from (8) admit the representations:

$$g_k=\sum_{|\alpha|<r}\lambda'_{\alpha,k}(\lambda u)^\alpha+G_k,\quad h_k=\sum_{|\alpha|<r}\lambda''_{\alpha,k}(\lambda u)^\alpha+H_k,$$

where $|\lambda'_{\alpha,k}|\leq C_\alpha$, $|\lambda''_{\alpha,k}|\leq C_\alpha$ for any $\alpha$ and $k$, and $|\varphi(G_k)|\leq\rho^r$, $|\varphi(H_k)|\leq\rho^r$ for every $k$. We can rewrite (8) as follows: $f=\Sigma_{|\alpha|\leq r}\lambda_\alpha(\lambda u)^\alpha+F$, where $\lambda_\alpha=1$ for $|\alpha|=1$ and $\lambda_\alpha=\Sigma_{k=1}^\infty\mu_k\Sigma_{\beta+\gamma=\alpha\,|\beta|,|\gamma|<r}\lambda'_{\beta,k}\cdot\lambda''_{\gamma,k}$ for $2\leq|\alpha|\leq r$, and $F=\Sigma_{k=1}^\infty\mu_k F_k$ with

$$F_k=G_kH_k+G_k\sum_{|\alpha|<r}\lambda''_{\alpha,k}(\lambda u)^\alpha+H_k\sum_{|\alpha|<r}\lambda'_{\alpha,k}(\lambda u)^\alpha+\sum_{\substack{\beta+\gamma=\alpha\\ r<|\alpha|<2r\\ |\beta|,|\gamma|<r}}\lambda'_{\beta,k}\lambda''_{\gamma,k}(\lambda u)^\alpha.$$

Consequently $|\lambda_\alpha|=1\leq C_\alpha$ for $|\alpha|=1$, and

$$|\lambda_\alpha|\leq\sum_{k=1}^\infty|\mu_k|\sum_{\beta+\gamma=\alpha}C_\beta C_\gamma\leq\frac{1}{2\rho}\sum_{\beta+\gamma=\alpha}C_\beta C_\gamma\leq C_\alpha$$

for $2\leq|\alpha|\leq r$. In addition, for every $\varphi\in\operatorname{sp}A$,

$$|\varphi(F_k)| \leqq |\varphi(G_k)\varphi(H_k)| + (|\varphi(G_k)| + |\varphi(H_k)|) \sum_{|\alpha|<r} C_\alpha |\varphi(\lambda u)|^\alpha$$
$$+ \sum_{r<|\alpha|<2r} \Big( \sum_{\beta+\gamma=\alpha} C_\beta C_\gamma \Big) |\varphi(\lambda u)|^\alpha .$$

Because

$$\| \varphi(\lambda u) \|_{l_1} = \Sigma_1^\infty |\varphi(\lambda_j u_j)| \leqq \| \varphi - \theta \| \, \Sigma_1^\infty \| \lambda_j u_j \| \leqq \varepsilon \Sigma_1^\infty |\lambda_j| \leqq \rho^5 \frac{1}{2\rho} \leqq \frac{\rho^4}{2},$$

then the sequence $\{\varphi(\lambda_j u_j)\}_1^\infty$ belongs to $w(\frac{\rho^3}{2}) \subset w(\rho^2)$, where the corresponding function $P(z)$ is defined. Now $|P(\varphi(\lambda u))| = |P(\{\varphi(\lambda_j u_j)\}_1^\infty)| \leqq \Sigma_{\alpha \in I} C_\alpha |\varphi(\lambda u)|^\alpha = P(|\varphi(\lambda u)|)$, which yields

$$|\varphi(F_k)| \leqq |\varphi(G_k)| \, |\varphi(H_k)| + (|\varphi(G_k)| + |\varphi(H_k)|) P(|\varphi(\lambda u)|) + 2\rho \sum_{|\alpha|>r} C_\alpha |\varphi(\lambda u)|^\alpha$$
$$\leqq \rho^{2r} + 2\rho^r P(|\varphi(\lambda u)|) + 2\rho \, . \, P_r(|\varphi(\lambda u)|)$$
$$\leqq \rho^{2r} + 2\rho^r \frac{\rho}{2} \frac{\| \varphi(\lambda u) \|_{l_1}}{\rho^2 - \| \varphi(\lambda u) \|_{l_1}} + \frac{2\rho \, \| \varphi(\lambda u) \|_{l_1}^{r+1}}{2\rho^{2r-1} (\rho^2 - \| \varphi(\lambda u) \|_{l_1})}$$
$$\leqq \rho^{2r} + \rho^{r+1} \frac{\| \varphi(\lambda u) \|_{l_1}}{\rho^2 - \| \varphi(\lambda u) \|_{l_1}} + \frac{2\rho \, \| \varphi(\lambda u) \|_{l_1}^{r+1}}{2\rho^{2r+1} (\rho^2 - \| \varphi(\lambda u) \|_{l_1})}$$
$$\leqq \rho^{2r} + \frac{\rho^{r-1} \, . \, \rho^4}{2 - \rho^2} + \frac{2\rho \, . \, \rho^{4r+4} / 2^{r+1}}{2\rho^{2r-1} (\rho^2 - \frac{\rho 4}{2})} \leqq \rho^{2r} + \frac{\rho^{r+3}}{2 - \rho^2} + \frac{\rho^{4r+4} \, . \, 2\rho}{2^{r+2} \rho^{2r-1} (\rho^2 - \frac{\rho^4}{2})}$$
$$\leqq \rho^{2r} + \frac{2}{3} \rho^{r+2} + \frac{2\rho \, . \, \rho^{2r+3}}{2^{r+2} (1 - \frac{\rho^2}{2}) \, . \, \rho^2} \leqq \rho^{r+2} \Big( \rho^{r-2} + \frac{2}{3} + \frac{\rho^r}{3.2^{r-1}} \Big)$$
$$\leqq \rho^{r+2} \Big( 1 + \frac{2}{3} + \frac{1}{3} \Big) \leqq 2\rho^{r+2}.$$

Now $|\varphi(F)| \leqq \Sigma |\mu_k| \, |\varphi(F_k)| \leqq \frac{1}{2\rho} 2\rho^{r+2} = \rho^{r+1}$, Q. E. D.

Note, that according to the remark after the case $r=1$, $N=[M^2]$ and hence the space $M/[M^2] = M/N$ of continuous point derivations at the point $\theta$ is not finite-dimensional.

If $E \subset \overline{\Delta}_{\widehat{R}}(\eta)$, then $A_G^\vee(E)$ will denote the algebra of uniform limits of finite linear combinations over $\mathbb{C}$ of functions $\chi^p$, with $p$ belonging to $\Gamma_+ = Q^\nu = \mathrm{Rat}\,[\nu, +\infty) \cup \{0\}$. By $\widehat{u}$: $\mathrm{sp}\, A \to \mathbb{C}^\infty$ we denote in the following the function: $\widehat{u}(\varphi) = (\widehat{u_1}(\varphi), \widehat{u_2}(\varphi), \ldots) = (\varphi(u_1), \varphi(u_2), \ldots)$, $u_j \in A$.

Theorem 1. *Let* $A$ *be a uniform algebra and* $\theta$ *be a fixed linear multiplicative functional on* $A$. *Suppose, that there exists a multiplicative subsemigroup* $\mathfrak{U} = \{u_{p(j)}\}_{j=1}^\infty$ *in* $M = \mathrm{Ker}\,\theta$, $\| u_{p(j)} \| \leqq 1$, $u_{p(i)} - u_{p(j)} \notin [M^2]$ *for* $i \neq j$, *isomorphic to the additive semigroup* $Q^\nu = \mathrm{Rat}\,[\nu, +\infty) \cup \{0\}$, $0 < \nu < 1$, *such that any function* $f \in A$ *is presentable uniquely in the form*

$$f = \theta(f) + \sum_{p(j) \in [\nu, 2\nu]} {}_{j=1}^{\infty} \lambda_j u_{p(j)} + g,$$

*where* $g \in N = \{ f \in M \mid f = \Sigma_1^\infty \mu_k g_k h_k,\ g_k, h_k \in B_M,\ \| \{\mu_k\} \|_{l_1} < \infty \}$. *Then there exists a non-isolated point according to the metric topology in* $\mathrm{sp}\, A$, *there exists* *a set* $U \subset \operatorname{sp} A$, *containing* $\theta$ *as an inner point in the metric topology of* $\operatorname{sp} A$, *a gen.-analytic set* $V$ *in some big disc* $\Delta_G(d)$, *where* $G = \widehat{Q}$, $d > 0$, *and a homeomorphism* $\tau: U \to V$, $\tau(\theta) = \{*\}$, *such that* $\widehat{f} \circ \tau^{-1}$ *is a gen.-analytic function from the algebra* $A_G^\vee(d) = A_G^\vee(\Delta_G(d))$ *for any* $f \in A$.

It is clear that under the above conditions the space $M/N$, and consequently the space $M/[M^2]$ of continuous point derivations at $\theta$ too, have countable many dimensions.

P r o o f. If $\psi \in \operatorname{sp} A$, for every $p = n/m \in R^\nu$ it holds that: $|\psi(u_p)|^m = |\psi(u_{n/m})|^m = |\psi(U_m)| = |\psi(u_1)|^n$, i. e. $|\psi(u_p)| = |\psi(u_1)|^{n/m} = |\psi(u_1)|^p$. Let $\rho$ be the positive number, defined in Lemma 1, $\varepsilon = \rho^5$, $\delta = 2\rho^7/1 + \rho^4 < \varepsilon$, $\eta = \delta^{1/\nu} < \delta$ and $U_1 = \{\varphi \in \operatorname{sp} A \mid \|\varphi - \theta\| \leq \varepsilon, \ |\varphi(u_1)| < \eta\} \supset \{\varphi \in \operatorname{sp} A \mid \|\varphi - \theta\| < \eta\} \ni \theta$, considered with the metric topology, induced on $U_1$ from $A^*$. We claim that for every $\varphi \in U_1$ there exists a point $\tau(\varphi)$ in $\overline{\Delta}_G(\eta)$, where $G = \widehat{R}$, such that $\chi^p(\tau(\varphi)) = \varphi(u_p) = \widehat{U}_p(\varphi)$ for any $p \in \operatorname{Rat}[\nu, 2\nu)$. Let $\lambda_\varphi = |\varphi(u_1)| \leq 1$ and $\gamma_\varphi(p) = \varphi(u_p)/|\varphi(u_p)|$ for any $p \in \operatorname{Rat}[\nu, 2\nu)$, $\gamma_\varphi(p) = \overline{\gamma_\varphi(-p)}$ for $p \in \operatorname{Rat}[\nu, 2\nu)$. The function $\gamma_\varphi$ is authomatically extendable on $R$, so that we can assume that $\gamma_\varphi \in G = \widehat{R}$. For any $\varphi \in U_1$ the point $\tau(\varphi) = (\lambda_\varphi, \gamma_\varphi)$ belongs to $\overline{\Delta}_G(\eta)$ and $\chi^p(\tau(\varphi)) = \chi_\varphi^p \chi^p(\gamma_\varphi) = \lambda_\varphi^p \gamma_\varphi(p) = |\varphi(u_p)|(\varphi(u_p)/|\varphi(u_p)|) = \varphi(u_p) = \widehat{u}_p(\varphi)$. Consequently $\tau(U_1) \subset \overline{\Delta}_G(\eta)$ and $(\lambda_\varphi, \gamma_\varphi)$ satisfies the sought properties. The point $\tau(\varphi)$ is defined uniquely, because the functions $X^p$, $p \in \operatorname{Rat}[\nu, 2\nu)$, separate the points of $\overline{\Delta}_G(\eta)$. If $\varphi_\alpha \to \varphi_0$ is a strongly convergent sequence of elements of $U_1 \subset A^*$, then $\varphi_\alpha(u_p) \to \varphi_0(u_p)$ for every $p \in \operatorname{Rat}[\nu, 2\nu)$. Then $\lambda_\alpha \to \lambda_0$, $\gamma_\alpha \to \gamma_0$. If $t_0$ is a point of accumulation for $\{\tau(\varphi_\alpha)\} = \{(\lambda_\alpha, \gamma_\alpha)\}$, then $\tau(\varphi_{\alpha_\beta}) \to t_0$ for some subsequence $\alpha_\beta$, from where $\tau(\varphi_0) = t_0$. From $\chi^p(\tau(\varphi)) = \widehat{u}_p(\theta) = 0$ for $p \in \operatorname{Rat}[\nu, 2\nu)$, we obtain that $\tau(\theta) = \{*\}$. Applying Lemma 1 to the algebra $A$, we see that given a function $f \in M$, there exists a sequence $\{p_r \mid p_r(\lambda z) = \sum_{|\alpha| \leq r} \xi_{\alpha,r} \lambda^\alpha z^\alpha, \ |\xi_{\alpha,r}| \leq 2\|f\| C_\alpha\}$ of polynomials of countable many-dimensional arguments and the functions $p_r(\lambda \widehat{U})$ approximate the function $\widehat{f} - \theta(f)$ uniformly on $U_1$. Consequently every $\varphi \in U_1$ is uniquely determined by its values on the elements $u_p$, $p \in Q^\nu$. A consequence from this is that the mapping $\tau: U_1 \to \tau(U_1) \subset \overline{\Delta}_G(\eta)$ is one-to-one. As a one-to-one and continuous mapping from a locally compact set $U_1 \subset \operatorname{sp} A$ into a Hausdorff space, $\tau$ is a homeomorphism. Consequently the peak $\{*\}$ is an inner point of $V_1 = \tau(U_1)$. On the other hand, we obtain that for any $f \in A$ the function $\widehat{f} \circ \tau^{-1} \in A_G^\vee(V_1)$, i. e. that the function $\widehat{f} \circ \tau^{-1}$ belongs to the algebra $A_G^\vee(V_1)$. Moreover, the function $\widehat{f} \circ \tau^{-1}$ can be extended from $V_1$ up to the big disc $\Delta_G(d)$ as an element of $A_G^\vee(d)$. In fact, the functions $q_r = p_r \circ \lambda \widehat{u} \circ \tau^{-1}$ are defined not only on $V_1$, but also on $\overline{\Delta}_G(\eta)$, and present the partial sums of a gen.-power series, converged on any big disc $\overline{\Delta}_G(d)$ with $0 < d < \max\{\lambda \mid (\lambda, g) \in V_1 \subset \overline{\Delta}_G(\eta)\}$, according to a generalized version of Abel's theorem for power series. Let $V$ denote the set $\Delta_G(d) \cap V_1$, where $d$ is as above, and let $U = \tau^{-1}(V)$. In order to prove that $V$ is a gen.-analytic subset of $\Delta_G(d)$, we consider the family $\mathscr{F} = \{F \in A_G^\vee(d) \mid F(\lambda, g) \equiv 0$ on $V\}$ of gen.-analytic functions, and the set $\widetilde{V} = \{(\lambda, g) \in \Delta_G(d) \mid F(\lambda, g) = 0, F \in \mathscr{F}\}$. It is clear that $V \subset \widetilde{V}$. Let $(\gamma_0, g_0) \in \widetilde{V}$. Let $A_f = \{F \in A_G^\vee(d) \mid \widehat{f} \circ \tau^{-1} = F\}$, where $f$ is the given element of $A$. If $F_1$ and $F_2$ are elements of $A_f$, then $F_1 - F_2 \in \mathscr{F}$ and $F_1(\lambda_0, g_0) = F_2(\lambda_0, g_0)$ for the point

$(\lambda_0, g_0)$ from $\widetilde{V}$, Now the linear and multiplicative functional $\varphi_0$: $\varphi_0(f)=F(\lambda_0, g_0)$ belongs to the spectrum $\operatorname{sp} A$ (see [1]). According to Lemma 1,

$$|\varphi_0(f)-\theta(f)|=|F(\lambda_0, g_0)-F(*)|=\lim_r |q_r(\lambda_0, g_0)|=|\sum_\alpha a_\alpha \lambda^\alpha(\mathbf{X}^{p(j)})^\alpha|$$

$$\leqq 2\|f\| \Sigma C_\alpha |\lambda^\alpha(\mathbf{X}^{p(j)})^\alpha| \leqq 2\|f\| p(\{|\lambda_j \mathbf{X}^{p(j)}|\}) \leqq 2\|f\| \frac{\rho}{2} \frac{\Sigma|\lambda_j||\mathbf{X}^{p(j)}|}{\rho^2-\Sigma|\lambda_j||\mathbf{X}^{d(j)}|}$$

$$\leqq \rho\|f\| \frac{\frac{1}{2\rho} d^\nu}{\rho^2-\frac{1}{2\rho} d^\nu} \leqq \rho\|f\| \frac{n^\nu}{2\rho^3-\eta^\nu} \leqq \rho\|f\| \frac{\rho}{2\rho^3-\delta} \leqq \frac{\rho \frac{2\rho^7}{1+\rho^4}}{2\rho^3-\frac{2\rho^7}{1+\rho^4}} \|f\| = \rho^5\|f\| = \varepsilon\|f\|$$

for $(\lambda_0, g_0) \in \Delta_G(d) \subset \Delta_G(\eta)$ and $f \in A_0$. Consequently $\|\varphi_0-\theta\| \leqq \varepsilon$, and $\varphi(u_1) = \widehat{u}_1(\varphi_0) = \mathbf{X}^1(u_1) \in V$, since $|\varphi_0(u_1)| = |\widehat{u_1}(\varphi_0)| = |\mathbf{X}^1(u_1)| \leq d \leq \eta$. Now $(\lambda_0, g_0) = \tau(\varphi_0) \in V$, because $\varphi_0 \in U_1$, $\tau(\varphi_0) \in \Delta_G(d)$. Hence, $\widetilde{V}=V$, i. e. $V$ is a gen.-analytic subset of $\Delta_G(d)$. The theorem is proved.

As an application of Theorem 1 we give another proof of the result of [6].

Corollary 1. *Let $\theta \in \operatorname{sp} A$ be such that in $M=\operatorname{Kev}\theta$ there exists a sequence $\{g_{p(j)}\}_{j=1}^\infty$, $\|g_{p(j)}\| \leqq 1$, for which:*

*1) $\{g_{p(j)}\}$ is a multiplicative subsemigroup, isomorphic to the additive semigroup $Q^\nu=\operatorname{Rat}[\nu, +\infty) \vee \{0\}$, $0<\nu<1$;*

*2) any element $f$ of $M$ admits a unique representation of the form, where $f_j \in A$, $\Sigma_{j=1}^\infty \|f_j\| < \infty$. Then there exist a number $d>0$, a neighbourhood $\mathsf{U}$ of $\theta$ of the type $\mathsf{U}=\{\varphi \in \operatorname{sp} A \mid |\varphi(g_1)|<\eta, \eta>0\}$, a gen.-analytic set $V$ in some big disc $\Delta_G(d)$ with $G=\widehat{Q}$ and a homeomorphism $\tau: \mathsf{U} \to V$, $\tau(\theta)=\{*\}$ such that $\widehat{f} \circ \tau^{-1}$ is a gen.-analytic function from the algebra $A_G^\nu(d)$ for any $f \in A$.*

Proof. Let $f \in M$, $f=\Sigma f_j g_{p(j)}$, $f_j \in A$, $\|f\|_1=\Sigma\|f_j\|<\infty$. For every $j=1, 2, \ldots, f_j-\theta(f_j)=\sum_k f_{jk} g_{p(k)}$, with $\sum_k |f_{jk}\|<\infty$, and hence $f=\sum_j \theta(f_j) g_{p(j)} + \sum_{jk} f_{jk} g_{p(k)} g_{p(l)}$, where $\Sigma|\theta(f_j)| \leqq \Sigma\|f_j\|<\infty$, so that the kernel $M$ satisfies the conditions of Theorem 1. We shall show that now any metric neighbourhood of $\theta$ contains a set of the type $\{\varphi \in \operatorname{sp} A \mid |\varphi(g_1)|<\varepsilon\}$ for some $\varepsilon>0$. In fact, if $S=\{f \in A \mid f=\Sigma f_j g_{p(j)}, \|f\|_1 \leq 1\}$, then $M=\bigcup_m mS=\bigcup_m [mS]$. Applying the Baire category theorem, we can see that $[S]$ contains a $\|\ \|_1$-neighbourhood of 0, say $\{f \in M \mid \|f\|_1=\Sigma\|f_j\| \leqq 4\sigma\}$. It is easy to see that similarly to the situation in Lemma 1, $B_M \subset \frac{1}{2\sigma} S$. Consequently for any $f \in A$ there exist elements $f_j \in A$, with $\Sigma\|f_j\| \leq \frac{1}{2\sigma}\|f\|$, for which $f=\Sigma_1^\infty f_j g_{p(j)}$. If now $\|f\| \leq 1$, $f \in A_0$, then... Now the metric neighbourhood $\{\varphi \in \operatorname{sp} A \mid \|\varphi-\theta\|<\varepsilon\}$ of $\theta$ contains the weak*-neighbourhood $\{\varphi \in \operatorname{sp} A \mid |\varphi(g_1)|<d=(\varepsilon\sigma)^{1/\nu}\}$, Q. E. D.

Remark 1. With obvious modifications Lemma 1 holds also in the case of uncountable subsemigroups $\mathfrak{U}$ of $\mathbf{R}^1$.

Remark 2. The right inequality for $\nu$: $0<\nu<1$ in Theorem 1 and in Corollary 1 can be dropped without loss of generality.

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