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ON A CLASS OF GENERALIZED RIEMANN ENTIRE FUNCTIONS

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A class of generalized Riemann entire functions is considered. Some results about the distribution of the zeros of such functions are obtained.

In [1] Iliev introduced entire functions of the kind

$$\int_{-\infty}^{\infty} F(t)g(z+t)dt \tag{1}$$

and

$$\int_{-\infty}^{\infty} F(t)\,p(itz)\,dt, \tag{2}$$

where $g(z)$ is a polynomial with zeros in a strip $\alpha \leqq Rez \leqq \beta$ or a limit of such polynomials in every bounded domain, and $p(z)$ is a polynomial with real non-positive zeros only or a limit of such polynomials in every bounded domain. The functions (1) and (2) are called generalized Riemann entire functions [1].

In [2] Bozhorov examined generalized Riemann entire functions of the kind (1), where the zeros of $g(z)$ lie in the half-plane $Rez \leqq \gamma$, $\gamma \in \mathbf{R}$.

Let L_1^φ denote the class of entire functions which are polynomials with zeros in A_1^φ, A_1^φ: $|\arg z - \pi| \leqq \varphi$, $0 \leqq \varphi \leqq \pi/2$ or in every bounded domain are limits of such polynomials.

In this paper entire functions of the kind

$$\int_{-\infty}^{\infty} F(t)p(tz)\,dt \tag{3}$$

are considered where $p(z) \in L_1^{\varphi-\infty}$. According to [1] the functions of the kind (3) will be called generalized Riemann entire functions of the class $R(L_1^\varphi)$, $0 \leqq \varphi \leqq \pi/2$.

Let us represent the entire functions $p(z)$ as a series

$$p(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k. \tag{4}$$

Then we get formally for (3)

$$\int_{-\infty}^{\infty} F(t)p(zt)dt = \sum_{k=0}^{\infty} \frac{a_k b_k}{k!} z^k,$$

where $b_k = \int_{-\infty}^{\infty} t^k F(t)dt$, $k = 0, 1, 2, \ldots$

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Let $g(S)$ be a linear transformation defined on entire functions by

$$g(S)p=\sum_{k=0}^{\infty}\frac{a_k b_k}{k!}z^k, \tag{5}$$

where $p(z)=\sum_{k=0}^{\infty}\frac{a_k}{k!}z^k$ and $g(z)=\sum_{k=0}^{\infty}\frac{b_k}{k!}z^k=\int_{-\infty}^{\infty}F(t)e^{zt}dt$. It is natural to call (5) transformations of Shur. Thus we obtain for (3)

$$\int_{-\infty}^{\infty}F(t)p(zt)dt=g(S)p.$$

This representation (in the domain where the integrals are convergent) provides a possiblilitary to connect the distribution of the zeros of (3) with the operation of the transformation (5) on the functions of the class L_1^{φ}.

1. The following statements hold.

Theorem 1. *For every function $p(z)\in L_1^{\varphi}$ the function $g(S)p$ belongs to the class L_1^{φ} if and only if $g(z)\in L_1^0$: $L_1^0\equiv L_1$ is the class of entire functions which are polynomials with real non-positive zeros or limits of such polynomials in every bounded domain.*

Proof. Let $g(z)=\sum_{k=0}^{\infty}\frac{b_k}{k!}z^k\in L_1$ and $p(z)=\sum_{k=0}^{\infty}\frac{a_k}{k!}z^k\in L_1^{\varphi}$. It follows from the transcendental criteria for α-sequences and I^{φ}-sequences ([1], p. 18 and p. 39) that $\{b_k\}_{k=0}^{\infty}$ is α-sequence and $\{a_k\}_{k=0}^{\infty}$ is I^{φ}-sequence. Consequently $\{a_k b_k\}_{k=0}^{\infty}$ is I^{φ}-sequence. Then

$$\sum_{k=0}^{\infty}\frac{a_k b_k}{k!}z^k=g(S)p\in L_1^{\varphi}.$$

Conversely, suppose that for every $p(z)\in L_1^{\varphi}$

$$g(S)p=\int_{-\infty}^{\infty}F(t)p(zt)dt\in L_1^{\varphi}.$$

In particular, when $p(z)=(1+z)^n\in L_1$ we obtain

$$g(S)p=\int_{-\infty}^{\infty}F(t)(1+zt)^n dt=J_n(g;z)\in L_1,$$

where $J_n(g;z)$ is the Jensen polynomial for the function $g(z)$. Consequently $g(z)\in L_1$. This completes the proof of the theorem.

In particular, if $\varphi=0$ we obtain that for every function $p(z)\in L_1$ the function $g(S)p$ belongs to the class L_1 if and only if $g(z)\in L_1$.

Theorem 2. *For every function $p(z)\in L_1$ the function $g(S)p$ belongs to the class L_1^{φ} if and only if $g(z)\in L_1^{\varphi}$.*

Theorem 3. *For every function $p(z)\in L_1$ the function $g(S)p$ belongs to the class L_2 if and only if $g(z)\in L_2$: L_2 is the class of entire functions which are polynomials with real zeros only or limits of such polynomials in every bounded domain.*

The proofs of Theorems 2 and 3 are analogous to that of Theorem 1. As an application of Theorem 1 we get the following

Theorem 4. *Let* $g_k(z) = \int_{-\infty}^{\infty} F_k(t)\, e^{tz}\, dt \in L_1$, $k=1, 2, \dots, n$ *and* $p(z) \in L_1^{\varphi}$. *Then the function*

$$\Phi(z) = \int_{-\infty}^{\infty} F_1(t_1) dt_1 \int_{-\infty}^{\infty} F_2(t_2) dt_2 \dots \int_{-\infty}^{\infty} F_n(t_n) p(z t_1 t_2 \dots t_n) dt_n$$

belongs to the class L_1^{φ}.

In particular, if $p(z) = (1+z)^n \in L_1$, *then*

$$\int_{-\infty}^{\infty} F_1(t_1) dt_1 \int_{-\infty}^{\infty} F_2(t_2)\, dt_2 \dots \int_{-\infty}^{\infty} F_n(t_n)\ (1 + t_1 t_2 \dots t_n z)^n dt_n \in L_1$$

and therefore

$$\int_{-\infty}^{\infty} F_1(t_1) dt_1 \int_{-\infty}^{\infty} F_2(t_2) dt_2 \dots \int_{-\infty}^{\infty} F_n(t_n) \exp(t_1 t_2 \dots t_n z) dt_n \in L_1.$$

2. We have shown that the distribution of the zeros of the generalized Riemann entire functions of the kind (3) can be connected with the distribution of the zeros of functions of the kind $g(z) = \int_{-\infty}^{\infty} F(t)\, e^{zt} dt$. With these functions are associated the polynomials

$$J_n(z) = \int_{-\infty}^{\infty} F(t) (1 + tz)^n dt.$$

The polyonmials $J_n(z)$ are the so-called Jensen polynomials of the function $g(z)$. Let us notice that $J_n(z)$ can be considered as generalized Riemann entire functions of the class $R(L_1^{\varphi})$

We introduce the following classes of functions

$$H^{\varphi}: \{F(t) \mid \int_{-\infty}^{\infty} F(t)\ (1+tz)^n dt \in L_1^{\varphi}\},\quad n = 1, 2, \dots,\ 0 \leqq \varphi \leqq \pi/2;$$

$$S^{\varphi}: \ = \{\alpha(t) \mid \forall F(t) \in H^{\varphi} \Rightarrow \alpha(t) F(t) \in H^{\varphi}\}.$$

According to Bozhorov [2] the functions $\alpha(t)$ of the class S^{φ} will be called multiplier functions of the class H^{φ}.

It is evident that if $\alpha(t) \in S^{\varphi}$, $\beta(t) \in S^{\varphi}$ then $\alpha(t)\beta(t) \in S^{\varphi}$. We will prove the following

Theorem 5. *Every function which belongs to the class* L_1^{φ} *is a multiplier function of the class* H^{φ}, *i. e.* $L_1^{\varphi} \subset S^{\varphi}$.

Proof. It is sufficient to prove that every polynomial of L_1^{φ} is a multiplier function of H^{φ}.

Let B^{φ} denote the domain: $|\arg z| \leqq \varphi$. We will show that if $\xi \in B^{\varphi}$ then $1 + t\xi \in S^{\varphi}$.

We suppose that $J(z) = \int_{-\infty}^{\infty} F(t)\ (1+tz)^n\, dt \in L_1^{\varphi}$, $n = 1, 2, \dots$ Let

$$\tilde{J}_{\xi}(z) = nJ(z) - (z - \xi)\ J'(z) = n \int_{-\infty}^{\infty} F(t)\ (1 + t\xi)\ (1 + tz)^{n-1}\, dt$$

be the polar derivative of $J(z)$. Since the zeros of $J(z)$ belong to A_1^φ and A_1^φ is the section of the half-planes $\varphi \leq \arg z \leq \pi+\varphi$, $\pi-\varphi$ $\arg z \leq 2\pi-\varphi$ then from the theorem of Laguerre ([3], p. 120) it follows that the zeros of $J\tilde{\xi}(z)$ also belong to this section. Consequently

$$\int_{-\infty}^{\infty} (1+t\xi)F(t)\ (1+tz)^n dt \in L_1^\varphi.$$

This implies $1+t\xi \in S^\varphi$.

Let us now suppose that the polynomial $Q(z) \in L_1^\varphi$ and $Q(z)=az^k \prod_{n=1}^{m}(1-z/z_n)$, where $z_n \in A_1^\varphi$ and $k \geq 0$. Then the points $-1/z_n \in B^\varphi$. In view of the above result we have $1+t(-1/z_n) \in S^\varphi$ for $n=1, 2, \ldots, m$. Since $t^k \in S^\varphi$ we obtain $Q(t) \in S^\varphi$.

Theorem 6. *If* $g(z)=\int_{-\infty}^{\infty} F(t)e^{zt}dt$ *is an entire function of the class* L_1^φ *and the function* $\alpha(t) \in S^\varphi$, *then*

$$g_1(z)=\int_{-\infty}^{\infty} \alpha(t)F(t)e^{zt}\, dt \in L_1^\varphi.$$

Proof. Let $g(z) \in L_1^\varphi$. Then

$$J_n(g\,;\,z)=\int_{-\infty}^{\infty} F(t)\ (1+tz)^n dt \in L_1^\varphi, \quad n=1,\ 2,\ 3,\ldots$$

and therefore $F(t) \in H^\varphi$. Since $\alpha(t) \in S^\varphi$ we have $\alpha(t)F(t) \in H^\varphi$, i. e.

$$\int_{-\infty}^{\infty} \alpha(t)F(t)\ (1+tz)^n dt=J_n(g_1\,;\,z) \in L_1^\varphi.$$

It is sufficient to conclude that $g_1(z) \in L_1^\varphi$.

As an application of Theorem 5 and Theorem 6 we obtain that if $g(z)=\int_{-\infty}^{\infty} F(t)e^{tz}\,dt \in L_1^\varphi$ and $\alpha(t) \in L_1^\varphi$, then $\int_{-\infty}^{\infty} \alpha(t)\,F(t)e^{tz}dt \in L_1^\varphi$. In this case the following problem arises. Let us suppose that for every entire function $g(z) = \int_{-\infty}^{\infty} F(t)e^{zt}dt \in L_1^\varphi$, the function $\int_{-\infty}^{\infty} \alpha(t)F(t)e^{zt}dt \in L_1^\varphi$. Then whether the function $\alpha(t)$ belongs to the class L_1^φ. When $\varphi=\pi/2$ the problem is solved by Bozhorov [2].

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