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NOTE ON A PAPER OF GROSSWALD AND SCHNITZER

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Let M be a compact Riemann surface of genus $g\geq 2$, l_n, $n=1, 2, 3, \ldots$ be the lengths of the primitive closed geodesics and $N(P_n)=\exp(l_n)$. The Selberg zeta function is defined by an infinite product

$$Z(s)=\prod_{\{p\}}\prod_{k=0}(1-N(P_n)^{-s-k}),\quad Re(s)>1.$$

Here and in what it follows $\prod_{\{p\}}$ denotes $\prod_{n=1}^{\infty}$ and $\sum_{\{p\}}$ denotes $\sum_{n=1}^{\infty}$. Let $\lambda_n=1/4+r_n^2$ be the eigenvalues of the Laplace-Beltrami operator on M, such that $\lambda_n\geq 1/4$. It is known (see [2]) that $Z(s)$ can be continued as an entire function and all its zeroes γ with $\operatorname{Im}\gamma\neq 0$ have the form $1/2+i.r_n$, i. e. lie on the line $Re(s)=1/2$.

Let we select any real numbers $N(Q_n)$, so that $N(P_n)\leq N(Q_n)$ and $N(Q_n)\leq N(P_{n+1})$ for $n>n_0$, where n_0 is arbitrary large. Then we form

$$Z^*(s)=\prod_{n=1}^{\infty}\prod_{k=1}^{\infty}(1-N(Q_n)^{-s-k}),\quad s=\sigma+i.t,\quad \sigma>1.$$

Theorem 1. $Z^*(s)$ can be continued as a meromorphic function in $\sigma>0$, where it has the same zeroes as $Z(s)$.

For the Riemann zeta function this theorem was proved by Grosswald and Schnitzer, whose proof we follow.

Proof. Let $Z^*(s)=\varphi(s).Z(s)$, $\sigma>1$, where

$$\varphi(s)=\prod_{\{p\}}\prod_{k=0}^{\infty}(1-N(Q_n)^{-s-k}).(1-N(P_n)^{-s-k})^{-1}.$$

We shall verify the absolutely and uniform convergence of the above infinite product on any compact subset of $\sigma>0$. It is easy to see that this is true for $\sigma>1$. Hence, it is sufficient to prove the absolutely and uniform convergence of the series for $\log\varphi(x)$ in $|t|\leq T$, $0<\sigma_0\leq\sigma\leq 1$. From this also follows that $\varphi(x)\neq 0$.

We assume that $N(P_n)>2$ and consider

$$\log\varphi(x)=\sum_{\{p\}}\sum_{k=0}^{\infty}\sum_{m=1}^{\infty}(1/m).(N(P_n)^{-m\cdot(s+k)}-N(Q_n)^{-m(s+k)}).$$

We split the sum over k and m into three parts

$$\sum_{\substack{m=1\\k=0}}^{m_0}+\sum_{\substack{m=m_0+1\\k=0}}^{\infty}+\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}=\Sigma^1+\Sigma^2+\Sigma^3,$$

where $m_0=[\sigma^{-1}]+1$ and $[x]$ stands for the greatest integer function.

For the second and third sum we easily estimate

$$|\Sigma^3|\leqq 2.\sum_{p}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}(m.N(P_n)^{m(\sigma+k)})^{-1}\leqq 2.\sum_{\{p\}}\sum_{k=1}^{\infty}N(P_n)^{-\sigma-k}.(1-N(P_n)^{-\sigma-k})^{-1}$$

$$\leqq 2.(1-N(P_1)^{-\sigma_0})^{-1}\sum_{\{p\}}\sum_{k=1}^{\infty}N(P_n)^{-\sigma-k}\leqq C_1\sum_{\{p\}}N(P_n)^{-\sigma_0-1}.(1-N(P_n)^{-1})^{-1}$$

$$\leqq C_1.(1-N(P_1)^{-1})^{-1}.\sum_{\{p\}}N(P_n)^{-1-\sigma_0}.$$

Consider the first sum. Let $N(Q_n)=N(P_n)+s_n$.
From the prime number theorem for $N(P_n)$ (see Huber [3]) it follows that $s_n\leqq N(P_n)^{3/4+\varepsilon}$, where $\varepsilon>0$, n is a sufficiently large number and from this $0\leq s_n\leq N(P_n)^{7/8}$ for $n>n_1$. If we set $x_n=N(P_n)/N(Q_n)$, we have $x_n^2>1-N(Q_n)^{-1/8}$ and consequently $x_n^2>1/3$, for $n>N$. Thus we obtain

$$|N(P_n)^{-m\sigma}-N(Q_n)^{-m\sigma}|=N(P_n)^{-m\sigma}|1-x^{m\cdot\sigma}.\exp(itm\log^{‘}(x)|.$$

Applying estimates from [1], we set

$$|1-x^{m\sigma}\exp(\text{i. t. m.}\log(x))|\leq C_2.(1-x^{m\sigma})\ \text{for}\ 1\leq m\leqq m_0.$$

Here C_2 depends only on σ_0 and T. From this

$$\sum_{n\geq N}|N(P_n)^{-ms}-N(Q_n)^{-m\cdot s}|\leq C_2.\sum_{n\geq N}(N(P_n)^{-m\sigma}-N(Q_n)^{-m\sigma})$$

$$\leq C_2.\sum_{n\geq N}(N(P_n)^{-m\sigma}-N(P_{n+1})^{-m\sigma})=C_2.N(P_N)^{-m\sigma}\leq C_2.N(P_N)^{-\sigma_0}$$

and $|\Sigma^1|\leq C_2.m_0.N(P_N)^{-\sigma_0}$.

We note that knowing the eigenvalues of the Laplace-Beltrami operator on M we can find the lengths of the primitive closed geodesics. Thus, we have the following

Corollary. Let l_n be the lengths of the primitive closed geodesics on M and $\tilde{l}_n\in[l_n, l_{n+1}]$ for $n>N$. There is no other Riemann surface $\tilde{M}$ of the same genus g with lengths of primitive closed geodesics equal to l_n.

Proffesor L. Keen has kindly pointed out to me that the corollary essentially follows from the so-called collar lemma (see [4]).

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