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NOTE ON A PAPER OF GROSSWALD AND SCHNITZER

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Let M be a compact Riemann surface of genus $g \geq 2$, l_n , $n=1, 2, 3, \dots$ be the lengths of the primitive closed geodesics and $N(P_n) = \exp(l_n)$. The Selberg zeta function is defined by an infinite product

$$Z(s) = \prod_{\{p\}} \prod_{k=0}^{\infty} (1 - N(P_n)^{-s-k}), \quad \operatorname{Re}(s) > 1.$$

Here and in what it follows $\prod_{\{p\}}$ denotes $\prod_{n=1}^{\infty}$ and $\sum_{\{p\}}$ denotes $\sum_{n=1}^{\infty}$. Let $\lambda_n = 1/4 + r_n^2$ be the eigenvalues of the Laplace-Beltrami operator on M , such that $\lambda_n \geq 1/4$. It is known (see [2]) that $Z(s)$ can be continued as an entire function and all its zeroes γ with $\operatorname{Im} \gamma \neq 0$ have the form $1/2 + i \cdot r_n$, i. e. lie on the line $\operatorname{Re}(s) = 1/2$.

Let us select any real numbers $N(Q_n)$, so that $N(P_n) \leq N(Q_n)$ and $N(Q_n) \leq N(P_{n+1})$ for $n > n_0$, where n_0 is arbitrary large. Then we form

$$Z^*(s) = \prod_{n=1}^{\infty} \prod_{k=1}^{\infty} (1 - N(Q_n)^{-s-k}), \quad s = \sigma + i \cdot t, \quad \sigma > 1.$$

Theorem 1. $Z^*(s)$ can be continued as a meromorphic function in $\sigma > 0$, where it has the same zeroes as $Z(s)$.

For the Riemann zeta function this theorem was proved by Grosswald and Schnitzer, whose proof we follow.

Proof. Let $Z^*(s) = \varphi(s) \cdot Z(s)$, $\sigma > 1$, where

$$\varphi(s) = \prod_{\{p\}} \prod_{k=0}^{\infty} (1 - N(Q_n)^{-s-k}) \cdot (1 - N(P_n)^{-s-k})^{-1}.$$

We shall verify the absolutely and uniform convergence of the above infinite product on any compact subset of $\sigma > 0$. It is easy to see that this is true for $\sigma > 1$. Hence, it is sufficient to prove the absolutely and uniform convergence of the series for $\log \varphi(x)$ in $|t| \leq T$, $0 < \sigma_0 \leq \sigma \leq 1$. From this also follows that $\varphi(x) \neq 0$.

We assume that $N(P_n) > 2$ and consider

$$\log \varphi(x) = \sum_{\{p\}} \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} (1/m) \cdot (N(P_n)^{-m \cdot (s+k)} - N(Q_n)^{-m(s+k)}).$$

We split the sum over k and m into three parts

$$\sum_{\substack{m=1 \\ k=0}}^{m_0} + \sum_{\substack{m=m_0+1 \\ k=0}}^{\infty} + \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} = \Sigma^1 + \Sigma^2 + \Sigma^3,$$

where $m_0 = [\sigma^{-1}] + 1$ and $[x]$ stands for the greatest integer function.

For the second and third sum we easily estimate

$$\begin{aligned} |\Sigma^3| &\leq 2 \cdot \sum_p \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} (m \cdot N(P_n)^{m(\sigma+k)})^{-1} \leq 2 \cdot \sum_{\{p\}} \sum_{k=1}^{\infty} N(P_n)^{-\sigma-k} \cdot (1 - N(P_n)^{-\sigma-k})^{-1} \\ &\leq 2 \cdot (1 - N(P_1)^{-\sigma_0})^{-1} \sum_{\{p\}} \sum_{k=1}^{\infty} N(P_n)^{-\sigma-k} \leq C_1 \sum_{\{p\}} N(P_n)^{-\sigma_0-1} \cdot (1 - N(P_n)^{-1})^{-1} \\ &\leq C_1 \cdot (1 - N(P_1)^{-1})^{-1} \cdot \sum_{\{p\}} N(P_n)^{-1-\sigma_0}. \end{aligned}$$

Consider the first sum. Let $N(Q_n) = N(P_n) + s_n$. From the prime number theorem for $N(P_n)$ (see Huber [3]) it follows that $s_n \leq N(P_n)^{3/4+\epsilon}$, where $\epsilon > 0$, n is a sufficiently large number and from this $0 \leq s_n \leq N(P_n)^{7/8}$ for $n > n_1$. If we set $x_n = N(P_n)/N(Q_n)$, we have $x_n^2 > 1 - N(Q_n)^{-1/8}$ and consequently $x_n^2 > 1/3$, for $n > N$. Thus we obtain

$$|N(P_n)^{-m\sigma} - N(Q_n)^{-m\sigma}| = N(P_n)^{-m\sigma} |1 - x^{m \cdot \sigma} \cdot \exp(itm \log^t(x))|.$$

Applying estimates from [1], we set

$$|1 - x^{m\sigma} \exp(i \cdot t \cdot m \cdot \log(x))| \leq C_2 \cdot (1 - x^{m\sigma}) \text{ for } 1 \leq m \leq m_0.$$

Here C_2 depends only on σ_0 and T . From this

$$\begin{aligned} \sum_{n \geq N} |N(P_n)^{-ms} - N(Q_n)^{-m \cdot s}| &\leq C_2 \cdot \sum_{n \geq N} (N(P_n)^{-m\sigma} - N(Q_n)^{-m\sigma}) \\ &\leq C_2 \cdot \sum_{n \geq N} (N(P_n)^{-m\sigma} - N(P_{n+1})^{-m\sigma}) = C_2 \cdot N(P_N)^{-m\sigma} \leq C_2 \cdot N(P_N)^{-\sigma_0} \end{aligned}$$

and $|\Sigma^1| \leq C_2 \cdot m_0 \cdot N(P_N)^{-\sigma_0}$.

We note that knowing the eigenvalues of the Laplace-Beltrami operator on M we can find the lengths of the primitive closed geodesics. Thus, we have the following

Corollary. Let l_n be the lengths of the primitive closed geodesics on M and $\tilde{l}_n \in [l_n, l_{n+1}]$ for $n > N$. There is no other Riemann surface \tilde{M} of the same genus g with lengths of primitive closed geodesics equal to \tilde{l}_n .

Professor L. Keen has kindly pointed out to me that the corollary essentially follows from the so-called collar lemma (see [4]).

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