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SHAPE-PRESERVING INTERPOLATION AND APPROXIMATION

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Interpolation of k-convex data ($k\in N$) by functions with non-negative k-th derivative is considered. Necessary and sufficient conditions for existence of such kind interpolation is given when $k\geq 3$.

1. Introduction. Let the points $X=\{x_i : i=1, 2, \dots, N\}$ be given and let $x_1<x_2<\dots<x_N$. For each $y\in R^N$ one may consider the data (x_i, y_i), $i=1, 2, \dots, N$ as a graph of the discrete function $y: X\to R$.

Definition 1. *The k-th divided difference is given by the quantity* $y[x_i, \dots, x_{i+k}]=(x_{i+k}-x_i)^{-1}(y[x_{i+1}, \dots, x_{i+k}]-y[x_i, \dots, x_{i+k-1}])$ *determining by induction, where* $y[x_i]=y_i$.

For each $i=1, 2, \dots, n=N-k$ we denote

$$\Delta_i=y[x_i, \dots, x_{i+k}]. \tag{1.1}$$

In the sequel the numbers $x_1, \dots, x_N$, k and n $(n\geq 1)$ are fixed and the connection between y and Δ is always given by (1.1).

Definition 2. *The data* (x_i, y_i) *are called* ***k-convex*** *(k-strictly convex) if* $\Delta_i\geq 0$ $(\Delta_i>0)$ *for each* $i=1, 2, \dots, n$.

In particular the 1-convex data are the non-decreasing data and the 2-convex data are simply the convex data.

For given $\{(x_i, y_i) : i=1, 2, \dots, N\}$ and $p\in[1, \infty]$ we define

$$F_p^k(y)=\{f\in C[x_1, x_N] : f^{(k-1)}\in AC,\quad f^{(k)}\in L_p,\quad f^{(k)}\geq 0,\ f(x_i)=y_i,\ i=1, 2, \dots, N\}.$$

Problem 1. Find $f\in F_p^k(y)$ for fixed k-strictly convex data (x_i, y_i), $i=1, 2, \dots, N$ and $p\in[1, \infty]$.

Problem 1 can be divided into two subproblems:

Problem 1 a. To establish that $F_p^k(y)$ is non-empty.

Problem 1 b. To construct an algorithm for finding some $f\in F_p^k(y)$.

There are many papers on Problem 1 for $k=1$ or $k=2$. We shall mention only few of them: McAllister, Passow and Roulier [5, 6, 7], Hornung [1, 2], Iliev and Pollul [3, 4]. For $k=1$ the existence and the construction of an increasing interpolating function is trivial. The case for $k=2$ is more complicated. However, $F_p^2(y)\neq\emptyset$ when the data are 2-strictly convex. Having in mind that Problem 1a is solved positively, different kind of restrictions are posed on the solution of Problem 1, e. g. f should minimize a given functional on $F_p^2(y)$ or f should belong to a given class of functions, say spline functions.

It is known that $F_p^k(y)$ may be empty when $k \geqq 3$. The aim of this paper is the investigation of the set $Y_p^k = \{y \in \mathsf{R}^N : F_p^k(y) \neq \varnothing\}$. For $y \in Y_p^k$ we present an algorithm for finding some $f \in F_p^k(y)$.

The paper is organized as follows: In Section 2 some preliminaries are given. In Section 3 we characterize the set Y_p^k. Section 4 is devoted to Problem 1 b when the function solved an additional extremal problem. An algorithm for checking whether $y \in Y_p^3$ is given in Section 5. In Section 6 the condition $\Delta_i > 0$ is replaced by $\Delta_i \geqq 0$. The problems arising in connection with the replacement of the condition "$f^{(k)} \geqq 0$" by "$f^{(k-1)}$ is non-decreasing" are considered in Section 7. In Section 8 the case $y \notin Y_p^k$ is considered and the problem for approximation of these by by means of elements of Y_p^k is formulated. Section 9 is devoted to some generalizations and non-solved problems.

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2. Preliminaries. For $\alpha \in \mathsf{R}$, $y, z \in \mathsf{R}^N$ we use the notations $y = (y_1, \dots, y_n)$,

$$\alpha y = (\alpha y_1, \dots, \alpha y_n), \quad y + z = (y_1 + z_1, \dots, y_n + z_n),$$

$$(y, z) = y_1 z_1 + \dots + y_n z_n, \quad \|y\| = \sqrt{(y, y)}.$$

If f is defined and integrable in the interval $[\xi, \eta]$, then $\int f = \int_\xi^\eta f(t)dt = \int_{-\infty}^{\infty} f(t)dt$, where we assume that $f(t) = 0$ for $t < \xi$ or $t > \eta$. The notation $g\uparrow$ means that the function g is non-decreasing. We set $(a)_+ = a$ if $a \geqq 0$ and $(a)_+ = 0$ if $a < 0$. The closure of $A \subset \mathsf{R}^n$ is denoted by $\bar{A}$.

Definition 3. *The functions* $B_i(t) = B_{i,k}(t) = k f_t[x_i, \dots, x_{i+k}]$ $(t \in \mathsf{R})$ *are called B-splines of the k-th order, where* $f_t(x) = (x - t)_+^{k-1}$.

The following properties of B-splines are well known: $B_i(t) > 0$ for $t \in (x_i, x_{i+k})$ and $B_i(t) = 0$ for $t \,\bar{\in}\, (x_i, x_{i+k})$, $\int B_i = 1$, $B_i \in C^{k-2}(\mathsf{R})$, B_i is a polynomial of $k-1$-st degree on each of the intervals $[x_i, x_{i+1}], \dots, [x_{i+k-1}, x_{i+k}]$,

$$f[x_i, \dots, x_{i+k}] = \int B_i f^{(k)} \quad \text{if} \quad f^{(k-1)} \in AC[x_i, x_{i+k}]. \tag{2.1}$$

For given k we set $B : [x_1, x_N] \to \mathsf{R}^n$, $B(t) = (B_1(t), \dots, B_n(t))$.

Lemma 1. *Let* φ_j $(j = i-k+1, \dots, i)$ *denote the polynomial coinciding with* $B_{j,k}$ *on the interval* $[x_i, x_{i+1}]$. *Then the polynomials* $\{\varphi_j : j = i-k+1, \dots, i\}$ *are linear independent.*

Lemma 2. *The following two conditions are equivalent:*

a) $y \in Y_p^k$;

b) *there is* $g \in L_p[x_1, x_N]$, $g \geqq 0$, *such that* $\int B_i g = \Delta_i$ *for* $i = 1, 2, \dots, n$.

Proof. a) $\Rightarrow$ b). Let $f \in F_p^k(y)$. We set $g = f^{(k)}$. Then $g \geqq 0$, $g \in L_p$ and (2.1) gives $\int B_i g = \int B_i f^{(k)} = f[x_i, \dots, x_{i+k}] = y[x_i, \dots, x_{i+k}] = \Delta_i$.

b) $\Rightarrow$ a). We set $f_k(t) = g(t)$, $f_j(t) = \int_{x_1}^t f_{j+1}(t)dt$ for $j = k-1, k-2, \dots, 0$ and each $t \in [x_1, x_N]$. Let Q be this polynomial of $k-1$-st degree for which $Q(x_i) = y_i - f_0(x_i)$ for $i = 1, 2, \dots, k$. We define $f = f_0 + Q$ and it is easy to see that $f \in F_p^k(y)$.

In view of Lemma 2 Problem 1 can be replaced with its equivalent.

Problem 2. For given $\Delta_i>0$, $i=1, 2, \ldots, n$ find $g\in L_p[x_1, x_N]$, $g\geq 0$, such that $\int B_i g=\Delta_i$ for $i=1, 2, \ldots, n$.

Problem 2 can be devided into two subproblems.

Problem 2a. Check whether $\Delta\in D_p$, where

(2.2) $D_p=D_p^k=\{\Delta\in \mathsf{R}^n:\Delta_i>0$ and there is $g\in L_p[x_1, x_N]$, such that $g\geq 0$ and $\int B_i g=\Delta_i$, $i=1, 2, \ldots, n\}$.

Problem 2b. Construct a solution g of the system $\int B_i g=\Delta_i$, $i=1, 2, \ldots, n$, if $\Delta\in D_p$.

Let $1<g<\infty$. We define the functional $\Phi:\mathsf{R}^n\rightarrow\mathsf{R}$ by $\Phi(a)=\int_{x_1}^{x_N}(a, B(t))_+^q dt$ for each $a\in\mathsf{R}^n$.

Lemma 3. a) *Φ is convex;*

b) *if $a_i\leq b_i$, then $\Phi(a)\leq\Phi(b)$.*

Proof. a) follows from the convexity of the functions $(x)_+$ and $x^q (q>1)$. If $a_i\leq b_i$ then $(a, B(t))\leq(b, B(t))$ because of $B_i(t)\geq 0$. Therefore $(a, B(t))_+^q\leq(b, B(t))_+^q$ and $\Phi(a)\leq\Phi(b)$.

Now we shall prove a lemma for the quadratic forms.

Lemma 4. *If $\gamma>0$ and*

$$\delta_2=\delta_1+\delta_3+2\sqrt{(1+\gamma)\delta_1\delta_3},\quad \delta_1,\ \delta_2,\ \delta_3>0. \tag{2.3}$$

$$a_1<0,\ a_2>0,\ a_3<0,\ 2\gamma a_2-a_1-a_3\leq\sqrt{(a_1+a_3)^2+4\gamma a_1a_3}, \tag{2.4}$$

then $(a, \delta)\leq 0$. Also we have $(a, \delta)=0$ only for

$$a_1=-\alpha(\sqrt{\delta_1\delta_3}+\delta_3\sqrt{1+\gamma}),\quad a_2=\alpha\sqrt{\delta_1\delta_3},\ a_3=-\alpha\sqrt{\delta_1\delta_3}+\delta_1(\sqrt{1+\gamma}),\ \alpha>0. \tag{2.5}$$

Proof. Let δ and a satisfy (2.3), (2.4) and be such that $(a, \delta)\geq 0$. Then

$$a_2\delta_2\geq -a_1\delta_1-a_3\delta_3. \tag{2.6}$$

From (2.4) and (2.6) we get $-a_1(2\gamma\delta_1+\delta_2)-a_3(2\gamma\delta_3+\delta_2)\leq\delta_2\sqrt{(a_1+a_3)^2+4\gamma a_1a_3}$ and

$$A_1a_1^2+A_2a_1a_3+A_3a_3^2\leq 0,\quad \text{where} \tag{2.7}$$

$$A_1=\delta_1(\gamma\delta_1+\delta_2),\quad A_2=2\gamma\delta_1\delta_3+\delta_1\delta_2+\delta_2\delta_3-\delta_2^2,\quad A_3(\gamma\delta_3+\delta_2). \tag{2.8}$$

From (2.8) and (2.3) it follows that $A_2^2-4A_1A_3=0$ and $A_2<0$. From here and (2.7) we get $(a_1\sqrt{A_1}-a_3\sqrt{A_3})^2\leq 0$ and this is possible only if

$$a_1=-\alpha\sqrt{A_3},\quad a_3=-\delta\sqrt{A_1}\quad(\alpha>0). \tag{2.9}$$

From (2.9) and (2.3) we get the prescribed values of a_1 and a_3. (2.9), (2.3) and (2.6) give $a_2\geq\alpha\sqrt{\delta_1\delta_3}$ and (2.9), (2.3) and (2.4) give $a_2\leq\alpha\sqrt{\delta_1\delta_3}$. This proves the lemma.

Lemma 4′. *If $\gamma>0$, a satisfies (2.4), $\delta_1\geq\delta_2>0$ and $\delta_3=0$, then $(a, \delta)<0$.*

Proof. From (2.4) if follows that $-a_1>a_2$.

3. Interpolation of k-strictly convex data. Instead of Problem 2 we consider

Problem 3. For $\Delta_i>0$, $i=1, 2, \ldots, n$ find $a\in\mathsf{R}^n$ such that

$$\int B_i(a, B)_+^{q-1}=\Delta_i\quad\text{for}\quad i=1, 2, \ldots, n. \tag{3.1}$$

Obviously if a is a solution of Problem 3 then $g=(a, B)_+^{q-1}$ is a solution of Problem 2.

Theorem 1. *For given* $\{x_i\}$, k *and* Δ *the following are equivalent*:

a) $\Delta\in D_p$ *for each* $p\in[1,\infty]$;

b) *there is* $p\in[1,\infty]$ *such that* $\Delta\in D_p$;

c) *for each* $b\in \mathrm{R}^n$ *such that* $(b,\Delta)=0$ *and* $b\neq0$ *there is* $t\in[x_2, x_{N-1}]$ *such that* $(b, B(t))>0$;

d) *for each* $q\in(1,\infty)$ *there is* $a_q\in\mathrm{R}^n$ *such that* $(a_q,\Delta)=1$ *and* $\inf\{\Phi(a): (a,\Delta)=1\}=\Phi(a_q)>0$;

e) *there are* $q\in(1,\infty)$ *and* $a_q\in\mathrm{R}^n$ *such that* $(a_q,\Delta)=1$ *and* $\inf\{\Phi(a): (a,\Delta)=1\}=\Phi(a_q)>0$;

f) *for each* $q\in(1,\infty)$ *Problem 3 has a solution*;

g) *there is* $q\in(1,\infty)$ *such that Problem 3 has a solution.*

Proof. We shall prove the theorem following the scheme: 1. a) $\Rightarrow$ b); 2. b) $\Rightarrow$ c); 3. c) $\Rightarrow$ d); 4. d) $\Rightarrow$ e); 5. e) $\Rightarrow$ g); 6. g) $\Rightarrow$ a); 7. d) $\Rightarrow$ f); 8. f) $\Rightarrow$ g). The implications 1., 4. and 8. are obvious and the proof of 7. is the same as the proof of 5.

2. The statement b) means that there is $g\in L_p\subset L_1$, $g\geq0$ such that

$$\int B_i g=\Delta_i \quad \text{for} \quad i=1,2,\dots,n. \tag{3.2}$$

Let $b\neq0$ and $(b,\Delta)=0$. We assume that for each $t\in[x_1,x_N]$ the inequality $Q(t)=(b,B(t))\leq0$ holds true. This and (3.2) give $\int Qg=(b,\int Bg)=(b,\Delta)=0$, where $Q\leq0$ and $g\geq0$. Because of $b\neq0$ Lemma 1 provides that Q is not the zero polynomial in each of the intervals $[x_i,x_{i+1}]$ for $i=j, j+1,\dots,j+k-1$. If we assume that $g\neq0$ a. e. in $[x_j, x_{i+k}]$, then $\int Qg\leq\int_{x_j}^{x_{j+k}}Qg<0$. So $g=0$ a. e. in $[x_j, x_{j+k}]$. Hence $\int B_j g=0$ which contradicts with (3.2). Therefore there is $t\in[x_1,x_N]$ such that $Q(t)>0$. This proves c) because of $\operatorname{sign}(b,B(t)) =\operatorname{sign}(b,B(x_2))$ for each $t\in(x_1,x_2]$ and $\operatorname{sign}(b,B(t))=\operatorname{sign}(b,B(x_{N-1}))$ for each $t\in[x_{N-1},x_N)$.

3. Let $q\in(1,\infty)$. We denote $H_0=\{a\in\mathrm{R}^n:(a,\Delta)=0\}$, $H_1=\{a\in\mathrm{R}^n:(a,\Delta)=1\}$ and $d=(d_1,\dots,d_n)$, where $d_i=(\Delta_1+\dots+\Delta_n)^{-1}$. If $a\in H_1$ then $b=a-d\in H_0$ and Lemma 3 gives $\Phi(b)\leq\Phi(a)$. Let $\bar b\in H_0$ be such that $\|\bar b\|=1$ and $\Phi(\bar b) =\inf\{\Phi(b): b\in H_0, \|b\|=1\}$ (Φ is continuous). Condition c) yields that $\Phi(\bar b)>0$. We set $R=\Phi((d)/\Phi(\bar b))^{1/q}>0$. For each $a\in H_1$, $\|a-d\|>R$ Lemma 3 provides $\Phi(a)\geq\Phi(a-d)=\|a-d\|^q\,.\,\Phi((a-d)/\|a-d\|)>R^q\Phi(\bar b)=\Phi(d)$. Hence there is $a_q\in H_1$ such that

$$\inf\{\Phi(a): a\in H_1\}=\inf\{\Phi(a): a\in H_1,\ \|a-d\|\leq R\}=\Phi(a_q).$$

If $a_q=d$ then obviously $\Phi(a_q)>0$. If $a_q\neq d$ then condition c) and Lemma 3 give $\Phi(a_q)\geq\Phi(a_q-d)>0$.

5. In g) we choose the same q as in e). Applying Lagrange theorem we get a real λ such that $\frac{\partial}{\partial a_i}(\Phi+\lambda(1-(a_q\Delta)))|_{a=a_q}=0$ for $i=1,2,\dots,n$. Hence $q\int B_i(t)(a_q,B(t))_+^{q-1}dt=\lambda\Delta_i$. Because of $\Phi(a_q)>0$ we have $(a_q,B(t))\neq0$ and $\int B_i(a_q,B)_+^{q-1}\neq0$ for some i. Then the condition $\Delta_i>0$ provides $\lambda>0$. Hence we get the solution of Problem 3 in the form $a=a_q(q/\lambda)^{1/(q-1)}$.

6. We set $g=(a,B)_+^{q-1}$. Obviously $g\in C[x_1,x_N]$ and $g\geq0$. Therefore Problem 2 has a solution for $p=\infty$, i. e. $\Delta\in D_\infty\subset D_p$.

The following two corollaries are an immediate consequence of Theorem 1

Corollary 1. $D_1 = D_p$ for each $p \in [1, \infty]$.

In view of this corollary we shall denote in the sequel D_p as D_1.

Corollary 2. Problem 2 and Problem 3 are equivalent. If Problem 2 has a solution then it has a continuous solution of the type (3.1).

Corollary 3. For $k=2$ we have $D_1^2 = \{\Delta \in \mathrm{R}^n : \Delta_i > 0\}$. Proof. Let $b \neq 0$, $(b, \Delta) = 0$. Hence there is $b_j > 0$. Then Theorem 1 a) and c) proves the corollary, because of $(b, B(x_{j+1})) > 0$.

Corollary 3 simply says that for any strictly convex data there exists a convex interpolating function. This fact is well known — see the papers cited in Section 1.

There are many possible generalizations of Corollary 3 for the case $k \geq 3$. The strongest of them — "Each $\Delta > 0$ belongs to D_1^k" is false as we shall see later. However, one can prove

Proposition 1. *D_1 is an open set in* R^n.

Proof. We consider the uniform norm $\| a \|_c = \max\{| a_i | : i\}$ in R^n. Theorem 1 b) and c) and the continuity of $(b, B(t))$ give that $\inf\{\| (b, B) \|_{C[x_1, x_N]} : \| b \| = 1, (b, \Delta) = 0\} = \beta > 0$. We take $\gamma = \| \Sigma_{i=1}^n B_i \|_{C[x_1, x_N]} > 0$. Now our aim is to prove that $\Delta' \in D_1$ if $\Delta' > 0$ and $\| \Delta - \Delta' \|_C < 1/3 . \min\{1, \beta\} . \min\{1, \gamma^{-1}\} . \min\{\Delta_i : i\}$. Let $\| a \|_C = 1$ and $(a, \Delta') = 0$. We define b by $b_i = \Delta_i' a_i / \Delta_i$. Then

$$\| b - a \|_C = \max_i | a_i | \, | 1 - \Delta_i'/\Delta_i | \leq \max_i | 1 - \Delta_i'/\Delta_i | < 1/3 . \min\{1, \beta\} \min\{1, \gamma^{-1}\}. \tag{3.3}$$

Hence $\| b \|_C > 2/3$ and the equality $(b, \Delta) = (a, \Delta') = 0$ provides $t_0 \in [x_2, x_{N-1}]$ such that $(b, B(t_0)) > 2\beta/3$. Applying (3.3) we get $|(b, B(t_0)) - (a, B(t_0))| \leq \| a - b \|_C \Sigma_{i=1}^n B_i(t_0) < \beta\gamma^{-1} . \gamma/3 = \beta/3$. Hence $(a, B(t_0)) > \beta/3$. Now Theorem 1 a) and c) gives $\Delta' \in D_1$.

We define

$$D_p^* = \{\Delta \in \mathrm{R}^n : \Delta > 0, \ \Delta_i = \int B_i g, \ i = 1, 2, \ldots, n, \ g > 0, \ g \in L_p[x_1, x_N]\}.$$

Obviously $D_p^* \subset D_p = D_1$.

Proposition 2. $D_p^* = D_1$.

Proof. Let $\Delta \in D_1$. Then Proposition 1 provides an $\varepsilon > 0$ such that $\Delta^\varepsilon \in D_1$, where $\Delta_i^\varepsilon = \Delta_i - \varepsilon$. Then Corollary 1 gives $\Delta^\varepsilon \in D_1$. Let $f \in L_p$ be such that $f \geq 0$ and $\Delta_i^\varepsilon = \int B_i f$. Then for $g = f + \varepsilon > 0$ we have $\Delta_i = \int B_i g$ because of $\int B_i = 1$. Therefore $\Delta \in D_p^*$.

Proposition 2 and Corollary 1 give

Corollary 4. If Problem 2 has a solution then it has a continuous positive solution.

Proposition 3. *D_1 is convex.*

The proof follows immediately from (2.2).

We can summarize some of the results of this section as follows: The set D_p of all Δ for which Problem 2 has a solution does not depend on p and it is an open convex cone with the vertex in the origin.

4. An extremum problem. The following extremal problem is a strengthened form of Problem 1 b.

Problem 4. For $y \in Y_p^k$ find $f_0 \in F_p^k(y)$ such that

$$\| f_0^{(k)} \|_p = \inf \{ \| f^{(k)} \|_p : f \in F_p^k(y) \}.$$

This problem is solved in the case $k=2$ for $p=2$ in [2], for $p=\infty$ in [3] and for $1<p<\infty$ in [4].

In this section we restrict ourselves to $1<p<\infty$ and K arbitrary. The result — Theorem 2 — is obtained as a consequence of Theorem 1 and the following lemma proved in [4].

Lemma 5. *Let S be a finite-dimensional subspace of L_q, λ be a linear functional on S, G be the set of all $g \in L_p$ $(1/p+1/q=1)$ such that $g \geqq 0$ and the functional $\langle \cdot, g \rangle$ coincides with λ on S. If there exists $h \in S$ such that $g_0=(h)_+^{q-1} \in G$, then $\| g_0 \|_p = \inf \{ \| g \|_p : g \in G \}$.*

In our case $S=\{(b, B(t)) : b \in \mathbb{R}^n\}$ and λ is given by $\lambda(B_i)=\Delta_i$ for a fixed $\Delta \in D_1$. Then G is the set of the solutions of Problem 2.

Theorem 2. *Let $1<p<\infty$ and $y \in Y_p^k$. Then Problem 4 has unique solution f_0 determining by $f_0^{(k)}=(a, B)_+^{q-1}$, where a is the solution of Problem 3.*

Proof. The uniqueness follows from the rotund of L_p. Theorem 1 b) and f) gives that Problem 3 has a solution a. Then the condition of Lemma 5 is fulfilled for $h=(a, B)$ and we get the conclusion of the theorem.

It follows from the above considerations that the main difficulty in the solution of Problem 4 is concentrated in Problem 3. Theorem 1 b), d) and f) provides a good tool for solving Problem 3.

Corollary 5. For $1/p+1/q=1$, $\Delta \in D_p$ the solution a of Problem 3 has the form $a=\alpha a_q$, where $\alpha=\text{const}>0$ and a_q is the point minimizing the functional Φ on the hyperplane $\{b \in \mathbb{R}^n : (b, \Delta)=1\}$.

A variety of gradient methods can be used for minimizing Φ on $\{b : (b, \Delta)=1\}$.

5. Case $k=3$. We saw in Section 4 that Problem 1 b) can be solved under the condition $y \in Y_p^k$ using Theorem 1 b), d) and f). In this section we shall show how to check whether y belongs to Y_p^3 using Theorem 1 c). This is equivalent (see Sections 2 and 3) to finding if $\Delta \in D_1$. In this section we suppose that $k=3$.

It is more convenient in the beginning to characterize the condition $\Delta \bar{\in} D_1$ (but $\Delta>0$). Theorem 1 gives

Corollary 6. Let $\Delta>0$. Then $\Delta \bar{\in} D_1$ iff there exists $b \neq 0$ such that $(\Delta, b)=0$ and $Q(t)=(b, B(t)) \leqq 0$ for each $t \in [x_2, x_{N-1}]$.

First we observe that if $\Delta \bar{\in} D_1$ and b is the vector given in the corollary then necessarily $b_1 \leqq 0$ and $b_n \leqq 0$ $(Q(x_2)=b_1 B_1(x_2)$ and $B_1(x_2)>0)$.

Now let us see what Corollary 6 gives if $n=2$, 3 or 4.

For $n=2$ the simultaneous satisfying of the inequalities $\Delta_1>0$, $\Delta_2>0$, $b_1 \leqq 0$, $b_2 \leqq 0$, $|b_1|+|b_2|>0$, $\Delta_1 b_1+\Delta_2 b_2=0$ is impossible and so we always have $\Delta \in D_1$ (this is true for each k and $n=2$).

Let $n=3$. It is necessary at least one of b_i to be positive. Therefore $b_2>0$. If $b_1=0$, then $Q(x_3)=b_2 B_2(x_3)>0$ which contradicts the corollary. Hence $b_1<0$ and in the same way $b_3<0$. Under the determined values of the signs of b_i the inequality $\max \{Q(t) : t \in [x_2, x_5]\} \leqq 0$ is equivalent to the inequality $\max \{Q(t) : t \in [x_3, x_4]\} \leqq 0$. Definition 1 and 3 give that in the interval $[x_3, x_4]$ B-splines have the form

$$B_1(t)=3(t-x_4)^2[(x_4-x_3)(x_4-x_2)(x_4-x_1)]^{-1};$$

$$B_2(t)=3(t-x_5)^2[(x_5-x_4)(x_5-x_3)(x_5-x_2)]^{-1}$$
$$-3(t-x_4)^2[(x_5-x_4)(x_4-x_3)(x_4-x_2)]^{-1};$$
$$B_3(t)=3(t-x_3)^2[(x_6-x_3)(x_5-x_3)(x_4-x_3)]^{-1}.$$

It follows from the above formuli that B_2 is concave and B_1 and B_3 are convex. Hence Q is concave in $[x_3, x_4]$ and it has unique maximum. This maximum is achieved at the point $\bar{t}$ which is the solution of the equation $Q'(t)=0$, i. e.

$$\bar{t}=\{-b_1x_4[(x_4-x_3)(x_4-x_2)(x_4-x_1)]^{-1}+b_2x_4[(x_5-x_4)(x_4-x_3)(x_4-x_2)]^{-1}$$
$$-b_2x_5[(x_5-x_4)(x_5-x_3)(x_5-x_2)]^{-1}-b_3x_3[(x_6-x_3)(x_5-x_3)(x_4-x_3)]^{-1}\}$$
$$\times\{-b_1[(x_4-x_3)(x_4-x_2)(x_4-x_1)]^{-1}+b_2[(x_5-x_4)(x_4-x_3)(x_4-x_2)]^{-1}$$
$$-b_2[(x_5-x_4)(x_5-x_3)(x_5-x_2)]^{-1}-b_3[(x_6-x_3)(x_5-x_3)(x_4-x_3)]^{-1}\}^{-1}.$$

The restrictions over b_i give $x_3<\bar{t}<x_4$. Therefore the condition $Q(t)\leqq 0$ for each $t\in[x_3, x_4]$ is equivalent to the condition $Q(\bar{t})\leqq 0$. After some elementary transformations we get the equivalent inequality

$$\frac{(x_5-x_2)(x_4-x_3)}{(x_5-x_1)(x_3-x_2)}\, a_2^2-a_1a_2-a_2a_3-a_1a_3\leqq 0,$$

where

(5.1) $$a_i=b_i[(x_{i+3}-x_i)(x_{i+2}-x_{i+1})]^{-1}.$$

In this section we shall use the notations

(5.2) $$\delta_i=\Delta_i(x_{i+3}-x_i)(x_{i+2}-x_{i+1})$$

and

(5.3) $$\gamma_i=(x_{i+3}-x_i)(x_{i+2}-x_{i+1})[(x_{i+3}-x_{i+2})(x_{i+1}-x_i)]^{-1}.$$

From the above considerations and Corollary 6 we get

Lemma 6. *Let $n=3$ and $\Delta>0$. Then $\Delta\in D_1$ iff there is a such that $a_1<0$, $a_2>0$, $a_3<0$, $(a, \delta)=0$ and*

(5.4) $$2\gamma_2a_2\leqq a_1+a_3+\sqrt{(a_1+a_3)^2+4\gamma_2a_1a_3}.$$

The comparison of Lemma 6 with Lemma 4 gives

Proposition 4. *Let $n=3$. Then $\Delta\in D_1$ iff*

(5.5) $$\delta_2<\delta_1+\delta_3+2\sqrt{(1+\gamma_2)\delta_1\delta_3}.$$

If $\{x_i\}_{i=1}^6$ are equidistant then

$$\Delta\in D_1 \Leftrightarrow \Delta_2<\Delta_1+\Delta_3+4\sqrt{\Delta_1\Delta_3}.$$

These results show that there are 3-strictly convex data (x_i, y_i), $i=1, \ldots, 6$ for which there are not any interpolating function with non-negative third derivative. For example we may choose $\Delta_1=1$, $\Delta_2\geqq 6$, $\Delta_3=1$ if the points $\{x_i\}$ are equidistant.

Let $n=4$. It is necessary $b_1\leqq 0$ and $b_4\leqq 0$. So at least one of b_2 and b_3 should be positive. But if $b_2\geqq 0$, $b_3\geqq 0$ and $b_2+b_3>0$ then $(b, B(x_4))=b_2B_2(x_4)+b_3B_3(x_4)>0$. Hence either $b_2>0$ and $b_3<0$ or $b_2<0$ and $b_3>0$. Thus the following two cases are possible: 1) $b_1<0$, $b_2>0$, $b_3<0$, $b_4\leqq 0$; 2) $b_1\leqq 0$, $b_2<0$, $b_3>0$, $b_4<0$. The same reasons as in the case $n=3$ give: $\Delta\notin D_1$ iff either $\delta_2\geqq\delta_1+\delta_3+2\sqrt{(1+\gamma_2)\delta_1\delta_3}$ or $\delta_3\geqq\delta_2+\delta_4+2\sqrt{(1+\gamma_3)\delta_2\delta_4}$. So we get

Proposition 5. *Let $n=4$. Then $\Delta\in D_1$ iff $\delta_2<\delta_1+\delta_3+2\sqrt{(1+\gamma_2)\delta_1\delta_3}$ and $\delta_3<\delta_2+\delta_4+2\sqrt{(1+\gamma_3)\delta_2\delta_4}$.*

These cases are typical and we are ready to begin with the case of arbitrary n.

Lemma 7. *Let $\Delta>0$. Then $\Delta\notin\overline{D}_1$ iff there is $a\neq 0$ such that $(a, \delta)=0$, $a_1\leqq 0$, $a_n\leqq 0$, and if $a_i>0$ then $a_{i-1}<0$, $a_{i+1}<0$ and*

$$(5.4')\qquad 2\gamma_i a_i\leqq a_{i-1}+a_{i+1}+\sqrt{(a_{i-1}+a_{i+1})^2+4\gamma_i a_{i-1}a_{i+1}}.$$

Proof. The "only if" part. Corollary 6 provides $b\neq 0$ such that $(\Delta, b)=0$ and $Q(t)\leqq 0$ for each $t\in[x_2, x_{N-1}]$. We get a from (5.1). From $(b, \Delta)=0$, (5.1) and (5.2) we have $(a, \delta)=0$. From $Q(x_2)\leqq 0$ and $Q(x_{N-1})\leqq 0$ we obtain $a_1\leqq 0$ and $a_n\leqq 0$. If $a_i>0$ then the inequalities $Q(x_{i+1})\leqq 0$ and $Q(x_{i+2})\leqq 0$ give $a_{i-1}<0$ and $a_{i+1}>0$. Now we get (5.4′) as in Lemma 6.

The "if" part.

We shall apply Corollary 6 and so we need to prove that $Q(t)\leqq 0$ for each $t\in[x_2, x_{N-1}]$. For simplicity we set $a_0=a_{n+1}=0$. Let $t\in[x_{i+1}, x_{i+2}]$ for some $i=1, 2, \ldots, n$. If $a_i>0$ then (5.4′) gives $Q(t)\leqq 0$ as in Lemma 6. If $a_i=0$ then $a_{i-1}\leqq 0$, $a_{i+1}\leqq 0$ and obviously we have $Q(t)\leqq 0$. Let $a_i<0$. Then we have three possible cases; 1) $a_{i-1}>0$, $a_{i+1}>0$; 2) $a_{i-1}\leqq 0$, $a_{i+1}\leqq 0$; 3) either $a_{i-1}\leqq 0$, $a_{i+1}>0$ or $a_{i-1}>0$, $a_{i+1}\leqq 0$. If case 1) holds then (5.4′) gives $Q(x_{i+1})\leqq 0$, $Q(x_{i+2})\leqq 0$, and in view of the convexity of Q we have $Q(t)\leqq 0$. In case 2) we obviously have $Q(t)\leqq 0$. In view of the symmetry in the case 3) we shall consider only $a_{i-1}\leqq 0$, $a_{i+1}>0$. We define $p(\tau)=b_iB_i(\tau)+b_{i+1}B_{i+1}(\tau)$ for $\tau\in[x_{i+1}, x_{i+2}]$. Then p is convex, $p(x_{i+1})=b_iB_i(x_{i+1})<0$ and (5.4′) gives $p(x_{i+2})=Q(x_{i+2})\leqq 0$. Hence $p(t)\leqq 0$. Finally $Q(t)=b_{i-1}B_{i-1}(t)+p(t)\leqq 0$.

Lemma 8. *Let $\Delta>0$. Then the statement $\Delta\notin D_1$ is equivalent to the existence of $j, s\in\mathbf{N}$ $(1\leqq j<j+2s\leqq n)$ with the following property. If we define by induction $\delta'_j=\delta_j$ and*

$$(5.6)\qquad \delta'_{j+2i}=\delta_{j+2i}-(1+2\gamma_{j+2i-1})\delta'_{j+2i-2}-\delta_{i+2i-1}+2\sqrt{(1+\gamma_{j+2i-1})\delta'_{j+2i-2}(\gamma_{j+2i-1}\delta'_{j+2i-2}+\delta_{j+2i-1})}$$

for $i=1, 2, \ldots, s$, then $\delta_{i+2i+1}>\delta'_{j+2i}>0$ for $i=0, 1, \ldots, s-1$ and $\delta'_{j+2s}\leqq 0$.

Proof. We set $\delta''_{j+2i}=\delta_{j+2i}-\delta'_{j+2i}$ for $i=1, 2, \ldots, s$.

1. Let j, s exist with the given property. Then (5.6) gives $\delta''_{i+2i}>0$ for $i=1, 2, \ldots, s$. We set $a_j=-1$. Now we define by induction the next "a"-s as follows. If for some $i=0, 1, \ldots, s-1$ the number a_{j+2i} is define then we get from (2.5) $a_{j+2i+1}=a_2$ and $a_{j+2i+2}=a_3$, where $\delta_1=\delta'_{j+2i}$, $\delta_2=\delta_{j+2i+1}$, $\delta_3=\delta''_{j+2i+2}$, $\gamma=\gamma_{j+2i-1}$ and $\alpha=-a_{j+2i}(\sqrt{\delta_1\delta_3}+\delta_3\sqrt{1+\gamma})^{-1}$. So defined $\{a_m\}$ satisfy $\operatorname{sign} a_{j+l}=(-1)^{l+1}$ for $l=0, 1, \ldots, 2s$. Applying Lemma 4 we obtain

$$\sum_{m=j}^{j+2s}\delta_m a_m=\sum_{i=1}^{s}(\delta'_{j+2i-2}a_{j+2i-2}+\delta_{j+2i-1}a_{j+2i-1}+\delta''_{j+2i}a_{j+2i})+\delta'_{j+2s}a_{j+2s}$$

$$=\delta'_{j+2s}a_{j+2s}\geqq 0.$$

On the other hand,

$$\sum_{m=j}^{j+2s-2}\delta_m a_m+\delta_{j+2s}a_{j+2s}=\delta'_{j+2s-2}a_{j+2s-2}+\delta_{j+2s}a_{j+2s}<0.$$

Therefore there is $\bar{a}_{j+2s-1}$ such that $0<\bar{a}_{j+2s-1}\leqq a_{j+2s-1}$ and $\sum_{m=j}^{j+2s-2}\delta_m a_m+\bar{a}_{j+2s-1}\delta_{j+2s-1}+a_{j+2s}=0$. We also set $\bar{a}_m=0$ for $m<j$ and $m>j+2s$. The vector a (with $\bar{a}_{j+2s-1}$ instead of a_{j+2s-1}) satisfies the condition of Lemma 7 and hence $\Delta\notin D_1$.

2. Let $\Delta\notin D_1$. Let a be the vector from Lemma 7. We separate the integers from 1 to n to the parts $I_0, I_1,\dots, I_r$ as follows: 1) $i\in I_0$ iff $a_{i-1}\leqq 0$, $a_i\leqq 0$, $a_{i+1}\leqq 0$ (we assume that $a_0=a_{n+1}=-1$); 2) $I_\rho(\rho=1,2,\dots,r)$ is of the type $\{j, j+1,\dots, j+2s\}$ and the following conditions are satisfied a) $\operatorname{sign} a_{j+m}=(-1)^{m+1}$ for $m=0,1,\dots,2s$); b) either $j=1$ or $j-1\in I_0$; c) either $j+2s=n$ or $j+2s+1\in I_0$. Because of $a\neq 0$ and $(a,\delta)=0$ we have $r\geqq 1$. Therefore $0=(a,\delta)\leqq\sum_{\rho=1}^{r}\sum_{m\in I_\rho}a_m\delta_m$ and there is ρ such that $\sum_{m\in I_\rho}a_m\delta_m\geqq 0$. Let $I_\rho=\{l, l+1,\dots, l+2s\}$. Now we decompose $\delta'_{l+2i}=\delta'_{l+2i}+\delta''_{l+2i}$ in turn for $i=0,1,\dots,s$ using the formuli: a) $\delta''_l=0$; b) if $\delta'_{l+2i}\geqq\delta_{l+2i+1}$ then $\delta''_{l+2i+2}=0$; c) if $\delta'_{l+2i}<\delta_{l+2i+1}$ then

$$\delta''_{l+2i+2}=\delta_{l+2i+1}+(1+2\gamma_{l+2i+1})\delta'_{l+2i}$$

$$-2\sqrt{(1+\gamma_{l+2i+1})\delta'_{l+2i}(\gamma_{l+2i+1}\delta'_{l+2j}+\delta_{l+2i+1})}.$$

Now Lemma 4 and Lemma 4′ give

$$0\leqq\sum_{m\in I_\rho}a_m\delta_m=\sum_{i=1}^{s}(a_{l+2i-2}\delta'_{l+2i-2}+a_{l+2i-1}\delta_{l+2i-1}+a_{l+2i}\delta''_{l+2i})$$

$$+\delta'_{l+2s}a_{l+2s}\leqq\delta'_{l+2s}a_{l+2s}.$$

Therefore $\delta'_{l+2s}\leqq 0$. Let $j=\max\{m:\delta''_m=0\}$ and $s'=(l+2s-j)/2$. We have $s'\geqq 1$ because of $\delta''_{l+2s}>0$. Then the numbers j and s' have the property given in the condition of the lemma.

Now we are ready to construct an algorithm for solving Problem 2 a if $k=3$.

Algorithm I.

Step 0. $m:=0$ and δ is given by (5.2).

Step 1. $m:=m+1$, $i:=m$.

Step 2. If $m>2$, then STOP — $\Delta\in D_1$.
If $m\leqq 2$, then $\delta'_i:=\delta_i$. Go to Step 3.

Step 3. If $i<n-1$, then go to Step 4.
If $i\geqq n-1$, then go to Step 1.

Step 4. If $\delta'_i\geqq\delta_{i+1}$, then $i:=i+2$ and $\delta'_i:=\delta_i$. Go to Step 5.
If $\delta'_i<\delta_{i+1}$, then $i:=i+2$,

$\gamma := (x_{i+2}-x_{i-1})(x_{i+1}-x_i)/[(x_{i+2}-x_{i+1})(x_i-x_{i-1})]$ and

$$\delta'_i := \delta_i - \delta_{i-1} - (1+2\gamma)\delta'_{i-2} + 2\sqrt{(1+\gamma)\delta'_{i-2}(\gamma\delta'_{i-2}+\delta_{i-1})}.$$

Go to Step 5.

Step 5. If $\delta'_i \leq 0$, then Stop — $\Delta \bar{\in} D_1$.

If $\delta'_i > 0$, then go to Step 3.

Theorem 3. *Let* $\Delta > 0$. *Then* $\Delta \bar{\in} D_1$ *iff there is* $i \in \mathrm{N}$ *such that* $\delta'_i \leq 0$ in Algorithm I.

Proof. The "if" part follows immediately from Lemma 8. Now we shall prove the "only if" part. There are numbers j and s with the property given in Lemma 8. If in Algorithm I $\delta'_i \leq 0$ before i takes value j, then Theorem 3 is proved. Now let us have $0 < \delta'_j \leq \delta_j$. If $\delta'_j = \delta_j$ then Lemma 8 gives $\delta'_i \leq 0$ for $i = j+2s$ and the theorem is proved. If $\delta'_j < \delta_j$ then all values of δ'_i from Algorithm I are less than these ones from Lemma 8. Hence we shall have in Algorithm I $\delta'_i \leq 0$ *for* $i = j+2m$ for some $m \leq s$. This completes the proof.

Let us note that Algorithm I is very fast — it terminates after n times taking a square root and $O(n)$ times multiplication or division.

6. Interpolation of k-convex data. In the sequel we assume that $\Delta \geqq 0$ instead of $\Delta > 0$.

We have required the strictly convexity of the data ($\Delta > 0$) in Problem 1, while the condition $f^{(k)} \geq 0$ has been posed in the definition of $F^k_p(y)$. Proposition 2 shows that the replacement of this condition by $f^{(k)} > 0$ does not lead to any interchanges in the set of all data for which Problem 1 has a solution. Moreover the strictly convexity gives that D_1 is open but the replacement of $\Delta > 0$ with $\Delta \geq 0$ in (2.2) does not provide that D_1 is closed. Nevertheless it is natural to combine $f^{(k)} \geq 0$ with $\Delta \geq 0$.

Problem 5. For $\Delta_i \geq 0, i = 1, 2, \ldots, n$ check whether $\Delta \in D_0$, where $D_0 = \{\Delta \in \mathrm{R}^n : \Delta_i = \int B_i g,\ i = 1, 2, \ldots, n,\ g \in L_1[x_1, x_N],\ g \geq 0\}$.

We do not consider the problem for constructing of an $f \in Y^k_1(y)(\Delta \geq 0)$ because it is reduced to solving some times Problem 1 b if Problem 5 has a positive answer.

Having in mind the results of Section 3 we see that D_0 does not depend on p if we require $g \in L_p$ instead of $g \in L_1$.

If $\Delta > 0$ then Problem 5 coincides with Problem 2 a. So we assume that there is i such that $\Delta_i = 0$. We introduce the following notations: $I_0 = \{i : \Delta_i = 0\} (I_0 \neq \emptyset)$. The other natural numbers less than $n+1$ form some non-intersecting sets $I_1, I_2, \ldots, I_r$ of the type $I_\rho = \{i_\rho + 1, \ldots, i_\rho + j_\rho\}$, where $i_1 = 0$ or $i_1 \in I_0$, $i_\rho \in I_0$ for $\rho \geq 2$, $i_\rho + j_\rho + 1 \in I_0$ for $\rho \leq r-1$, $i_2 + j_r + 1 \in I_0$ or $i_r + j_r = n$.

If $i_1 = 0$ then I_1 will be called a left interval, if $i_r + j_r = n$ then I_r will be called a right interval and all other $I_\rho (\rho \neq 0)$ will be called inner intervals.

Let $\alpha_1 = x_1$ if I_1 is a left interval and $\alpha_\rho = x_{i_\rho + k}$ otherwise, $\beta_r = x_N$ if Ir is a right interval and $\beta\rho = x_{i\rho + \rho j + 1}$ otherwise. If $\Delta \in D_0$ then necessarily $g(t) = 0$ if $t \bar{\in} [\alpha_\rho, \beta_\rho]$ for any $\rho = 1, 2, \ldots, r$. Therefore Problem 5 decomposes to r independent problems: For $\rho = 1, 2, \ldots, r$ to determine if there is $g_\rho \in L_1[\alpha_\rho, \beta_\rho]$ such that $g_\rho \geq 0$ and $\Delta_i = \int_{\alpha_\rho}^{\beta_\rho} B_i g$ for $i \in I_\rho$.

The problems for the right and left intervals (in view of the symmetry) are of the type

Problem 2′. Check whether $\Delta\in D_p$, where

(2.2′) $D_p=\{\Delta\in \mathrm{R}^n$: $\Delta_i>0$ and there is $g\in L_p[x_1, x_{n+1}]$ such that $g\geqq 0$, $\int_{x_1}^{x_{n+1}} B_i g=\Delta_i$, $i=1,\ldots,n\}$.

The problems for the inner intervals are of the type

Problem 2″. Check whether $\Delta\in D_p$, where

$$D_p=\{\Delta\in \mathrm{R}^n: \Delta_i>0 \text{ and there is } g\in L_p[x_k, x_{n+1}] \text{ such that } g\geqq 0, \int_{x_k}^{x_{n+1}} B_i g=\Delta_i,\ i=1,\ldots,n\}. \tag{2.2″}$$

Now we give two analogs of Theorem 1.

Theorem 1′. *Theorem 1 is true for Problem 1′ under the following replacements: in c) $[x_2, x_{N-1}]$ is replaced by $[x_2, x_{n+1}]$; $\Phi(a)=\int_{x_1}^{x_{n+1}}(a, B(t))_+^q dt$ and in Problem 3 (3.1) is replaced by*

$$\Delta_i=\int_{x_1}^{x_{n+1}} B_i(t)(a, B(t))_+^{q-1}dt \quad \text{for} \quad i=1, 2, \ldots, n. \tag{3.1′}$$

Theorem 1″. *Theorem 1 holds true for Problem 1″ under the following replacements: in c) $[x_2, x_{N-1}]$ is replaced by $[x_k, x_{n+1}]$, $n\geqq k$: $\Phi(a)=\int_{x_k}^{x_{n+1}}(a, B(t))_+^q dt$ and in Problem 3 (3.1) is replaced by*

$$\Delta_i=\int_{x_k}^{x_{n+1}} B_i(t)(a, B(t))_+^{q-1}dt \quad \text{for} \quad i=1, 2, \ldots, n. \tag{3.1″}$$

The proofs of Theorem 1′ and Theorem 1″ repeat the proof of Theorem 1. The corollaries of Theorems 1′ and 1″ are similar to these of Theorem 1.

The results of Section 4 have the following analog. We consider

Problem 4′. For y such that $\Delta\in D_0$ find $f_0\in F_p^k(y)$ such that $\|f_0^{(k)}\|_p=\inf\{\|f^{(k)}\|_p: f\in F_p^k(y)\}$.

Then from the above considerations, Theorems 1′ and 1″ and Lemma 5 we get

Theorem 4. *Let $1<p<\infty$ and y be such that $\Delta\in D_0$. Then Problem 4′ has unique solution f_0 determining by $f_0^{(k)}=\chi(t)(a, B(t))_+^{q-1}$, where $\chi(t)=0$ for $t\in \bigcup_{i\in I_0}[x_i, x_{i+k}]$ and $\chi(t)=1$ for $t\in[x_1, x_N]\setminus \bigcup_{i\in I_0}[x_i, x_{i+k}]$ and the components of the vector a are determined by (3.1′) if $i\in I_p$ and I_p is a right or left interval, by (3.1″) if $i\in I_p$ and I_p is an inner interval and $a_i=0$ if $i\in I_0$.*

Now we shall apply Theorems 1′ and 1″ for solving Problem 5.

Theorem 5. *Let $k=2$. Then $\Delta\in D_0$ iff each inner interval has at least two integers.*

The proof is similar to the proof of Corollary 3.

The following two algorithms solve Problem 2′ and Problem 2″ respectively if $k=3$.

Algorithm II.

Step 0, Step 1 and Step 2 are these ones from Algorithm I.

Step 3. If $i<n-1$, then go to Step 4.
If $i=n-1$, then go to Step 6.
If $i=n$, then go to Step 1.

Step 4 and Step 5 — from Algorithm I.

Step 6. If $\delta_i' > \delta_{i+1}$, then go to Step 1.

If $\delta_i' \leq \delta_{i+1}$, then STOP — $\Delta \bar{\in} D_0$.

Algorithm III.

Step 0. $m := 0$, $\delta_0 := 0$ and δ_i are given by (5.2) for $i=1, 2, \ldots, n$.

Step 1. $m := m+1$, $i := m-1$.

Step 2, Step 3, Step 4, Step 5 and Step 6 are these ones from Algorithm II.

Using Theorems 1′ and 1″ we get the proofs of these two algorithms and of the following theorem in the same way as we get the proof of Theorem 3 with the help of Theorem 1.

Theorem 6. *Let $k=3$. Then $\Delta \in D_0$ iff the following four conditions are fulfilled:*

a) *each inner interval has at least three integers;*

b) *if a left interval has at least two integers then Algorithm II terminates at Step 2;*

c) *if a right interval has at least two integers then Algorithm III applied for $\bar{x}$ and $\bar{\Delta}$ terminates at Step 2, where $\bar{x}_j = x_{N+1-j}$, $j=1, 2, \ldots, N$ and $\bar{\Delta}_j = \Delta_{n+1-j}$, $j=1, 2, \ldots, n$;*

d) *for each inner interval Algorithm III terminates at Step 2.*

An algorithm for solving Problem 5 in case $k=3$ follows immediately from Theorem 6.

7. A generalization of Problem 1

The following problem generalizes Problem 1,

Problem 1′. For k-convex data (x_i, y_i), $i=1, 2, \ldots, N$ find $f \in F^k(y)$, where

$F^k(y) = \{f \in C[x_1, x_N] : f^{(k-1)}$ is non-decreasing and bounded$\}$.

Problem 1′ is equivalent to

Problem 2‴. For $\Delta \geq 0$ find a non-decreasing bounded function g such that $\Delta_i = \int_{-\infty}^{\infty} B_i(t)dg(t)$, $i=1, 2, \ldots, n$.

Here $\int B_i dg$ is an Stieltjes integral. We can also consider dg as a positive measure. We denote

$$D = D^k = \{\Delta \in \mathrm{R}^n : \Delta_i = \int B_i dg, \quad i=1, 2, \ldots, n, \quad g\uparrow\}.$$

Having in mind F. Riesz theorem D can be consider as the image in R^n of the set C_+^* of all positive linear functionals on $C[x_1, x_N]$ under the maping $L : C_+^* \to \mathrm{R}^n$, $L(l) = (l(B_1), \ldots, l(B_n))$.

Lemma 9. $D \subset \bar{D}_1$.

Proof. Let $\Delta \in D$ and $\Delta_i = \int B_i dg$, $g\uparrow$. For $m \in \mathrm{N}$ we set $g_m(x) = m\int_0^{m^{-1}} g(x+t)dt + xm^{-1}$ (we assume $g(t) = g(x_N)$ for $t > x_N$) and $\Delta_i^m = \int B_i g_m'$. We have $\Delta^m \in D_1$ because of $g_m'(t) = m[g(t+m^{-1}) - g(t)] + m^{-1}$ a. e. Now the proof follows from the inequalities

$$|\Delta_i - \Delta_i^m| = |\int B_i dg - \int B_i dg_m| = | -\int B_i' g + \int B_i' g_m| \leq \int |g - g_m| \, |B_i'|$$

$$\leq \int_{x_1}^{x_N} |g(t) - m \int_0^{m^{-1}} g(t+\tau)d\tau| \, |B_i'(t)| \, dt + m^{-1} \int_{x_1}^{x_N} |tB_i'(t)| \, dt \underset{m\to\infty}{\longrightarrow} 0.$$

Lemma 10. *D is closed.*

Proof. Let $\Delta \in D$ and $\bar{g}\uparrow$ be such that $\Delta_i = \int B_i d\bar{g}, i=1, 2, \ldots, n$. We can define g as follows: $g(t)=\bar{g}(t)$ for $t \in (x_2, x_{N-1})$, $g(t)=g_1$ for $t \leq x_2$, $g(t)=g_N$ for $t \geq x_{N-1}$, where

$$B_1(x_2)[\bar{g}(x_2+0)-g_1] = \int_{x_1}^{x_2} B_1 d\bar{g},\ B_n(x_{N-1})[g_N-\bar{g}(x_{N-1}-0)] = \int_{x_{N-1}}^{x_N} B_n d\bar{g}.$$

Hence $\Delta_i = \int B_i dg$. Let $\Delta^m \in D$, $m=1, 2, \ldots$, $\Delta^m \to \Delta$ with respect to some norm in R^n and $\Delta_i^m = \int B_i dg_m$, where g_m are functions of the type given above. If $\beta>0$ denotes $\inf\{\Sigma_{i=1}^n B_i(t) : t \in [x_2, x_{N-1}]\}$, then $\beta[g_m(x_N)-g_m(x_1)] = \beta \int dg_m \leq \int \Sigma_{i=1}^n B_i dg_m = \Sigma_{i+1}^n \Delta_i^m \underset{m\to\infty}{\longrightarrow} \Sigma_{i=1}^n \Delta_i$. This inequality and the possibility to change g_m with an additive constant provide the existence of $M=$const such that $\|g_m\|_{C[x_1, x_N]} \leq M$ for each m. Now E. Helly theorem provides a function $g\uparrow$, $\|g\|_{C[x_1, x_N]} \leq M$ and a subsequence $\{m_r\}$ such that $g_{m_r}(t) \to g(t)$ for each $t \in [x_1, x_N]$. Then we have $\int B_i dg = -\int B_i' g = -\lim_{r\to\infty} \int B_i' g_{m_r} = \lim_{r\to\infty} \int B_i dg_{m_r} = \lim_{r\to\infty} \Delta_i^{m_r} = \Delta_i$. Hence $\Delta \in D$, i. e. D is closed.

Theorem 7. $\bar{D}_1 = D$.

The proof follows from $D_1 \subset D$, Lemma 9 and Lemma 10.

Let cone (B) denote the closed convex cone with the vertex in the origin generated by $\{B(t) : t \in [x_2, x_{N-1}]\}$, i. e.

$$\text{cone}(B) = \{a \in \mathrm{R}^n : a = \sum_{i=1}^r \alpha_i B(t_i),\quad t_i \in [x_2, x_{N-1}],\ \alpha_i \geq 0\}.$$

Theorem 8. *For* $\Delta \geq 0$, $\Delta \neq 0$ *the following are equivalent*:

a) $\Delta \in$ cone (B);

b) $\Delta \in D$;

c) *for each* b, $(b, \Delta)=0$ *there is* $t \in [x_2, x_{N-1}]$ *such that* $(B(t), b) \geq 0$.

Proof. a) $\Rightarrow$ b). Let $\Delta = \Sigma_{i=1}^s \alpha_i B(t_i)$ and $t_1 < t_2 < \ldots < t_s$. Then we set $g(t)=0$ for $t<t_1$, $g(t)=\Sigma_{i=1}^j \alpha_i$ for $t_j \leq t < t_{j+1}$ and $j=1, 2, \ldots, s-1$, $g(t) = \Sigma_{i=1}^s \alpha_i$ for $t \geq t_s$. Then $\Delta_i = \int B_i dg$ for $i=1, 2, \ldots, n$.

b) $\Rightarrow$ c). In the representation $\Delta_i = \int B_i dg$, $i=1, 2, \ldots, n$ we may assume that $g(t)=g(x_2)$ for $t \leq x_2$ and $g(t)=g(x_{N-1})$ for $t \geq x_{N-1}$ (see the proof of Lemma 10). Therefore $0=(b, \Delta)=\int_{x_2}^{x_{N-1}} (b, B(t)) dg(t)$ and hence there is $t \in [x_2, x_{N-1}]$ such that $(b, B(t)) \geq 0$ because of the inequalities $\int dg > 0$ and $\Delta \neq 0$.

c) $\Rightarrow$ a). Let us assume that $\Delta \notin$ cone (B). We set

$$\delta = \sum_{i=1}^n \Delta_i > 0,\quad H = \{a \notin \mathrm{R}^n : \sum_{i=1}^n a_i = \delta\} \quad\text{and}\quad K = H \cap \text{cone}\ (B).$$

The hyperplane H contains the point Δ and the closed convex set K. Then the separation theorems provide a hyperplane H_0 of H such that $H_0 \ni \Delta$ and $H_0 \cap K = \emptyset$. Let H_1 be this hyperplane of R^n for which $H_1 \cap H = H_0$ and $H_1 \ni 0$. Then $H_1 \ni \Delta$. If we assume that there is $a \neq 0$, $a \in H_1 \cap$ cone (B), then $\bar{a} = (\delta / \Sigma_{i=1}^n a_i) a \in$ cone (B) and $\bar{a} \in H$. So $\bar{a} \in K$ and $\bar{a} \in H \cap H_1 = H_0$, i. e. $K \cap H_0 \neq \emptyset$ which is a contradiction. Hence $H_1 \cap$ cone $(B) = \{0\}$. Let $b \neq 0$ be such that $H_1=\{a\in \mathrm{R}^n : (a, b)=0\}$. Let $(b, B(x_2))<0$ (the equation $(b, B(x_2))=0$ is impossible and if $(b, B(x_2))>0$, then we consider $-b$ instead of b). Then $(b, a)<0$ for each $a\in \operatorname{cone}(B)\setminus\{0\}$ because of the convexity of cone (B). On the other hand, $(b, \Delta)=0$ which contradicts with c).

Theorem 8 can be used for constructing an algorithm checking whether $\Delta\in D$ in the same way as Theorem 1 has been used for proving Corollary 3 and Theorem 3. For $k=2$ we get $D=\{\Delta\in \mathrm{R}^n: \Delta\geq 0\}$. For $k=3$ and $\Delta\geq 0$ the arguments of Section 5 give

Algorithm IV.

All steps are the same as in Algorithm I except

Step 5. If $\delta_i'<0$, then STOP — $\Delta\bar{\in} D$.

If $\delta_i'\geq 0$, then go to Step 3.

Of course the Stop in Step 2 gives $\Delta\in D$.

Let K denotes the set of all extremal directions of D, i. e. K is the set of all $a\in D$ such that $a\neq 0$ and if $a=\beta b+\gamma c$ for some $\beta, \gamma>0$, $b, c\in D\setminus\{0\}$, then there is $\alpha>0$ such that $a=\alpha b$. Theorem 8 says that

$$K\subset K_1=\{\alpha B(t) : \alpha>0, \quad t\in [x_2, x_{N-1}]\}.$$

Proposition 6. *For $k\geq 3$ we have $K=K_1$.*

Proof. Let $t_0\in[x_i, x_{i+1}]$, $2\leq i\leq N-2$ and $B(t_0)=\beta b+\gamma c$ for some $\beta, \gamma>0$, $b, c\in D\setminus\{0\}$. Then Theorem 8 gives

$$B(t_0)=\sum_{j=1}^{s} \alpha_j B(t_j), \quad t_j\in [x_2, x_{N-1}]. \quad \alpha_j>0. \tag{7.1}$$

Obviously $t_j\in [x_i, x_{i+1}]$ because of $B_r(t_0)=0$ for $r<i$ and $r>i+k$. Now Lemma 1, $k\geq 3$ and (7.1) give $\Sigma_{j=1}^s\alpha_j=1$, $\Sigma_{j=1}^s\alpha_j t_j=t_0$ and $\Sigma_{j=1}^s\alpha_j t_j^2=t_0^2$. Hence $\Sigma_{j=1}^s\alpha_j(t_j-t_0)^2=0$. Therefore $t_j=t_0$ because of $\alpha_j>0$. This completes the proof.

8. Approximation. In the previous sections it was shown that for $k\geq 3$ there are strictly convex data which do not belong to Y_p^k. Hence the problem for the best approximation of these data with elements of Y_p^k can be considered. But Proposition 1 says that D_1 and Y_p^k consequently are open sets. In Section 7 we have investigated the closure of D_1 — the set D — and now it is possible to formulate correctly the approximation problem.

For $y\in \mathrm{R}^n$ fo find $E^k(y)=\inf\{\|y-z\| : z\in Y^k\}$, where $Y^k=\{y\in \mathrm{R}^N : F^k(y)\neq\emptyset\}$ and $\|\,.\,\|$ is some norm in R^N. For example $\|a\|_c=\max\{|a_i| : i\}$ or $\|a\|_q=(\Sigma_{i=1}^N |a_i|^q)^{1/q}$ for $1\leq q<\infty$. We also may consider

$$E_*^k(y)=\inf\{\|\Delta-a\|' : a\in D\},$$

where y and Δ are connected with (1.1) and $\|\,.\,\|'$ is a norm in R^n.

The following equivalences are obvious

$$E^k(y)=0\Leftrightarrow y\in Y^k\Leftrightarrow \Delta\in D\Leftrightarrow E_*^k(y)=0.$$

Proposition 7. *If $\Delta_i=y[x_i, \ldots, x_{i+k}]=z[x_i, \ldots, x_{i+k}]$ for $i=1, 2, \ldots, n$, then $E^k(y)=E^k(z)$.*

Proof. We have $(y-z)[x_i, \ldots, x_{i+k}]=0$ and hence there exists a polynomial p of $k-1$-st degree such that $y_i=p(x_i)+z_i$ for $i=1, 2, \ldots, N$. Also there is $\bar{y} \in Y^k$ such that $E^k(y) = \| y - \bar{y} \|$ because Y^k is closed. We set $\bar{z}_i = \bar{y}_i - p(x_i)$. Then $\bar{z} \in Y^k$ and $\| z - \bar{z} \| = \| y - \bar{y} \| = E^k(y)$. Hence $E^k(z) \leq E^k(y)$. Analogously we get $E^k(y) \leq E^k(z)$.

It seems that in general the computation of $E^k(y)$ is more complicated than the computation of $E^k_*(y)$.

Finally we give an example. Let $n = k = 3$, $x_i = i$ for $i = 1, 2, \ldots, 6$ and y be such that $(\Delta_2 - \Delta_1 - \Delta_3)^2 - 16\Delta_1\Delta_3 = \delta > 0$, $\Delta_i > 0$ (cf. Proposition 4). Then $E^3_*(y)_C = \delta(\gamma + \sqrt{\gamma^2 + 7\delta})^{-1}$ and $E^3(y)_C = 3E^3_*(y)_C/4$, where $\gamma = 5\Delta_1 + 3\Delta_2 + 5\Delta_3$.

9. Remarks and non-solved problems

9.1. The restriction on the data ($\Delta > 0$ or $\Delta \geq 0$) and the restriction on the interpolating function ($f^{(k)} \geq 0$) can be generalized as follows: $\varphi_1 > \Delta > \varphi_2$, $\psi_1 > f^{(k)} > \psi_k$, where φ_i and ψ_i are functions from R^n and $[x_1, x_N]$, respectively, to the extended real line. For example Hornung [2] consider the case $k = 2$, $\varphi_1 = +\infty$, $\varphi_2 = \gamma = \mathrm{const}$, $\psi_1 = +\infty$, $\psi_2 = \gamma$ and called this γ-convexity. Under some consistency conditions on φ_i and ψ_i we can get results similar to the results of this paper.

9.2. A. Donchev pointed out that the subject of this paper can be treated of the optimal control theory point of view.

9.3. We have not cosidered in Section 4 the cases $p = \infty$ and $p = 1$. When $p = \infty$ the minimizing element for Problem 4 is not unique (see [3] for $k = 2$). When $p = 1$ in many cases we have not any minimizing element in L_1. It is not difficult to find an interpolating function with a minimal L_1 norm of the k-th derivative, considered as a positive measure, using the interpolating conditions $\Delta_i = \int B_i dg$ and the fact that $\Sigma^j_{i=j-k+1} \frac{x_{i+k} - x_i}{k} B_i(t) = 1$ for each $t \in [x_j, x_{j+1}]$.

9.4. The problem for constructing an algorithm solving Problem 2 a in the case $k \geq 4$ is open. In our point of view the best situation here will be to find an algorithm similar to Algorithm I in the case $k = 3$. Another possibility is to use Theorem 1 d) and to minimize the functional Φ. The main difficulties here come from the fact that D_1 is open and $\Phi(a_q)$ may be very closed to zero. Moreover if Δ belongs to the boundary of D_1 then $\Phi(a) > 0$ for each a but $\inf\{\Phi(a) : (a, \Delta) = 1\} = 0$.

9.5. Let $\{B_i\}$ be a given system of functions in Problem 2 (not necessarily B-splines). It is interesting to see which minimal conditions on this system ensure that a theorem similar to Theorem 1 can be proved using the same arguments.

9.6. There are many problems arising from Section 8. We mention here only one of the simplest: To construct an algorithm for computing $E^3_*(y)$ if $\| . \|'$ is the Euclidean norm.

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