Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

SOME ESTIMATE, EXISTENCE AND UNIQUENESS THEOREMS FOR EVOLUTION EQUATIONS

HENRYK UGOWSKI

We consider the Cauchy problem for a linear evolution equation in a Banach space. There are derived some estimates concerning a solution and a measure of noncompactness for the above problem. Next these estimates are applied in proving existence and uniqueness theorems for some system of semilinear evolution equations in a Banach space. The above existence and uniqueness theorems involve, as a particular case, a system of integro-differential equations with functional arguments.

1. Introduction. Let X be a complex Banach space with a norm $\|\cdot\|$. By $C([0, t_0]; X)$ we denote the Banach space of all strongly continuous functions $v: [0, t_0]\to X$ with the norm

$$\|v\|_{[0, t_0]} = \sup_{0\leq t\leq t_0} \|v(t)\|.$$

$C^1((0, t_0]; X)$ denotes the set consisting of all functions $v: (0, t_0]\to X$ possessing strong derivative $v': (0, t_0]\to X$ which is strongly continuous in $(0, t_0]$. Moreover, we shall use the Banach space $C^{(\varepsilon)}([0, t_0]; X)$($\varepsilon\in(0, 1)$ being a constant) consisting of all functions $v: [0, t_0]\to X$ with finite norm

$$\|v\|^{(\varepsilon)}_{[0, t_0]} = \|v\|_{[0, t_0]} + \sup_{t', t\in[0, t_0]} \{\|v(t)-v(t')\|\,|t-t'|^{-\varepsilon}\}.$$

In Section 2 we consider the problem

$$(1.1)\qquad v'(t)+A(t)v(t)=f(t),\quad 0<t\leq t_0,\quad v(0)=v_0,$$

where $A(t)$ are (in general) unbounded linear operators in X, f and v take values in X, and $v_0\in X$. We establish some estimates in the space $C^{(\varepsilon)}([0, t_0]; X)$ of a solution of the above problem. Next there is derived some estimate of a measure of noncompactness in the space $C([0, t_0]; X)$ of the set of all solutions of the problem (1.1) when f varies in a bounded set of the space $C^{(\varepsilon)}([0, t_0]; X)$ and v_0 is fixed.

Section 3 deals with the problem

$$(1.2)\qquad v_i'(t)+A_i(t)v_i(t)=(B_iv)(t),\quad 0<t\leq t_0,$$

$$(1.3)\qquad v_i(0)=v_{0i},\quad i=1,\dots, N,$$

where $A_i(t)$ are linear unbounded operators in X, v_i take values in X, B_i are nonlinear operators defined on some class of vector functions $v=(v_1,\dots, v_N)$, and $v_{0i}\in X$ are given elements. Using the results of Section 2 and applying some Darbo type fixed point theorem, we prove an existence theorem for the above problem. Next under assumption different from those of the above existence theorem, we prove an existence and uniqueness theorem for the problem (1.2, 1.3) with the aid of the Banach fixed point theorem.

In Section 4 the results of Section 3 are applied to a system of integro-differential evolution equations with functional arguments. Finally, in Section 5 there are given some examples illustrating Theorem 4.1 of Section 4.

The most important result of this paper is the derivation of a suitable estimate of the measure of noncompactness concerning the problem (1.1) and its application in solving the problem (1.2, 1.3). The Cauchy problem of type (1.2, 1.3) for an infinite system of equations was considered in paper [4]. There were obtained (under assumptions somewhat different from those of this paper) the results similar to those of Sections 3 and 4 of the present paper. The main difference consists in the fact that the complete continuity of some operators, assumed in [4], is now replaced by some conditions imposed on the measure of noncompactness concerning some other operators. Comparing considerations in the papers in question, we can see that theorems of Sections 3 and 4 of this paper can be extended to an infinite system of equations. In order to give a relatively simple and general example to Section 4 we restricted the present paper to a finite system.

2. The problem (1.1). We shall make use of the following Bielecki's norms

$$\|v\|_{[0,t_0],\mu} = \sup_{0\leq t\leq t_0} \|e^{-\mu t}v(t)\| \tag{2.1}$$

and

$$\|v\|^{(\varepsilon)}_{[0,t_0],\mu} = \|v\|_{[0,t_0],\mu} + \sup_{t',t\in[0,t_0]} \{[\exp(-\mu\max(t,t'))]\,\|v(t)-v(t')\|\,|t-t'|^{-\varepsilon}\}, \tag{2.2}$$

where $\mu\in R$ is a constant (R being the set of all real numbers). Obviously, for any $\mu\in R$ the norms $\|\cdot\|_{[0,t_0],\mu}$ and $\|\cdot\|^{(\varepsilon)}_{[0,t_0],\mu}$ are equivalent with the norms

$$\|\cdot\|_{[0,t_0],0} = \|\cdot\|_{[0,t_0]} \quad\text{and}\quad \|\cdot\|^{(\varepsilon)}_{[0,t_0],0} = \|\cdot\|^{(\varepsilon)}_{[0,t_0]},$$

respectively. Let us put

$$\mathscr{F} = \bigcup_{0<\beta<1} C^{(\beta)}([0,t_0];X). \tag{2.3}$$

Clearly, the set $\mathscr{F}$ is dense in $C([0,t_0];X)$.

As in [1] (Sec. 2.3), we introduce the following assumptions concerning the operators $A(t)$ appearing in the problem (1.1).

(2.I) $A(t)\colon D\to X$, $t\in[0,t_0]$ are closed linear operators with a domain D dense in X.

(2.II) For any $t\in[0,t_0]$ the resolvents $R(\lambda, A(t))$ exist for all λ with $\operatorname{Re}\lambda\leq 0$ and $\|R(\lambda, A(t))\|\leq K_1(|\lambda|+1)^{-1}$, K_1 being a positive constant.

(2.III) There are constants $K_2>0$ and $\alpha\in(0,1)$ such that

$$\|[A(t)-A(\tau)]A^{-1}(s)\|\leq K_2|t-\tau|^{\alpha},\quad t,\tau,s\in[0,t_0].$$

Moreover, we need the following assumptions.

(2.IV) $f\in C^{(\varepsilon)}([0,t_0];X)$ for some $\varepsilon\in(0,1)$.

(2.V) v_0 belongs to the domain of $A^{\gamma}(0)$ for some $\gamma\in(0,1)$.

(2.VI) $F\subset\mathscr{F}$ is a bounded set of the space $C([0,t_0];X)$.

Theorem 2.1. *If assumptions* (2.I—2.V) *are satisfied, then the problem (1.1) has a unique solution* $v \in C([0, t_0]; X) \cap C^1((0, t_0]; X)$. *Moreover, for any* $\alpha' \in [0, \gamma)$ *we have*

$$A^{\alpha'}(0)v \in C^{(\alpha'')}([0, t_0]; X), \quad \alpha'' = \gamma - \alpha' \tag{2.4}$$

and there is a constant $K > 0$ *depending only on* K_1, K_2, t_0, α' *and* γ, *such that*

$$\| A^{\alpha'}(0)v \|^{(\alpha'')}_{[0,t_0],\mu} \leq K(\mu^{\gamma-1} \| f \|_{[0,t_0],\mu} + \| A^{\gamma}(0)v_0 \|) \tag{2.5}$$

for any $\mu > 0$.

Proof. The first assertion is proved in [1] under assumptions (2.I—2.IV). The solution v of the problem (1.1) is given by the formula

$$v(t) = U(t, 0)v_0 + \int_0^t U(t, s)f(s)ds, \tag{2.6}$$

where $U(t, s)$ is the fundamental solution of the equation $z'(t) + A(t)z(t) = 0$. Further we proceed like in the proof of Theorem 1 of [4]. Namely, let us put

$$y(t) = A^{\alpha'}(0)U(t, 0)v_0. \tag{2.7}$$

Writing $y(t) = [A^{\alpha'}(0)A^{-\gamma}(t)][A^{\gamma}(t)U(t, 0)A^{-\gamma}(0)][A^{\gamma}(0)v_0]$ and using (2.1) and inequalities (2.1 4.14) and (2.14.12) of [1], we get

$$\| y \|_{[0,t_0],\mu} \leq K_3 \| A^{\gamma}(0)v_0 \|. \tag{2.8}$$

If $0 \leq \tau < t \leq t_0$, then the relation

$$v(t) - y(\tau) = \{A^{\alpha'}(0)[U(t, 0) - U(\tau, 0)]A^{-\gamma}(0)\}[A^{\gamma}(0)v_0]$$

implies, by inequality (2.14.15) of [1], that

$$\| e^{-\mu t} [y(t) - y(\tau)] \| \leq K_4(t-\tau)^{\alpha''} \| A^{\gamma}(0)v_0 \|. \tag{2.9}$$

Now let us put

$$z(t) = A^{\alpha'}(0) \int_0^t U(t, s) f(s)ds = \int_0^t A^{\alpha'}(0)U(t, s) f(s)ds. \tag{2.10}$$

Hence, taking into account (2.1), the estimate

$$\| A^{\alpha'}(s)U(t, \tau) \| \leq K_5(t-\tau)^{-\alpha'}, \quad 0 \leq \tau < t \leq t_0, \quad 0 \leq s \leq t_0,$$

and Hölder's inequality, it follows that

$$\| z \|_{[0,t_0],\mu} \leq K_6 \mu^{\gamma-1} \| f \|_{[0,t_0],\mu}. \tag{2.11}$$

If $0 \leq \tau < t \leq t_0$, then (2.10) and Corollary of [1] (p. 164) imply the inequality

$$\| e^{-\mu t} [z(t) - z(\tau)] \| \leq K_7(t-\tau)^{\alpha''} [\int_0^t e^{\mu t/(\gamma-1)} \| f(s) \|^{1/(1-\gamma)} ds]^{1-\gamma} \tag{2.12}$$

$$\leq K_8 \mu^{\gamma-1}(t-\tau)^{\alpha''} \| f \|_{[0,t_0],\mu}.$$

Note that each constant $K_i (i = 3, \ldots, 8)$ depends, at most, on the constants K_1, K_2, t_0, α' and γ. Therefore relations (2.6—2.12), (2.2) yield (2.4) and (2.5), which completes the proof.

Let m be the Hausdorff's measure of noncompactness in X, i. e. $m(E) = \inf\{r>0 : E$ can be covered by a finite number of balls of radius $r\}$ for any bounded set $E\subset X$. This definition and some properties of m can be found, for instance, in [2]. By M_μ we denote the Hausdorff's measure of noncompactness in the space $C([0, t_0]; X)$ with respect to the norm

$$\|\cdot\|_{[0,t_0],\mu} \ (\mu\in R \text{ being a constant}).$$

Suppose assumptions (2.I—2.III) and (2.V) are satisfied and fix arbitrarily $\alpha'\in[0, \gamma)$. Then Theorem 2.1 implies that for any $f\in\mathcal{F}$ there exists a unique solution v of the problem (1.1) such that $A^{\alpha'}(0)v\in C^{(\alpha'')}([0, t_0]; X)$. Setting $Zf = A^{\alpha'}(0)v$ we define on the set $\mathcal{F}\subset C([0, t_0]; X)$ an operator taking values in $C^{(\alpha'')}([0, t_0]; X)\subset C([0, t_0]; X)$. With the aid of Theorem 2.1 we conclude that

$$\|Zf_1 - Zf_2\|_{[0,t_0],\mu} \leqq K\mu^{\gamma-1}\|f_1 - f_2\|_{[0,t_0],\mu} \tag{2.13}$$

for any $f_1, f_2\in\mathcal{F}$ and $\mu>0$.

Now suppose additionally that assumption (2.VI) is satisfied. Then it follows from Lemma 4.4 of [6] and (2.13) that

$$M_\mu(ZF)\leqq K\mu^{\gamma-1}M_\mu(F), \quad \mu>0,$$

where $ZF=\{Zf : f\in F\}$. Thus we have proved the following theorem.

Theorem 2.2. *Let assumptions* (2.I—2.III), (2.V) *and* (2.VI) *be satisfied. Fix arbitrarily* $\alpha'\in[0, \gamma)$ *and denote by* $V^{(\alpha')}$ *the set of all functions* $v^{(\alpha')} = A^{\alpha'}(0)v$ *such that* v *is a solution of the problem* (1.1) *for some* $f\in F$. *Then for any* $\mu>0$ *we have*

$$M_\mu(V^{(\alpha')})\leqq K\mu^{\gamma-1}M_\mu(F),$$

K being the constant appearing in Theorem 2.1.

3. The probleme (1.2, 1.3). We need the following assumption.

(3.I) The operators $A_i(t)$ $(t\in[0, t_0],\ i=1, \ldots, N)$ satisfy assumptions (2.I)—(2.III).

In order to formulate further assumptions we introduce some notation. By $C_N([0, t_0]; X)$ we denote the set of all functions $v=(v_1, \ldots, v_N)$, $v_i\in C([0, t_0]; X)$, $i=1, \ldots, N$. Providing $C_N([0, t_0]; X)$ with the norm

$$|\!|\!| v |\!|\!|_{[0,t_0]} = \sum_{i=1}^{N}\|v_i\|_{[0,t_0]} \tag{3.1}$$

we get a Banach space. Bielecki's norms

$$|\!|\!| v |\!|\!|_{[0,t_0],\mu} = \sum_{i=1}^{N}\|v_i\|_{[0,t_0],\mu} \ (\mu\in R \text{ being a constant})$$

will be used as well. Obviously, each norm $|\!|\!|\cdot|\!|\!|_{[0,t_0],\mu}$ is equivalent with

$$|\!|\!|\cdot|\!|\!|_{[0,t_0],0} = |\!|\!|\cdot|\!|\!|_{[0,t_0]}.$$

By $C_N^{(\varepsilon)}([0, t_0]; X)$ ($\varepsilon\in(0, 1)$ being a constant) we denote the set of all functions $v=(v_1, \ldots, v_N)$, $v_i\in C^{(\varepsilon)}([0, t_0]; X)$, $i=1, \ldots, N$.

Let us put $W(a, \delta, \mu)=\{v\in C_N^{(\delta)}([0, t_0]; X) : \|v_i\|^{(\delta)}_{[0,t_0],\mu}\leqq a,\ i=1, \ldots, N\}$, $a>0$, $\delta\in(0, 1)$ and $\mu\geqq 0$ being constants. We abbreviate $W(a, \delta, 0) = W(a, \delta)$.

Finally, denote by $D_i (i=1, \ldots, N)$ the domain of $A_i^{\alpha'}(0)$, $\alpha' \in [0, 1)$ being a constant.

Now we introduce the following assumptions.

(3.II) The operators $B_i (i=1, \ldots, N)$ are defined for $v \in C_N([0, t_0]; X)$ such that $v_j: [0, t_0] \to D_j (j=1, \ldots, N)$ and

(3.2) $$A^{\alpha'}(0)v = (A_1^{\alpha'}(0)v_1, \ldots, A_N^{\alpha'}(0)v_N) \in C_N([0, t_0]; X)$$

and they take values in $C([0, t_0]; X)$.

(3.III) For any $t \in [0, t_0]$, $v \in C_N([0, t_0]; X)$, $i=1, \ldots, N$ we have

$$\| [B_i A^{-\alpha'}(0)v](t) \| \leq K_9 (1 + ■ v ■_{[0,t]}),$$

K_9 being a positive constant. The norm $■ v ■_{[0,t]}$ is defined by (3.1) with t_0 replaced by t (for $t=0$ this norm equals $\Sigma_{i=1}^N \| v_i(0) \|$).

(3.IV) There is a constant $\beta \in (0, 1)$ and for any $a>0$ there is a positive constant $K_{10} = K_{10}(a)$ such that if $u, v \in C_N([0, t_0]; X)$ and

(3.3) $$\| u_j \|_{[0,t_0]}, \quad \| v_j \|_{[0,t_0]} \leq a, \quad j=1, \ldots, N,$$

then

$$\| [B_i A^{-\alpha'}(0)u](t) - [B_i A^{-\alpha'}(0)v](t) \| \leq K_{10} d_\beta (■ u - v ■_{[0,t]}), \quad t \in [0, t_0], \quad i=1, \ldots, N,$$

where

$$d_\beta(s) = \begin{cases} s^\beta, & 0 \leq s \leq 1, \\ s, & s > 1. \end{cases}$$

(3.V) There is a constant $\gamma \in (\alpha', 1)$ such that each element $v_{0i} (i=1, \ldots, N)$ belongs to the domain of $A_i^\gamma(0)$.

(3.VI) For any $a>0$ there is a constant $\beta' = \beta'(a) \in (0, 1)$ such that each operator $B_i A^{-\alpha'}(0) (i=1, \ldots, N)$ maps $W(a, \gamma - \alpha')$ into a bounded set of the space $C^{(\beta')}([0, t_0]; X)$.

(3.VII) For any $a>0$ there is a constant $K_{11} = K_{11}(a) > 0$ such that for any set $V \subset W(a, \gamma - \alpha')$ and any $t \in [0, t_0]$, $i=1, \ldots, N$ we have

$$m([B_i A^{-\alpha'}(0)V](t)) \leq K_{11} \sum_{j=1}^N \sup_{0 \leq \tau \leq t} [m(V_j(\tau))],$$

where V_j denotes the set of all j-th coordinates of elements of V and $V_j(\tau) = \{v_j(\tau) : v_j \in V_j\}$.

Note that

(3.4) $$\bar{M}_\mu(V) = \sum_{j=1}^N M_\mu(V_j) < \infty$$

for any bounded set $V \subset C_N([0, t_0]; X)$. Hence, taking $\mu=0$ and using Lemma 4.8 of [6], we find that the sum appearing in assumption (3.VII) is finite.

It follows from assumptions (3.VI) and (3.VII) that

(3.5) $$M_\mu(B_i A^{-\alpha'}(0)V) \leq K_{11} \bar{M}_\mu(V)$$

for any $V \subset W(a, \gamma - \alpha')$. Indeed, (3. VII) implies that

$$m(e^{-\mu t}[B_i A^{-\alpha'}(0)V](t)) \leq K_{11} \sum_{j=1}^{N} \sup_{0\leq\tau\leq t} [m(e^{-\mu\tau}V_j(\tau))].$$

Hence, according to Lemma 2(2°) of [2] and Sec. 4 of [2], we get (3.5).

Lemma 3.1. *For any bounded sets E, E_1, $E_2 \subset C_N([0, t_0]; X)$ and any $v \in C_N([0, t_0]; X)$ we have:*

1° $\bar{M}_\mu(E \cup \{v\}) = \bar{M}_\mu(E)$;

2° *if* $E_1 \subset E_2$, *then* $\bar{M}_\mu(E_1) \leq \bar{M}_\mu(E_2)$;

3° $\bar{M}_\mu(\mathrm{Conv}\, E) = \bar{M}_\mu(E)$;

4° *if* $\bar{M}_\mu(E) = 0$, *then the closure $\bar{E}$ of E is compact.*

The above lemma easily follows from (3.4), Lemma 2 of [2] and Lemma 4.7 of [6] with $(Y_\infty, |\cdot|_\infty)$ replaced by $(Y, |\cdot|)$, where

$$Y = \{y = (y_1, \ldots, y_N): y_i \in Y_i, \quad i = 1, \ldots, N\}$$

and $|y| = \sum_{i=1}^{N} |y_i|_i$.

Theorem 3.1. (*cf. Theorem 1 of* [4]). *If assumptions* (3.I—3.VII) *are satisfied, then the problem* (1.2, 1.3) *has a solution* $v \in C_N^{(\delta)}([0, t_0]; X) \cap C_N^1((0, t_0]; X)$, $\delta = \gamma - \alpha'$ *such that* $A^{\alpha'}(0)v \in C_N([0, t_0]; X)$, *where*

$$C_N^1((0, t_0]; X) = \{u = (u_1, \ldots, u_N): u_1 \in C^1((0, t_0]; X), \quad i = 1, \ldots, N\}.$$

Proof. We proceed like in the proof of Theorem 1 of [4]. Namely, let us take into consideration $W(a, \delta, \mu)$, where $a > 0$ and $\mu \geq 1$ are some constants which will be specified later. Note that $W(a, \delta, \mu)$ is a convex, closed and bounded subset of $C_N([0, t_0]; X)$. For any $v \in W(a, \delta, \mu)$ consider the problem

$$w_i'(t) + A_i(t)w_i(t) = [B_i A^{-\alpha'}(0)v](t), \quad 0 < t \leq t_0, \quad w_i(0) = v_{0i}.$$

In view of Theorem 2.1 there exists a unique solution $w_i \in C([0, t_0]; X) \cap C^1((0, t_0]; X)$ $(i = 1, \ldots, N)$ of the above problem and, moreover,

$$\| A_i^{\alpha'}(0)w_i \|_{[0,t_0],\mu}^{(\delta)} \leq K(\mu^{\gamma-1} \| B_i A^{-\alpha'}(0)v \|_{[0,t_0],\mu} + \| A_i^{\gamma}(0)v_{0i} \|).$$

Hence, by assumption (3.III), we get

$$\| A_i^{\alpha'}(0)w_i \|_{[0,t_0],\mu}^{(\delta)} \leq K(K_9 N a \mu^{\gamma-1} + K_9 + b_1) \tag{3.6}$$

$(i = 1, \ldots, N)$, where $b_1 = \max_{1\leq i\leq N} \| A_i^{\gamma}(0)v_{0i} \|$.

Now we define on $W(a, \delta, \mu)$ an operator T setting $Tv = A^{\alpha'}(0)w = (A_1^{\alpha'}(0)\, w_1, \ldots, A_N^{\alpha'}(0)w_N) = ((Tv)_1, \ldots, (Tv)_N)$. Choosing

$$a \geq 2K(b_1 + K_9), \tag{3.7}$$

$$\mu \geq \mu_1 = \max\{1, (2KNK_9)^{1/(1-\gamma)}\} \tag{3.8}$$

it follows from (3.6) that $\| (Tv)_i \|_{[0,t_0],\mu}^{(\delta)} \leq a$, $v \in W(a, \delta, \mu)$, $i = 1, \ldots, N$, i. e. $T: W(a, \delta, \mu) \to W(a, \delta, \mu)$.

Taking into consideration Theorem 2.1, assumption (3.IV), the monotonicity of d_β, and the inequality $d_\beta(s_1 s_2) \leq d_\beta(s_1) d_\beta(s_2)$, s_1, $s_2 \geq 0$, one can show that

$$\| (Tv)_i - (T\bar{v})_i \|_{[0,t_0],\mu} \leq K_{12} d_\beta (\blacksquare v - \bar{v} \blacksquare_{[0,t_0],\mu}), \quad i = 1, \ldots, N$$

for any $v, \bar{v} \in W(a, \delta, \mu)$, $K_{12} > 0$ being a constant. Hence we have

$$\blacksquare Tv - T\bar{v} \blacksquare_{[0,t_0],\mu} \leq K_{12} N d_\beta (\blacksquare v - \bar{v} \blacksquare_{[0,t_0],\mu}), \quad v, \bar{v} \in W(a, \delta, \mu).$$

This proves the continuity of T.

Now let us take any set $V \subset W(a, \delta, \mu)$. Then, in virtue of Theorem 2.2 and (3.5),

$$M_\mu((TV)_i) \leq K K_{11} \mu^{\gamma-1} \bar{M}_\mu(V), \quad i = 1, \ldots, N.$$

Hence, by (3.4), we get

$$\bar{M}_\mu(TV) \leq K K_{11} N \mu^{\gamma-1} \bar{M}_\mu(V). \tag{3.9}$$

Taking into consideration (3.8) choose

$$\mu \geq \max\{\mu_1, (2KK_{11}N)^{1/(1-\gamma)}\}. \tag{3.10}$$

Then it follows from (3.9) that $\bar{M}_\mu(TV) \leq \frac{1}{2} \bar{M}_\mu(V)$, $V \subset W(a, \delta, \mu)$.

According to Lemma 3.1 and the above considerations $\bar{M}_\mu$ and T satisfy all the assumptions of Lemma 4.2 of [6] (the Darbo type fixed point theorem). Consequently, T has a fixed point $\tilde{v} \in W(a, \delta, \mu)$ with a and μ defined by (3.7) and (3.10), respectively. The function $v = A^{-\alpha'}(0)\tilde{v}$ is the desired solution of the problem (1.2, 1.3).

Note that the main difference between Theorem 1 of [4] and Theorem 3.1 consists in the fact that the complete continuity of the operators $A_i^{-1}(0)$, assumed in [4], has been replaced in this paper by some condition imposed on the measure of noncompactness concerning the operators $B_i A^{-\alpha'}(0)$ (assumption (3.VII)).

Now we are going to prove an existence and uniqueness theorem for the problem (1.2, 1.3) with the aid of the Banach fixed point theorem.

Theorem 3.2 (*cf. Theorem 2 of* [4]). *Let assumptions* (3.I—3.III) *and* (3.V) *be satisfied. Suppose that the operators* $B_i A^{-\alpha'}(0)$ $(i = 1, \ldots, N)$ *map the product* $\mathscr{F}^N$ *into* $\mathscr{F}$, *where* $\mathscr{F}$ *is defined by* (2.3). *Moreover, suppose there are constants* $K_{13} > 0$, $\gamma' \in (0, 1-\gamma)$ *such that for any* $a \geq e$ *and any* $u, v \in C_N([0, t_0]; X)$ *satisfying* (3.3) *we have*

$$\| [B_i A^{-\alpha'}(0)u](t) - [B_i A^{-\alpha'}(0)v](t) \| \leq K_{13} (\ln a)^{\gamma'} \blacksquare u - v \blacksquare_{[0,t]}, \quad t \in [0, t_0], \quad i = 1, \ldots, N. \tag{3.11}$$

Then the problem (1.2, 1.3) *has a unique solution* $u \in \mathscr{F}^N \cap C_N^1((0, t_0]; X)$. *Moreover* v, $A^{\alpha'}(0)v \in C_N^{(\delta)}([0, t_0]; X)$, $\delta = \gamma - \alpha'$.

Proof. The first part of the proof is the same as that one of Theorem 3.1. Thus we have $T: W(a, \delta, \mu) \to W(a, \delta, \mu)$, $a \geq e$ and μ being defined by (3.7) and (3.8), respectively. Theorem 2.1 and (3.11) imply that

$$\|(Tv)_i-(T\bar{v})_i\|_{[0,t_0],\mu}\leq K_{14}\mu^{\gamma+\gamma'-1}\blacksquare v-\bar{v}\blacksquare_{[0,t_0],\mu}$$

for any $v, \bar{v}\in W(a, \delta, \mu)$, $i=1,\ldots, N$, where

$$K_{14}=K_{13}(\ln a+t_0)^{\gamma'}.$$

This inequality yields $\blacksquare Tv-T\bar{v}\blacksquare_{[0,t_0],\mu}\leq K_{14}N\mu^{\gamma+\gamma'-1}\blacksquare v-\bar{v}\blacksquare_{[0,t_0],\mu}$.

Hence, setting $\mu=\max\{\mu_1, (2K_{14}N)^{1/(1-\gamma'-\gamma)}\}$ (see (3.8)), we get

$$\blacksquare Tv-T\bar{v}\blacksquare_{[0,t_0],\mu}\leq\frac{1}{2}\blacksquare v-\bar{v}\blacksquare_{[0,t_0],\mu},\quad v, \bar{v}\in W(a, \delta, \mu).$$

Consequently, by the Banach fixed point theorem, T has a unique fixed point $\tilde{v}\in W(a, \delta, \mu)$. Obviously $\tilde{v}$ and $v=A^{-\alpha'}(0)\tilde{v}$ belong to $C_N^{(\delta)}([0,t_0]; X)$ and, moreover, $v\in C_N^1((0, t_0]; X)$ and v is a solution of the problem (1.2, 1.3). Since $a\geq e$ satisfying (3.7) can be chosen arbitrarily, therefore it follows that v is a unique solution of this problem in the set

$$C_N^{(\delta)}([0, t_0]; X)\cap C_N^1((0, t_0]; X).$$

The proof we have been doing up to the present can be repeated for any $\delta'\in(0, \delta)$. Consequently, for any $\delta'\in(0, \delta)$ there exists a unique solution

$$u\in C_N^{(\delta')}([0, t_0]: X)\cap C_N^1((0, t_0]; X)$$

of the problem in question such that $A^{\alpha'}(0)u\in C_N^{(\delta')}([0, t_0]; X)$. It follows from the unicity that $u=v$. This completes the proof.

4. Integro-differential equations. Now, like in [4], we consider the particular case of the operators B_i given by the formulas

$$(4.1)\qquad (B_iv)(t)=g_i\Big(t, v(t), v(\varphi_i(t)), \int_0^t h_i(t, s, v(s), v(\psi_i(s)))ds\Big),$$

for $t\in[0, t_0]$, $i=1,\ldots, N$, where $v(\varphi_i(t))=(v_1(\varphi_{i1}(t)),\ldots, v_N(\varphi_{iN}(t)))$ and $v(\psi_i(t))$ is defined likewise. We use the notation introduced in the previous sections. Moreover, let us put

$$D'=D_1\times\ldots\times D_N,\quad X^N=\{x=(x_1,\ldots, x_N): x_j\in X, j=1,\ldots, N\},$$

$$\blacksquare x\blacksquare=\sum_{j=1}^{N}\|x_j\|,\quad x\in X^N,\quad E=\{(t, s): 0\leq t\leq t_0,\ 0\leq s\leq t\},$$

$$W_a=\{x\in X^N: \|x_j\|\leq a,\ j=1,\ldots, N\},\quad X_a=\{x\in X: \|x\|\leq a\}\ (a>0).$$

We introduce the following assumptions.

(4.I) The operators $g_i(i=1,\ldots, N)$ are defined for $t\in[0, t_0]$, $v, y\in X^N\cap D'$, $z\in X$ and take values in X. Moreover, g_i are continuous.

(4.II) For any $t\in[0, t_0]$, $v, y\in X^N$, $z\in X$ we have

$$\|g_i(t, A^{-\alpha'}(0)v, A^{-\alpha'}(0)y, z)\|\leq K_{15}(1+\blacksquare v\blacksquare+\blacksquare y\blacksquare+\|z\|),\quad i=1,\ldots, N,$$

$K_{15}>0$ being a constant.

(4.III) There are constants $\beta \in (0, 1)$, $\beta_1 \in [\beta, 1]$ and for any $a, b>0$ there are constants $K_{16}=K_{16}(a, b)>0$, $\beta_2=\beta_2(a, b) \in (0, 1)$ such that

$$\| g_i(t, A^{-\alpha'}(0)v, A^{-\alpha'}(0)y, z)-g_i(t', A^{-\alpha'}(0)v', A^{-\alpha'}(0)y', z')\|$$

$$\leq K_{16}[d_\beta(|||v-v'|||+|||y-y'|||)+d_{\beta_1}(\|z-z'\|)+|t-t'|)^{\beta_2}], \quad i=1, \ldots, N$$

for any $v, v', y, y' \in W_a$, $z, z' \in X_b$, $t, t' \in [0, t_0]$.

(4. IV) The operators $h_i (i=1, \ldots, N)$ are defined for $(t, s) \in E$, $v, y \in X^N \cap D'$ and take values in X. Moreover, h_i are continuous.

(4.V) For any $(t, s) \in E$, $v, y \in X^N$ we have

$$\| h_i(t, s, A^{-\alpha'}(0)v, A^{-\alpha'}(0)y)\| \leq K_{17}(1+|||v|||+|||y|||), \quad i=1, \ldots, N,$$

$K_{17}>0$ being a constant.

(4.VI) For any $a>0$ there are constants $K_{18}=K_{18}(a)>0$ and $\beta_3=\beta_3(a) \in (0, 1)$ such that

$$\| h_i(t, s, A^{-\alpha'}(0)v, A^{-\alpha'}(0)y)-h_i(t', s', A^{-\alpha'}(0)v', A^{-\alpha'}(0)y')\|$$

$$\leq K_{18}[d_{\beta/\beta_1}(|||v-v'|||+|||y-y'|||)+(|t-t'|+|s-s'|)^{\beta_3}]$$

for any $(t, s), (t', s') \in E$, $v, v', y, y' \in W_a$, $i=1, \ldots, N$.

(4.VII) The functions $\varphi_{ij}, \psi_{ij}: [0, t_0] \to R$, $i, j=1, \ldots, N$ satisfy the inequalities

$$0 \leq \varphi_{ij}(t) \leq t, \quad 0 \leq \psi_{ij}(t) \leq t.$$

Moreover, the functions ψ_{ij} are continuous, whereas φ_{ij} satisfy the uniform Hölder condition $|\varphi_{ij}(t)-\varphi_{ij}(s)| \leq K_{19}|t-s|^{\beta_4}$, $t, s \in [0, t_0]$, $K_{19}>0$ and $\beta_4 \in (0, 1)$ being some constants.

(4.VIII) For any $a, b>0$ there is a constant $K_{20}=K_{20}(a, b)>0$ such that for any $t \in [0, t_0]$, $V, Y \subset W_a$ and $Z \subset X_b$ we have

$$m(g_i(t, A^{-\alpha'}(0)V, A^{-\alpha'}(0)Y, Z)) \leq K_{20}[\bar{m}(V)+\bar{m}(Y)+m(Z)], \quad i=1, \ldots, N,$$

$$\text{where } \bar{m}(V)=\sum_{j=1}^{N} m(V_j) \quad \text{(see (3. VII))}.$$

(4.IX) For any $a>0$ there is a constant $K_{21}=K_{21}(a)>0$ such that for any $(t, s) \in E$, $V, Y \subset W_a$ we have

$$m(h_i(t, s, A^{-\alpha'}(0)V, A^{-\alpha'}(0)Y)) \leq K_{21}[\bar{m}(V)+\bar{m}(Y)], \quad i=1, \ldots, N.$$

Using Lemma 2(3°) of [2] one can easily show that assumptions (4.I—4.IX) imply assumptions (3.II — 3.IV), (3.VI, 3. VII) for the case (4.1). Therefore Theorem 3.1 yields the following one.

Theorem 4.1. *If assumptions* (3.I, 3.V) *and* (4.I — 4.IX) *are satisfied, then the assertion of Theorem* 3.1 *holds true in the case* (4.1).

Retaining assumptions (3.I, 3.V, 4.I, 4.II, 4.IV, 4.V, 4.VII), and suitably modifying assumptions (4.III) and (4.VI), one can find the validity of Theorem 3.2 in the case (4.1).

5. Example. We shall consider the Cauchy problem for a system of random integro-differential equations. We begin with some lemmas and definitions.

L e m m a 5.1. *Let A be a closed linear operator in X with a dense domain $D(A)$, and assume that the resolvent $R(\lambda, A)$ exists for all λ with* $\operatorname{Re}\lambda \leqq 0$ *and satisfies the inequality*

$$\| R(\lambda, A) \| \leqq N_1(1+|\lambda|)^{-1},$$

$N_1>0$ *being a constant. Let B be a closed linear operator in X with a domain $D(B)\supset D(A^\delta)$, $\delta\in(0, 1]$ being a constant. Then there exists a constant $N_2>0$ such that*

$$\| BA^{-\delta}u \| \leqq N_2 \| u \|, \quad u\in X.$$

P r o o f. By [1] (Sec. 2.14) the operator $A^{-\delta}$ is bounded. Since $D(B)\supset D(A^\delta)$, $BA^{-\delta}$ is defined on X and closed. Consequently, by the closed graph theorem, $BA^{-\delta}$ is bounded.

Let (Ω, Γ, P) be a complete probability space. In our further considerations X will denote the Banach space $L^r(\Omega)$ ($r\in[1, \infty)$ being a constant) consisting of all random variables $u:\Omega\to C$ with finite norm

$$\| u \| = [\int_\Omega | u(\omega) |^r P(d\omega)]^{1/r},$$

where C denotes the set of all complex numbers.

D e f i n i t i o n. *A random variable u: $\Omega\to C$ is said to be simple if there exists a finite number of disjoint sets $\Omega_i\in\Gamma$ $(i=1, \ldots, m)$ such that*

$$\Omega=\Omega_1\cup \ldots \cup\Omega_m \text{ and } u(\omega)=c_i, \ \omega\in\Omega_i \ (i=1, \ldots, m),$$

where $c_i\in C$ are constants depending on u.

Denote by X_0 the set of all simple random variables. Obviously, $X_0\subset X$ and $uv\in X$ for any $u\in X_0$ and $v\in X$.

L e m m a 5.2 (*see* [5] (*Sec. 3*)). *The set X_0 is dense in X.*

We make the following assumptions.

(5.I) S_i: $[0, t_0]\times\Omega\to R$ $(i=1, \ldots, N)$ are random functions (i. e. for any $t\in[0, t_0]$ the functions $S_i(t, \cdot)$ are random variables) such that for any $t\in[0, t_0]$ $S_i(t, \omega)\geqq N_3$ almost everywhere (shortly a. e.) in Ω (i. e. for any $\omega\in\Omega_t$, where $P(\Omega_t)=1$) and $S_i(t, \cdot)\in X$, where $N_3>0$ is a constant. Moreover, there are constants $N_4>0$, $\alpha\in(0, 1)$ such that

$$\left\| \frac{S_i(t_1, \cdot)}{S_i(t_2, \cdot)} - 1 \right\| \leqq N_4 | t_1-t_2 |^\alpha, \quad t_1, t_2\in[0, t_0].$$

(5.II) The set $D=\{u\in X: S_i(t, \cdot)u\in X\}$ is independent of $t\in[0, t_0]$ and $i\in\{1, \ldots, N\}$.

Note that in general D is a proper subset of X. For instance, if $S_i(t, \cdot)$ does not belong to $L^{2r}(\Omega)$, then $D\neq X$.

Assumptions (5.I, 5.II) will be satisfied if we take, for instance,

$$S_i(t, \omega)=s_i(t)s(\omega), \quad t\in[0, t_0], \quad \omega\in\Omega, \quad i=1, \ldots, N,$$

where $s\in X$, $s(\omega)\geqq N_5$ a. e. in Ω($N_5>0$ being a constant), and $s_i:[0, t_0]\to(0, \infty)$ are uniformly Hölder continuous functions with exponent α. The set D is then defined as $D=\{u\in X: su\in X\}$.

Now we introduce operators $A_i(t)$, $t\in[0, t_0]$, $i=1, \ldots, N$ defined by the formula

(5.1) $$A_i(t)u = S_i(t, \cdot)u, \quad u \in D.$$

Operators of similar type were considered in [3] (*Sec.* 4.6).

Lemma 5.3. *If assumptions* (5.I, 5.II) *are satisfied, then the operators* $A_i(t)$ *defined by* (5.1) *satisfy assumption* (3.I).

Proof. It is clear that D is a linear set of X and $A_i(t)$ are linear operators. The relation $X_0 \subset D$ and Lemma 5.2 imply that D is dense in X. Moreover, $A_i(t)$ are closed operators. Indeed, under arbitrarily fixed i and t take a sequence (u_k) of elements of D such that $u_k \to u$ and $A_i(t)u_k \to v$ in X. Then there is a subsequence (also denoted by u_k)) such that

$$u_k(\omega) \to u(\omega), \quad S_i(t, \omega)u_k(\omega) \to v(\omega) \text{ a. e. in } \Omega.$$

Hence it follows that $S_i(t, \omega)u_k(\omega) \to S_i(t, \omega)u(\omega)$ a. e. in Ω.
Thus we have proved that $S_i(t, \cdot)u = v \in X$ and consequently $u \in D$ and $A_i(t)u = v$.

One can easily find that the resolvent $R(\lambda, A_i(t))$ exists for any $\lambda \in C$ with $\operatorname{Re}\lambda \leq 0$ and is defined by the formula $[R(\lambda, A_i(t))]v = [S_i(t, \cdot) - \lambda]^{-1}v$. Hence we have

$$\| R(\lambda, A_i(t)) \| \leq N_3^{-1}(1 + N_3^2)^{1/2}(1 + |\lambda|)^{-1}$$

and $\| A_i(t_1)A_i^{-1}(t_2) - I \| \leq N_4 |t_1 - t_2|^\alpha$, $t_1, t_2 \in [0, t_0]$, where I is the identity operator in X. The last inequality implies, by [1] (Problem on p. 110), the following one

$$\| [A_i(t_1) - A_i(t_2)]A_i^{-1}(t_3) \| \leq N_6 |t_1 - t_2|^\alpha, \quad t_1, t_2, t_3 \in [0, t_0],$$

where $N_6 > 0$ is a constant. This completes the proof.

Note that in general $A_i(t)$ are unbounded operators. Moreover, the powers $A_i^{-\delta}(t)$, $\delta \geq 0$ are bounded operators but not completely continuous.

Now we shall consider the problem

(5.2) $$v_1'(t) + A_i(t)v_i(t) = G_i(\cdot, t, v(t), v(\varphi_i(t)), K_i(v(t)),$$

$$K_i(v(\varphi_i(t))), Z_i(v(t)), Z_i(v(\varphi_i(t))), \int_0^t H_i(\cdot, t, s, v(s),$$

$$v(\psi_i(s)), K_i(v(s)), K_i(v(\psi_i(s))), Z_i(v(s)), Z_i(v(\psi_i(s))))ds),$$

$$0 < t \leq t_0, \quad i = 1, \ldots, N,$$

(5.3) $$v_i(0) = v_{0i}, \quad i = 1, \ldots, N,$$

where $v(\varphi_i(t))$, $v(\psi_i(s))$ are defined in Sec. 4 and

$$K_i(v(t)) = (K_{i1}(v_1(t)), \ldots, K_{iN}(v_N(t))), \quad Z_i(v(t)) = (Z_{i1}(v_1(t)), \ldots, Z_{iN}(v_N(t))),$$

and the remaining components depending on K_i and Z_i are defined likewise

The following assumptions will be needed.

(5.III) $$G_i : \Omega \times [0, t_0] \times C^{6N+1} \to C \text{ and}$$

$$H_i : \Omega \times E \times C^{6N} \to C \quad (i = 1, \ldots, N)$$

are random functions, where E is defined in Sec. 4.

(5.IV) There exists a nonnegative random variable $\xi_0 \in X$ such that

$$|G_i(\omega, 0, \ldots, 0)|, \ |H_i(\omega, 0, \ldots, 0)| \leqq \xi_0(\omega) \text{ a. e. in } \Omega.$$

(5.V) There are constants $\varepsilon, \varepsilon_1 \in (0, 1)$, $N_7 > 0$ such that for any $a > 0$ we have

$$|G_i(\omega, t, u_1, \ldots, u_6, z) - G_i(\omega, t', u_1', \ldots, u_6', z')| \leqq N_7[|t-t'|^{\varepsilon_1} + (1+a^{1-\varepsilon})(|u_3-u_3'|$$

$$+|u_4-u_4'|)^{\varepsilon} + |u_1-u_1'| + |u_2-u_2'| + |u_5-u_5'| + |u_6-u_6'| + |z-z'|], \quad i=1, \ldots, N$$

for any $\omega \in \Omega_0$, $t, t' \in [0, t_0]$, $u_3, u_3', u_4, u_4' \in C_a^N$, $u_j, u_j' \in C^N$, $j=1, 2, 5, 6$, $z, z' \in C$, where

$$|u_k - u_k'| = \sum_{j=1}^{N} |u_{kj} - u_{kj}'|, \quad C_a^N = \{w \in C^N : |w| \leqq a\},$$

and $\Omega_0 \in \Gamma$ is such a set that $P(\Omega_0) = 1$.

(5.VI) There are constants $\varepsilon_2, \varepsilon_3 \in (0, 1)$, $N_8 > 0$ such that for any $a > 0$ we have

$$|H_i(\omega, t, s, u_1, \ldots, u_6) - H_i(\omega, t', s', u_1', \ldots, u_6')| \leqq N_8[(|t-t'| + |s-s'|)^{\varepsilon_2}$$

$$+(1+a^{1-\varepsilon_3})(|u_3-u_3'| + |u_4-u_4'|)^{\varepsilon_3} + |u_1-u_1'| + |u_2-u_2'| + |u_5-u_5'| + |u_6-u_6'|]$$

$$i=1, \ldots, N$$

for any (t, s), $(t', s') \in E$ and any remaining arguments as in (5.V).

(5.VII) Each element v_{0i} $(i=1, \ldots, N)$ belongs to the domain of $A_i^{\gamma}(0)$ where $\gamma \in (0, 1)$ is a constant.

(5.VIII) The operators $K_{ij}: X \to X$ $(i, j=1, \ldots, N)$ are completely continuous, whereas $Z_{ij}: D_{ij} \to X$ $(i, j=1, \ldots, N)$ are linear closed operators, where $D_{ij} \subset X$ include the domain D_j of $A_j^{\alpha'}(0)$, $\alpha' \in (0, \gamma)$ being a constant.

Note that assumptions (5.IV—5.VI) imply the inequalities

$$(5.4) \quad |G_i(\omega, t, u_1, \ldots, u_6, z)| \leqq \xi_1(\omega) + N_9(|u_1| + \ldots + |u_6| + |z|), \quad i=1, \ldots, N$$

for any $(\omega, t, u_1, \ldots, u_6, z) \in \Omega_0 \times [0, t_0] \times C^{6N+1}$, and

$$(5.5) \quad |H_i(\omega, t, s, u_1, \ldots, u_6)| \leqq \xi_1(\omega) + N_9(|u_1| + \ldots + |u_6|), \quad i=1, \ldots, N$$

for any $(\omega, t, s, u_1, \ldots, u_6) \in \Omega_0 \times E \times C^{6N}$, where $N_9 > 0$ is a constant and $\xi_1 \in X$ is a nonnegative random variable.

Now take into consideration the complete continuity of the operators $K_{ij} A_j^{-\alpha'}(0)$, the boundedness of the operators $Z_{ij} A_j^{-\alpha'}(0)$ (following from Lemma 5.1), and the definition and properties of the Hausdorff's measure of noncompactness. It is not difficult to prove that assumptions (5.I—5.VI), (5.VIII) and inequalities (5.4, 5.5) imply assumptions (4.I—4.VI), (4.VIII, 4.IX) for the operators

$$g_i(t, u, v, z) = G_i(\cdot, t, u, v, K_i(u), K_i(v), Z_i(u), Z_i(v), z),$$

$$h_i(t, s, u, v) = H_i(\cdot, t, s, u, v, K_i(u), K_i(v), Z_i(u), Z_i(v)).$$

Hence, taking into account Lemma 5.3, we conclude that if assumptions (5.I—5.VIII) are satisfied and the operators $A_i(t)$ are defined by (5.1), then Theorem 4.1 can be applied to the problem (5.2, 5.3).

In the considered problem K_{ij} may be continuous operators such that the sets $\{K_{ij}(u): u \in X\}$ are finite dimensional. For instance, we may take

$$K_{ij}(u) = \xi_{ij}(\| u \|), \quad u \in X,$$

where $\xi_{ij}: [0, \infty) \to C$ are contiouous functions. Finally, Z_{ij} may be defined, for instance, by the formula

$$Z_{ij}(u) = \eta_{ij} u, \quad u \in D_{ij},$$

where $\eta_{ij} \in X$ are such random variables that $D_{ij} = \{v \in X: \eta_{ij} v \in X\} \supset D_j$.

REFERENCES

1. A. Friedman. Partial Differential Equations. Holt, Rinehart and Winston, 1969.
2. K. Goebel, W. Rzymowski. An Existence Theorem for the Equation $x' = f(t, x)$ in a Banach Space. *Bull. Acad. Polon. Sci., Ser. Math. Astronom. Phys.*, **18**, 1970, 367-370.
3. M. Schechter. Spectra of Partial Differential Operators. Amsterdam, 1971.
4. H. Ugowski. On a Certain System of Semilinear Evolution Equations. *Ann. Mat. pura appl.*, **125**, 1980, 337-347.
5. H. Ugowski. On the First Fourier Problem for Random Parabolic Equations of the Second Order. *Demonstratio Math.*, **14**, 1981, 835-864.
6. H. Ugowski. An Application of Measures of Noncompactness to the Theory of Parabolic Equations in a Banach Space. *Demonstratio Math.*, **17**, 1984 (in press).

Department of Mathematics
Technical University of Gdansk
ul. Majakowskiego 11/13
80-952 Gdansk, Poland

Received 18. 11. 1983
Revised 29. 11. 1984