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ON THE BEHAVIOUR OF TWO MODULI OF FUNCTIONS. II.

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In this article we continue the investigations from [5, 6] on the relations between different characteristics of functions. Two type generalizations of the results in [6] are given. The mathematical facts involving such kind of results are emphasized. An answer to the question posed in [5] is given.

1. Notations. $\mathbb{R}^m$ in considered as a vector space over the field $\mathbb{R}$. D^β denotes as usual the differential operator $(\partial/\partial x_1)^{\beta_1}\ldots(\partial/\partial x_m)^{\beta_m}$ with multiindex $\beta=(\beta_1,\ldots,\beta_m)$, $\beta_i\geq 0$. $|\cdot|$ stands either for the length of β ($|\beta|=\Sigma\beta_i$) or for the Euclidean norm in $\mathbb{R}^m$ or $\mathbb{R}$. $\|\cdot\|_p$ is the L_p norm of a function if $1\leq p<\infty$ and $\|\cdot\|_\infty$ is the sup norm. k is natural, $1\leq p$, $q\leq\infty$, δ, θ, $\varepsilon>0$. x, y, z are points in the domain $\Omega\subset\mathbb{R}^m$ of the function considered, $h\in\mathbb{R}^m$. $V_\delta(x)=\{y\in\mathbb{R}^m: |x-y|\leq\delta\}$. M is the set of all bounded and measurable functions. We denote with c a positive constant although not always the same one. The exact dependence of c on the other parameters is explicitly given.

2. Functions of several variables. In this section we deal with real valued functions defined on a domain $\Omega\subset\mathbb{R}^m$. In Theorem 1, Corollaries 1 and 2 and Lemma 3 Ω is supposed to be: 1) $\Omega=\mathbb{R}^m$; 2) $\Omega=[0,\infty)^m$; 3) $\Omega=[0,2\pi)^m$ if f is 2π periodical with respect to each variables. Comments for Lipschitz-graph domains are given in Section 4.1.

For a function f we consider the moduli

$$\omega_k(f;\delta)_p=\sup\{\|\Delta_h^k f(\cdot)\|_p : |h|\leq\delta\} \tag{2.1}$$

and

$$\tau_k(f;\delta)_p=\|\omega_k(f,\cdot;\delta)\|_p, \tag{2.2}$$

(where $\omega_k(f,x;\delta)=\sup\{|\Delta_h^k f(y)|: y, y+kh\in V_{k\delta/2}(x)\}$ and $\Delta_h^k f(y)=\Sigma_{s=0}^k(-1)^{k+s}\binom{k}{s}f(y+sh)$ if f is defined on the segmet with the end-points at y and $y+\kappa h$ and $\Delta_h^k f(y)=0$ othewise.

For our purposes we also need the moduli

$$\tau_k(f;\delta)_{q,p}=\|\omega_k(f,\cdot;\delta)_q\|_p, \tag{2.3}$$

where $\omega_k(f,x;\delta)_q=[\delta^{-m}\int_{|h|\leq\delta}|\Delta_h^k f(x)|^q dh]^{1/q}$.

Moduli (2.1) and (2.3) are defined for each class of equivalent functions in L_p and L_r ($r=\max\{p,b\}$), respectively, and moduli (2.2) are defined for each individual function in M.

Moduli (2.1) generate Besov spaces ($k>\theta$) $B_{p,q}^\theta=\{f\in L_p: \|f\|_{B_{p,q}^\theta}=\|f\|_p+[\int_0^\infty[t^{-\theta}\omega_k(f;t)_p]^q t^{-1}dt]^{1/q}<\infty\}$ ($q<\infty$) and $B_{p,\infty}^\theta=\{f\in L_p: \|f\|B_{p,\infty}^\theta=\|f\|_p+\sup\{t^{-\theta}\omega_k(f;t)_p: t>0\}<\infty\}$ and moduli (2.2) generate the spaces (see [8, 12]) $A_{p,q}^\theta=\{f\in M: \|f\|_{A_{p,q}^\theta}=\|f\|_p+[\int_0^\infty[t^{-\theta}\tau_k$

$(f;\ t)_p]^q t^{-1}dt]^{1/q}<\infty\}$ $(q<\infty)$ and $A^\theta_{p,\infty}=\{f\in M: \|f\|_{A^\theta_{p,\infty}}=\|f\|_p+\sup\{t^{-\theta}\tau_k(f;\ t)_p:\ t>0\}<\infty\}$.

The properties of ω_k are assumed to be known and we give only the needed properties of moduli (2.2) and (2.3).

$$\tau_k(f+g;\ \delta)_p\leqq\tau_k(f;\ \delta)_p+\tau_k(g;\ \delta)_p \tag{2.4}$$

$$c(k,\ m)\tau_k(f;\ t)_{p,p}(V_\delta(x))\leqq\omega_k(f;\ t)_p(V_\delta(x))\leqq c(k,\ m)\tau_k(f;\ t)_{p,p}(V_\delta(x)) \tag{2.5}$$

for each $x\in\Omega$ and $t\in(0,\ \delta]$. (The indication of the domain $V_\delta(x)$ means that the norm is taken over this domain and the function may not be defined out of this domain.)

$$c(k,\ m)\,\tau_k(f;\ \delta)_{p,p}\leqq\omega_k(f;\ \delta)_p\leqq c(k,\ m)\tau_k(f;\ \delta)_{p,p} \tag{2.6}$$

(2.4) follows directly from (2.2). The first inequalities in (2.5) and (2.6) are gotten from (2.1) and (2.3). The proof of the second inequalities in (2.5) and (2.6) follows the proof for the one-dimensional case given in [3].

Now we turn our attention to the property

$$\tau_k(f;\ \delta)_p\leqq c(k,\ m)\ (\sum_{|\beta|=k}\delta^k\|D^\beta f\|_p+\sum_{\substack{k<|\beta|\leqq m\\ \beta_i=0,\ 1}}\delta^{|\beta|}\|D^\beta f\|_p). \tag{2.7}$$

(2.7) is a consequence of Theorem 1 in [13] and (2.7′) below. A weaker form of (2.7) is Lemma 4 in [12]. Let us compare (2.7) with the corresponding inequality for

$$\omega_k(f;\ \delta)_p\leqq c(k,\ m)\sum_{|\beta|=k}\delta^k\|D^\beta f\|_p. \tag{2.7′}$$

We see that the additional (in comparison with (2.7′)) sum in (2.7) vanishes either if $m=1$ (the one-dimensional case!) or if $k\geqq m$. But V. Popov [9] shows that (2.7) is not true without the second sum in the case $k=1,\ m=2,\ 1\leqq p<2$. So we have a pure several-dimensional effect which will bother the proof of the main result of this section.

Theorem 1. *If $f\in B^{m/p}_{p,1}$ then f coincides a. e. with a continuous function F such that* $(k>m/p)$

$$\tau_k(F;\ \delta)_p\leqq c(k,\ m,\ p)\delta^{m/p}\int_0^\delta\omega_k(f;\ t)_p t^{-1-m/p}dt. \tag{2.8}$$

Theorem 1 is a several-dimensional generalization of Theorem 1 in [6]. Two facts are basic in the proof of (2.8): the embedding $B^{m/p}_{p,1}\subset L_\infty$ (Lemma 1) and the equivalence of w_k with a property defined K-functional taking into account the semi-norm in the right-hand side of (2.7) (Lemma 3).

Lemma 1. *Let $f\in B^{m/p}_{p,1}([0,\ 1]^m)$. Then f coincides a. e. with a continuous functon F such that for $k>m/p$ we have*

$$\|F\|_\infty\leqq c(k,\ m,\ p)\ (\|f\|_p+\int_0^1\omega_k(f;\ t)_p t^{-1-m/p}\,dt). \tag{2.9}$$

This lemma is a consequence of Theorem 18.10 and Theorem 18.11 in [11, pp. 302, 303]. With a linear transformation from Lemma 1 we get

Lemma 2. *Let $z\in\mathrm{R}^m$, $\delta>0$, $\Omega=[z_1,\ z_1+\delta]\times\cdots\times[z_m,\ z_m+\delta]$ and $g\in B^{m/p}_{p,1}(\Omega)$. Then g coincides a. e. with a continous function G such that for $k>m/p$ we have*

$$\|G\|_\infty(\Omega)\leqq c(k,m,p)\,(\delta^{-m/p}\|g\|_p(\Omega)+\int_0^\delta\omega_k(f;t)_p(\Omega)t^{-1-m/p}\,dt). \tag{2.10}$$

The constants in (2.9) and (2.10) are one and the same.

Lemma 3. *Let $f\in L_p$. Then for each k and $\delta>0$ there exists $g_{k,\delta}$ such that*

$$\|f-g_{k,\delta}\|_p\le c(k, m)\omega_k(f;\delta)_p, \tag{2.11}$$

$$\delta^k\|D^\beta g_{k,\delta}\|_p\le c(k, m)\,\omega_k(f;\delta)_p \quad \text{for each } |\beta|=k \tag{2.12}$$

and

$$\delta^{|\beta|}\|D^\beta g_{k,\delta}\|_p\le c(k, m)\omega_{|\beta|}(f;\delta)_p \text{ for each } |\beta|>k,\ 0\le\beta_i\le k. \tag{2.13}$$

A weaker form of (2.13)

$$\delta^{|\beta|}\|D^\beta g_{k,\delta}\|_p\le c(k, m)\omega_k(f;\delta)_p \text{ for } \kappa<|\beta|\le m,\ \beta_i=0,\ 1. \tag{2.14}$$

will be used in the proof of Theorem 1.

Proof. We set $U_\delta=[0,\delta]^m$ and define a modified Steklov function $g_{k,\delta}$ as follows

$$g_{k,\delta}(x)=-\delta^{-km}\int_{U_\delta}\cdots\int_{U_\delta}\sum_{s=1}^{k}(-1)^s\binom{k}{s}f(x+sz)dh_1\dots dh_k, \tag{2.15}$$

where $z=(h_1+\cdots+h_m)/k$. Note that $h_i\in\mathbb{R}^m$ for $i=1, 2, \dots, k$! To prove (2.11) we use (2.1) and Minkovski's inequality

$$\|f-g_{k,\delta}\|_p=\delta^{-km}\Big\|\int_{U_\delta}\cdots\int_{U_\delta}\Delta_z^k f(\cdot)dh_1\dots dh_k\Big\|_p$$

$$\le\delta^{-km}\int_{U_\delta}\cdots\int_{U_\delta}\|\Delta_z^k f\|_p\,dh_1\dots dh_k\le\omega_k\ (f,\sqrt{m}\delta)_p\le c(k, m)\omega_k(f;\delta)_p.$$

In the proof of (2.12) and (2.13) we can consider separately each term in the sum in (2.15). We set $\Delta_\delta^\beta f(x)=\Delta_{\delta_1}^{\beta_1}\dots\Delta_{\delta_m}^{\beta_m}f(x)$ where $\delta_1=(\delta, 0,\dots 0),\dots,\delta_m=(0,\dots,0,\delta)$. Then for $k\le|\beta|\le km$, $0\le\beta_i\le k$ using Minkowski's inequality we get

$$\|D^\beta g_{k,\delta}\|_p\le c(k, m)\delta^{-|\beta|}\sum_{s=1}^{k}\|\Delta_{s\delta/k}^\beta f\|_p. \tag{2.16}$$

Now (2.16) proves (2.12) and (2.13) in view of Theorem 1 in [2] which gives a repesentation of the mixed finite difference Δ_β^β as a linear combination of forward finite differences of order $|\beta|$.

Proof of Theorem 1. Lemma 1 gives that f coincides a. e. with a continuous function F. Let $\delta>0$ and g be $g_{k,\delta}$ from Lemma 3. Then (2.4) gives

$$\tau_k(F;\delta)_p\le\tau_k(F-g;\delta)_p+\tau_k(g;\delta)_p. \tag{2.17}$$

From (2.7), (2.12) and (2.14) we get

$$\tau_k(g;\delta)_p\le c(k, m)\Big(\sum_{|\beta|=k}\delta^k\|D^\beta f\|_p+\sum_{\substack{k<|\beta|\le m\\ \beta_i=0,1}}\delta^{|\beta|}\|D^\beta f\|_p\Big) \tag{2.18}$$

$$\le c(k, m)\,\omega_k(f;\delta)_p\le c(k, m)\delta^{-m/p}\int_0^\delta\omega_k(f;t)_p t^{-1-m/p}\,dt.$$

From Lemma 2 and (2.5) we get

$$\omega_k(F-g, x;\delta)\le 2^k\sup\{|F(y)-g(y)|: y\in V_{k\delta/2}(x)\} \tag{2.19}$$

$$\le c(k, m, p)\,\Big(\delta^{-m/p}\|F-g\|_p(V_{k\delta/2}(x))+\int_0^\delta\tau_k(F-g, t)_{p,p}(V_{k\delta/2}(x))t^{-1-m/p}\,dt\Big).$$

Applying (2.2), (2.19), Minkowski's inequality, the inequality $\int_{V\delta(x)} du \leqq c(m)\delta^m$, (2.3), (2.11) and (2.6), we obtain

$$(2.20)\qquad \tau_k(F-g;\ \delta)_p \leqq c(k,\ m,\ p)[\delta^{-m/p}\ (\int_\Omega \int_{V_\delta(x)} |F(y)-g(y)|^p\, dy\, dx)^{1/p}$$

$$+\int_0^\delta t^{-1-m/p}(\int_\Omega \int_{V_\delta(x)} \omega_k(F-g,y;t)_p^p\, dy\, dx)^{1/p} dt]$$

$$\leqq c(k,\ m,\ p)\ [\|F-g\|_p + \delta^{m/p} \int_0^\delta \tau_k(F-g;\ t)_{p,p}\, t^{-1-m/p}\, dt]$$

$$\leqq c(k,\ m,\ p)\,[\omega_k(f;\delta)_p + \delta^{m/p} \int_0^\delta \omega_k(F-g;\ t)_p t^{-1-m/p}\, dt]$$

$$\leqq c(k,\ m,\ p)\ [\omega_k(f;\ \delta)_p + \delta^{m/p} \int_0^\delta (\omega_k(f;\ t)_p + \omega_k(g;\ t)_p) t^{-1-m/p}\, dt].$$

From (2.7′) and (2.12) we get

$$(2.21)\qquad \int_0^\delta \omega_k(g\ ;\ t)_p t^{-1-m/p} dt \leqq c(k,\ m) \sum_{|\beta|=k} \|D^\beta g\|_p \int_0^\delta t^{k-1-m/p} dt$$

$$= c(k,\ m,\ p)\,\delta^{k-m/p} \sum_{|\beta|=k} \|D^\beta g\|_p \leqq c(k,\ m,\ p)\delta^{-m/p}\omega_k(f;\ \delta)_p.$$

Now (2.17), (2.18), (2.20) and (2.21) prove (2.8).

From Theorem 1 we immediately get

Corollary 1. *$B^{m/p}_{p,1} \subset A^{m/p}_{p,\infty}$ with the corresponding inequality for the norms.*

Embeddings of the above type should be considered as follows: in each class of equivalent functions in $B^{m/p}_{p,1}$ there is a representative in $A^{m/p}_{p,\infty}$.

Using Theorem 1, we obtain repeating the arguments from the proof of Corollary 2 in [6].

Corollary 2. *If $\theta > m/p$, then $B^\theta_{p,q} = A^\theta_{p,q}$.*

Corollary 2 is stated without proof for $\Omega = \mathsf{R}^m$ in [12] and it is gotten with different methods by V. Hristov in [14] for $\Omega = \mathsf{R}^m$ and in [13] for $\theta > m$ and $\Omega = [0,\ 2\pi]^m$. For the case $m=1$ see [5, 6, 8].

3. An one-dimensional analog. What we use here from Section 2 is in the case $m=1$. In this section we replace the constant δ in the definition of $\omega_k(f;\ \delta)_p$ and $\tau_k(f;\ \delta)_p$ with a function depending on x. It is easy to do this for $\tau_k(f;\ \delta)_p$ — we consider in (2.2) δ as a function of the integration variable. In view of (2.6) it is convenient to consider (2.3) with $q=p$ as an equivalent definition of ω_k and to work with a non-constant δ.

In this section the functions are defined on $[-1,\ 1]$. We assume that $\Delta_h^k f(y) = 0$ if y or $y+kh \bar{\in} [-1,\ 1]$. We set $\Delta(d,\ x) = d\sqrt{1-x^2} + d^2$ for $x \in [-1,\ 1]$, $d = \text{const} > 0$ Then $\tau_k(f;\ \Delta(d))_p = \|\omega_k(f,\cdot;\ \Delta(d,\cdot))\|_p$ and $\tau_k(f;\ \Delta(d))_{p,p} = \|\omega_k(f,\cdot;\ \Delta(d,\cdot))_p\|_p$. The main result of the section is

Theorem 2. *Let $k > 2/p$, $f \in L_p[-1,\ 1]$ and $\int_0^1 \tau_k(f,\ \Delta(t))_{p,p} t^{-1-2/p} dt < \infty$. Then $f=F$ a. e., F is continuous and for $d < (4k+2)^{-1}$ we have*

$$(3.1)\qquad \tau_k(F;\ \Delta(d))_p \leqq c(k,\ p) d^{2/p} \int_0^{(4k+2)d} \tau_k(f;\ \Delta(t))_{p,p} t^{-1-2/p} dt.$$

Following the arguments of Section 2 to prove Theorem 2 we need the inequalities (see (4.1) and Corollary 4.2 or Corollary 4.4 in [3])

$$\tau_k(f+g;\ \Delta(d))_p \le \tau_k(f;\ \Delta(d))_p + \tau_k(g;\ \Delta(d))_p, \tag{3.2}$$

$$\tau_k(f+g;\ \Delta(d))_{p,p} \le \tau_k(f;\ \Delta(d))_{p,p} + \tau_k(f;\ \Delta(d))_{p,p}, \tag{3.3}$$

$$\tau_k(f;\ \Delta(d))_p \le c(\kappa)\,\|(\Delta(d))^k f^{(k)}\|_p \quad \text{if} \quad f^{(k)} \in L_p, \tag{3.4}$$

an embedding result (Lemma 2 is enough for our purposes) and a good intermediate function.

Lemma 4. *Let* $f \in L_p[-1, 1]$. *Then for each* k *and* $0<d\le(4k+2)^{-1}$ *there is* $g_{k,d}$ *such that*

$$\|f-g_{k,d}\|_p \le c(k)\tau_k(f;\ \Delta(d))_{p,p}, \tag{3.5}$$

$$\|(\Delta(d))^k g^{(k)}_{k,d}\|_p \le c(k)\tau_k(f;\ \Delta(d))_{p,p}. \tag{3.6}$$

This lemma follows from Theorem 3.1 in [4] if we put in it $w=1$ and $g_{k,d}=G_{k,n}$ where n is the integer part of $1/d$.

We set $\psi=\psi(d,\ x)=d(1-d)\sqrt{1-x^2}+d^2$ and $\gamma=\gamma(d,\ x)=\ln\psi(d,\ x)/\ln d$. It is easy to see that $\psi(d,\ x)\le\Delta(d,\ x)\le2\psi(d,\ x)$ and so

$$\tau_k(f;\ \Delta(d))_p \le \tau_k(f;\ 2\psi(d))_p. \tag{3.7}$$

The next three lemmas are technical.

Lemma 5. *Let* $k\ge2$, $x,\ y\in[-1,\ 1]$, $|x-y|\le k\psi(d,\ x)/2$. *Then*

$$((4k+3)\,(4k+2)^2)^{-1}\,\Delta((4k+2)d,\ y)\le\psi(d,\ x)\le\Delta((4k+2)\,d,\ y).$$

Proof. From (2.5) in [3] it follows that $(4k+2)^{-1}\Delta(d,\ x)\le\Delta(d,\ y)\le(4k+3)\Delta(d,\ x)$ for $|x-y|\le k/2\,.\,\Delta(d,\ x)$. Hence $\psi(d,\ x)\le\Delta(d,\ x)\le(4k+2)\Delta(d,\ y)\le\Delta((4k+2)d,\ y)$ and $\psi(d,\ x)\ge1/2\,.\,\Delta(d,\ x)\ge(4k+3)^{-1}\Delta(d,\ y)\ge(4k+3)^{-1}(4k+2)^{-2}\Delta((4k+2)d,\ y)$.

Lemma 6. *Let* $0<u\le d$. *Then* $u^{-\gamma(d,x)}\le d^2u^{-2}(\psi(d,\ x))^{-1}$.

Proof. $\dfrac{1}{\psi(d,\ x)}=\dfrac{1}{d^\gamma}=\left(\dfrac{d}{u}\right)^{2-\gamma}\ \dfrac{u^2}{d^2u^\gamma}\ge\dfrac{u^2}{d^2u^\gamma}$ because $\gamma(d,\ x)\le2$.

Lemma 7. *Let* $0<u\le d$. *Then* $u^{\gamma(d,x)}\le\psi(u,\ x)$.

Proof. We have $\ln u<0$ and hence Lemma 7 follows from the fact that for each fixed $x\in[-1,\ 1]$ $\ln\psi(u,\ x)/\ln u$ is non-decreasing function of u in $(0,\ 1)$.

Proof of Theorem 2. Let $0<a<1$. For $|x|\le a$ we have $d\sqrt{1-a^2}\le\Delta(d,x)\le2d$ and in view of (2.6), (2.1) and (2.3) the inequality $\omega_k(f;\ d)_p([-a,\ a])\le c(a,\ k)\tau_k(f;\ \Delta(d))_{p,p}$ holds true. Hence $\int_0^1\omega_k(f;\ t)_p([-a,\ a])t^{-1-2/p}\,dt<\infty$. So f coincides a. e. on $[-a,\ a]$ with a continuous function F. But a is an arbitrary number less than 1. So $f=F$ a. e. in $[-1,\ 1]$. Let g be $g_{k,\delta}$ from Lemma 4. From (3.2) we get

$$\tau_k(F;\ \Delta(d))_p\le\tau_k(F-g;\ \Delta(d))_p+\tau_k(g;\ \Delta(d))_p. \tag{3.8}$$

Using (3.4) and (3.6), we obtain

$$\tau_k(g;\ \Delta(d))_p\le c(k)\,\|(\Delta(d))^k g^{(k)}\|_p\le c(k)\tau_k(f;\ \Delta(d))_{p,p} \tag{3.9}$$

$$\le c(k)d^{2/p}\int_d^{2d}\tau_k(f;\ \Delta(t))_{p,p}t^{-1-2/p}dt\le c(k)d^{2/p}\int_0^{10d}\tau_k(f;\ \Delta(t))_{p,p}t^{-1-2/p}\,dt.$$

From (3.7) and (2.2) we get

$$\tau_k(F-g;\ \Delta(d))_p\le\tau_k(F-g;\ 2\psi(d))_p=[\int_{-1}^{1}\omega_k(F-g,\ x;\ 2\psi(d,\ x))^p dx]^{1/p} \tag{3.10}$$

$$\le2^k[\int_{-1}^{1}\sup\{|F(y)-g(y)|^p:\ |y|\le1,\ |y-x|\le k\psi(d,\ x)\}dx]^{1/p}.$$

Using Lemma 2 and (2.5), we get for each $x\in[-1,1]$

(3.11)
$$\sup\{|F(y)-g(y)|: |y|\le 1,\ |y-x|\le k\psi\}$$
$$\le c(k,p)\{\psi^{-1/p}\|f-g\|_p([x-k\psi, x+k\psi]\cap[-1,1])$$
$$+\int_0^{k\psi(d,x)}\omega_k(f-g;t)_p([x-k\psi, x+k\psi])t^{-1-1/p}dt\}$$
$$\le c(k,p)\{\psi^{-1/p}\|f-g\|_p([x-k\psi,x+k\psi]\cap[-1,1])+\int_0^{\psi}\omega_k(f-g;t)_p([x-k\psi, x$$
$$+k\psi])t^{-1-1/p}dt\}\le c(k,p)\{\psi^{-1/p}\|f-g\|_p([x-k\psi,x+k\psi]\cap[-1,1])+\int_0^d\tau_k(f-g;u^\gamma)_{p,p}([x$$
$$-k\psi, x+k\psi])u^{-1-\gamma/p}du\}.$$

Applying (3.11) in (3.10) and using Minkowski's inequality, Lemma 7, Lemma 6, (3.5), Lemma 5, (2.3) and (3.3), we get

(3.12)
$$\tau_k(F-g;\Delta(d))_p\le c(k,p)\{[\int_{-1}^{1}\psi(d,x)^{-1}\int_{[x-k\psi,x+k\psi]\cap[-1,1]}|f(y)-g(y)|^p\,dy\,dx]^{1/p}$$
$$+\int_0^d[\int_{-1}^{1}\tau_k(f-g;u^\gamma)_{p,p}^p([x-k\psi,x+k\psi])u^{-p-\gamma}dx]^{1/p}du\}$$
$$\le c(k,p)\{\|f-q\|_p+\int_0^d[\int_{-1}^{1}\int_{x-k\psi}^{x+k\psi}\omega_k(f-g,y;\psi(u,x))_p^p\,dy\,\frac{d^2}{u^2\psi(d,x)}\,dx]^{1/p}\frac{du}{u}\}$$
$$\le c(k,p)\{\tau_k(f;\Delta(d))_{p,p}+d^{2/p}\int_0^d[\int_{-1}^{1}\omega_k(f-g,y;w((4k+2)u,y))_p^p dy]^{1/p}u^{-1-2/p}du\}$$
$$\le c(k,p)\{\tau_k(f;\Delta(d))_{p,p}+d^{2/p}\int_0^{(4k+2)d}[\tau_k(f;\Delta(u))_{p,p}+\tau_k(g;\Delta(u))_{p,p}]u^{-1-2/p}du\}.$$

From (2.2), (2.3), (3.4) and (3.6) we obtain

(3.13)
$$\int_0^{(4k+2)d}\tau_k(g;\Delta(u))_{p,p}u^{-1-2/p}du\le\int_0^{(4k+2)d}\tau_k(g;\Delta(u))_p u^{-1-2/p}du$$
$$\le c(k)\int_0^{(4k+2)d}\|g^{(k)}(x)(\sqrt{1-x^2}+u)^k\|_p u^{k-1-2/p}du$$
$$\le c(k)d^{-k}\|(\Delta(d))^k g^{(k)}\|_p\int_0^{(4k+2)a}u^{k-1-2/p}du\le c(k,p)d^{-2/p}\tau_k(f;\Delta(d))_{p,p}.$$

Now (3.8), (3.9), (3.12) and (3.13) prove (3.1).

We shall formulate without introducing function spaces two corollaries of Theorem 2 which are analogs of Corollary 2.

Corollary 3. *If $k>2/p$, $\theta>2/p$ and $\int_0^1(t^{-\theta}\tau_k(f;\Delta(t))_{p,p})^q t^{-1}dt=J^q<\infty$ then there is a continuous function F, $F=f$ a. e., such that* $[\int_0^1[t^{-\theta}\tau_k(F;\Delta(t))_p]^q t^{-1}dt]^{1/q}\le c(k,p).J$.

Corollary 4. *If $k>2/p$, $\theta>2/p$ and $\sup\{d^{-\theta}\tau_k(f;\Delta(d))_{p,p}: 0<d<1\}=J<\infty$ then there is a continuous function F, $F=f$ a. e. such that $\tau_k(F;\Delta(d))_p\le c(k,p)\,J\,d^\theta$.*

4. Remarks and generalizations.

4.1. Theorem 1 is true not only for the domains mentioned in the beginning of Section 2. Without any changes in the proof we can get that Theorem 1 is valid for domains which are translations and rotations of domains of the type $\Omega=\Omega_1\times\Omega_2\times\cdots\times\Omega_m$, where Ω_i is either R or $[0, \infty)$ or $[0, 2\pi)$ if the function is 2π periodical with respect to x_i.

Let Ω possesses the following extension property:

(E) There is $\Omega_1\supset\Omega$ such that the Hausdorff distance between Ω and $R^m\setminus\Omega_1$ is positive and for each $f\in B_{p,1}^{m/p}(\Omega)$ there is $F\in L_p(\Omega_1)$ such that $F|_\Omega=f$ and

$$(4.1)\qquad \int_0^\delta \omega_k(F;\ t)_p(\Omega_1)t^{-1-m/p}dt\leq c(k,\ v,\ \Omega_1)\int_0^\delta \omega_k(f;\ t)_p(\Omega)t^{-1-m/p}\,dt$$

for each small positive δ.

For such kind domains we can prove (2.7) and Lemma 3 and therefore we can get Theorem 1. Some domains possessing (E) are given in Corollary 1, Section 4 in [7]. These domains even satisfy the condition $w_k(F;\ t)_p(\Omega_1)\leq c(k,\ \Omega_1)\,w_k(f;t)_p(\Omega)$ instead of (4.1).

Another approach in generalizing Theorem 1 is to get Lemma 3 for more common domains without extending the function and to apply Theorem 18.10 in [11] not only for hypercubes. In this way we can get Theorem 1 for Lipschitz-graph domains (see [7]) or which is the same for domains satisfying the cone-property (see [11], p. 118).

A third approach in generalizing Theorem 1 is to use if possible the inequality $\tau_k(f;\ \delta)_p\leq c(k,\ m)\delta^k\Sigma_{|\beta|=k}\|D^\beta f\|_p$ instead of (2.7). It is announced in [1] that this is true for $k>m/p$ and $\Omega=[0,1]^m$.

4.2. We set $K(f;\ t)_p=\inf\{\|f-g\|_p+t^k\Sigma_{|\beta|=k}\|D^\beta g\|_p : g\in W_p^k\}$ and $K^*(f;\ t)_p=\inf\{\|f-g\|_p+\Sigma_{\substack{|\beta|\geq k\\ 0\leq\beta_i\leq k}} t^{|\beta|}\|D^\beta g\|_p\}$, where the infimum is taken over all g for which the expression is finite. Obviously $K(f;\ t)_p\leq K^*(f;\ t)_p$. Theorem 1 in [7] says that for Lipschitz-graph domains we have

$$(4.2)\qquad c(\kappa,\ p,\ \Omega)\omega_k(f;\ t)_p\leq K(f;\ t)_p\leq c(k,\ p,\ \Omega)\omega_k(f;\ t)_p.$$

From Lemma 3 we get that $K^*(f;\ t)\leq c(k,\ m,\ p)\omega_k(f;\ t)_p$ for domains mentioned in the beginning of Section 2 and hence $F^*(f;\ t)_p\leq c(k,\ m,\ p)K(f;\ t)_p$. In other words the two K-functionals are equivalent.

4.3. Theorem 2 is stated for $\Delta(d,\ x)=d(\sqrt{1-x^2}+d)$. It is possible to prove analogous statements with functions of the type $d(1-x+d^{\gamma_1})^{\alpha_1}(1+x+d^{\gamma_2})^{\alpha_2}$ $(\alpha_i,\gamma_i>0)$ if we consider the interval $[-1,\ 1]$ and of the type $d(x+d^\gamma)^\alpha$ for the interval $[0,\ \infty)$ and so on. For this purpose one should have a "good" intermediate function (cf. Lemma 4). Such functions are constructed in [10] but the deviation is evaluated with another type moduli. The proof that those moduli are equivalent to (2.3) is complicated and will appear somewhere else. This is the only difficult step in the generalization of Theorem 2 on this way.

Let us also mention that if we have proved an analog of Theorem 2 with a function δ and if δ_1 is such that $c\delta_1(d,\ x)\leq\delta(d,\ x)\leq c\delta_1(d,\ x)$ for each $x\in\Omega$, $0\leq d\leq c$, then the same statement is true with δ_1 instead of δ.

4.4. In (3.1) we integrate up to $(4k+2)d$ instead of up to d because we do not use properties of the type

$$(4.3)\qquad \tau_k(f;\ \lambda\delta(d))_{p,p}\leq c(k,\ \lambda)\,\tau_k(f;\ \delta(d))_{p,p}\ \text{for each}\ \lambda\geq 1.$$

It is possible to prove (4.3) for different functions $\delta(d,\ x)$ if one has the equivalence of $\tau_k(f;\ \delta(d))_{p,p}$ with a properly defined K-functional. Results of this type will be given somewhere else.

4.5. Corollary 1 and Corollary 1 in [6] cannot be improved in the terms of Besov spaces. Here we give the examples only for $m=1$.

a) It is not true that $B^{m/p}_{p,q} \subset A^{m/p}_{p,\infty}$ for any $q>1$. This comes from the fact that for $q>1$ $B^{m/p}_{p,q}$ contains essentially unbounded functions. E. g. for $g_\varepsilon(x)=|\ln|x||^\varepsilon$ if $|x|<1$ and $g_\varepsilon(x)=0$ for $|x|\geqq 1$ we have $\omega_k(g_\varepsilon;\ \delta)_p \leqq c(k,\ p,\ \varepsilon)\delta^{1/p}|\ln\delta|^{\varepsilon-1}$ and hence $g_\varepsilon \in B^{1/p}_{p,q}$ if $q>(1-\varepsilon)^{-1}$.

b) It is not true that $B^{m/p}_{p,1} \subset A^{m/p}_{p,q}$ for any $q<\infty$. E. g. we set $f_\varepsilon(x)=1-|\ln|x||^{-\varepsilon}$ if $|x|\leqq 1/e$ and $f_\varepsilon(x)=0$ if $|x|\geqq 1/e$. Then for $k>1/p$ we have $\omega_k(g_\varepsilon;\ \delta)_p \leqq c(k,\ p,\ \varepsilon)\delta^{1/p}/|\log\delta|^{-\varepsilon-1}$ and $\tau_k(f_\varepsilon;\ \delta)_p \geqq c(k,\ p,\ \varepsilon)\delta^{1/p}|\ln\delta|^{-\varepsilon}$ and hence $f_\varepsilon \in B^{1/p}_{p,1}$ but $f\varepsilon \notin \mathrm{A}^{1/p}_{p,q}$ for $q\leqq\varepsilon^{-1}$.

4.6. Theorem 2 cannot be improved in the following senses:

a) The conclusion of the theorem is not true under the condition $\int_0^1(t^{-2/p}\tau_k(f;\ \Delta(t))_{p,p})^q t^{-1}\,dt<\infty$ for any fixed $q>1$. The example is the same as in point 4.5. a): we have $(k>2/p)$ $\tau_k(g_\varepsilon;\ \Delta(t))_{p,p} \leqq c(k,\ p,\ \varepsilon)\, t^{2/p}|\ln t|^{\varepsilon-1}$ and g_ε is essentially unbounded.

b) The statement $\int_0^1(t^{-2/p}\tau_k(f;\ \Delta(t))_p)^q t^{-1}dt<\infty$ does not follow from the condition ot Theorem 2 for any $q<\infty$. The example is similar to this one in point 4.5. b): For $f_\varepsilon(x)=|\ln(1-x)-\ln 2e|^{-\varepsilon}$ we have $(k>2/p)$ $\tau_k(f_\varepsilon;\ \Delta(t))_{p,p} \leqq c(k,\ p,\ \varepsilon)t^{2/p}|\ln t|^{-\varepsilon-1}$ and $\tau_k(f_\varepsilon;\ \Delta(t))_p \geqq c(k,\ p,\ \varepsilon)t^{2/p}|\ln t|^{-\varepsilon}$.

4.7. Let us compare Theorem 2 with Theorem 1 for $m=2$. In the both cases the cristal orders are $2/p$ and they can not be improved in view of points 4.5 and 4.6. This can be interpreted in the following way: moduli $\tau_k(f;\ \Delta(t))_{p,p}$ and $\tau_k(f;\ \Delta(t))_p$ posses some "twodimensional properties". Such conclusion is not very unexpected having in mind that $\tau_k(f;\ \Delta(t))_{p,p}$ characterizes the best approximation with algebraic polynomials (see [4]), from one hand, and, from the other hand, the connection between the algebraic approximations and the approximation on domains in the complex plane.

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