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POLYNOMIAL IDENTITIES OF FINITE DIMENSIONAL ALGEBRAS

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We study the varieties of unitary associative algebras over a field of characteristic zero and with a polynomial identity of degree four. The main result is that such a variety is generated by a finite dimensional algebra if and only if it satisfies a standard identity. Analogous results are obtained for the subvarieties of the variety defined by the commutator $[x_1, \ldots, x_d] = 0$ and for Lie algebra subvarietes of var sl_2.

1. Introduction. A class $\mathfrak{U}$ of associative algebras is called a variety if $\mathfrak{U}$ consists of all algebras satisfying a given system of polynomial identities. By the Birkhoff theorem, $\mathfrak{U}$ is a variety if and only if $\mathfrak{U}$ is closed with respect to the cartesian products (C), the subalgebras (S) and the quotient algebras (Q), i. e. $\mathfrak{U} = QSC\mathfrak{U}$. Two problems immediately arise: (i) Given an algebra R, find the polynomial identities of $\mathfrak{U} = \text{var}\, R = QSC(R)$; (ii) For a variety $\mathfrak{U}$ with a T-ideal of identities $U = T(\mathfrak{U})$, find an algebra R such that $\mathfrak{U} = \text{var}\, R$. The problem (ii) can be answered trivially, namely that $\mathfrak{U} = \text{var}\, F(\mathfrak{U})$, where $F(\mathfrak{U}) = K\langle X\rangle / T(\mathfrak{U})$ is the relatively free algebra of countable rank in $\mathfrak{U}$. We are interested when a variety of unitary algebras over a field of characteristic zero is generated by some finite dimensional algebra. Bearing in mind that any finite dimensional algebra satisfies a standard identity, we ask the converse question:

If a variety $\mathfrak{U}$ satisfies a standard indentity, is $\mathfrak{U}$ generated by a finite dimensional algebra?

That seems to be a very difficult problem. The purpose of this paper is to give an affirmative answer for the varieties with a polynomial identity of fourth degree. We prove the following main result:

Theorem A. *Let $\mathfrak{U}$ be a variety of unitary algebras over a field of characteristic* 0 *and satisfying a polynomial indentity of degree* 4. *Then $\mathfrak{U}$ is generated by a finite dimensional algebra if and only if a standard identity*

$$S_k(x_1, \ldots, x_k) = \Sigma(\text{sign}\,\sigma) x_{\sigma(1)} \cdots x_{\sigma(k)} = 0, \quad \sigma \in \text{Sym}\,(k),$$

holds for $\mathfrak{U}$.

The proof of this theorem allows to find the algebras generating the varieties. They are direct sums of some of the following algebras:

(i) the 2×2 upper triangular matrices;

(ii) the subalgebra $\langle e_{11}, e_{22}, te_{12}, te_{21}\rangle$ of the matrix algebra $M_2(K[t]/(t^s))$;

(iii) an algebra obtained by formal adjunction of 1 to a nilpotent algebra.

The method is based on the detailed description of T-ideals containing a polynomial of fourth degree [1, 2, 4, 8]. The idea is taken from [7], where an analogous result is obtained for non-unitary algebras satisfying an identity of degree three. We establish in the same way:

Theorem B. *Let $\mathcal{L}_d$ be the variety of unitary algebras in characteristic* 0 *defined by the left normed commutator $[x_1, \ldots, x_{d+1}] = 0$. Then $\mathfrak{U} \subset \mathcal{L}_d$ is generated by a finite dimensional algebra if and only if a standard identity vanishes on $\mathfrak{U}$.*

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Theorem C. *Let $sl_2(K)$ be the Lie algebra of 2×2 traceless matrices over a field K,* char $K=0$. *Then $\mathcal{U}\subset$ var $sl_2(K)$ is generated by a finite dimensional algebra.*

Recall that the problem of characterizing varieties of algebras over a finite field and varieties of rings, generated by a finite object, was solved by I. L'vov [5].

2. Preliminaries. We fix some notation: K is a field of characteristic 0; all associative algebras are unitary and over K; $A=A(X)=K\langle X\rangle$ is the free associative algebra generated by $X=\{x_1, x_2, \dots\}$; $A_m=K\langle x_1, \dots, x_m\rangle$ is the subalgebra of A of rank m; P_n is the set of multilinear elements in $x_1, \dots, x_n$ from A_n. Let $f(x_1, \dots, x_m)\in A$, R be an algebra and let $\mathcal{U}$ be a variety of algebras. We use the notation $T(f)$, $T(R)$, $T(\mathcal{U})$ respectively for the T-ideal generated by f or containing all the identities of R and $\mathcal{U}$. We denote by the same (script or ordinary) letters the varieties and their T-ideals: $U=T(\mathcal{U})$. The relatively free algebras of $\mathcal{U}$ are $F(\mathcal{U})=A/U$ and $F_m(\mathcal{U})=A_m/(U\cap A_m)$. For a subset S of A and the canonical homomorphism $\nu: A\to F(\mathcal{U})$ we denote $\nu(S)$ by $S(\mathcal{U})$. If S is a graded vector space (with respect to the natural grading of A or $F(\mathcal{U})$), then $S^{(n)}$ is used for the homogeneous component of degree n. Moreover, $S^n=\Sigma S^{(k)}$, $k\geqq n$.

The symmetric group Sym(n) and the general linear group GL_m act from the left respectively on P_n and A_m. This action plays an important role in the theory of PI-algebras. We use $N_m(\lambda)$ for the irreducible GL_m-module related to the partition $\lambda=(\lambda_1, \dots, \lambda_k)$ on n. On the other hand, $A_m^{(n)}$ is a right Sym(n)-module and Sym(n) permutes the positions of the variables in $x_{i_1}\dots x_{i_n}\in A_m^{(n)}$:

$$x_{i_1}\dots x_{i_n}\sigma^{-1}=x_{j_1}\dots x_{j_n},$$

where $\sigma\in$ Sym(n) and $j_s=i_{\sigma(s)}$, $s=1,\dots, n$. It is known that any irreducible GL_m-submodule $N_m(\lambda)$ of $A_m^{(n)}$ is generated by a nonzero element

$$f(x_1, \dots x_{k_1})=S_{k_1}(x_1, \dots, x_{k_1})\dots S_{k_r}(x_1, \dots, x_{k_r})\Sigma\, a_\alpha\sigma,$$

where $a_\sigma\in K$, $\sigma\in$ Sym(n) and $k_1, \dots, k_r$ are the lengths of the columns of the diagram $[\lambda]$. We call $f(x_1, \dots, x_{k_1})$ a standard generator of $N_m(\lambda)$.

Let B_m be the subspace of A_m, spanned by the products of commutators, $\Gamma_n=P_n\cap B_n$. Any subvariety of $\mathcal{U}$ is determined by its identities from $\Gamma_n(\mathcal{U})$, $n=2, 3\dots$. Moreover, Γ_n and $B_m^{(n)}$ are submodules of P_n and $A_m^{(n)}$. It is known that

$$B_m^{(2)}=N_m(1^2),\quad B_m^{(3)}=N_m(2, 1),$$

$$B_m^{(4)}=N_m(3, 1)+N_m(2^2)+N_m(2, 1^2)+N_m(1^4).$$

We fix $\mathcal{U}$ to be a variety with a multilinear polynomial identity of fourth degree. When $f\in\Gamma_4$, then f is equivalent to some of the identities

(1) $[x_2, x_1, x_1, x_1]=0$,

(2) $[x_1, x_2]^2=0$,

(3) $[[x_1, x_2], [x_3, x_4]]=0$,

(4) $S_4(x_1, x_2, x_3, x_4)=0$.

If $f\notin\Gamma_4$, then substituting 1 for some of the variables, we obtain $[x_1, x_2, x_3]=0$ or $[x_1, x_2]=0$. Hence, as a consequence of f we obtain some of the identities (1)—(4). We shall consider the cases (1)—(4) one by one.

3. Non-matrix identities. Let $\mathcal{V}_1$ be the variety of algebras, defined by the polynomial identity (1). It is known [2], that

$$B_m^{(n)}(\mathscr{V}_1)=B_m^{(n)} \quad \text{for} \quad n\leqq 3,$$

$$B_m^{(4)}(\mathscr{V})=N_m(2^2)+N_m(2,\ 1^2)+N_m(1^4),\ B_m^{(5)}(\mathscr{V}_1)=N_m(2,\ 1^3),$$

$$B_m^{(2n)}(\mathscr{V}_1)=N_m(1^{2n}),\quad B_m^{(2n+1)}(\mathscr{V}_1)=0,\quad n>2.$$

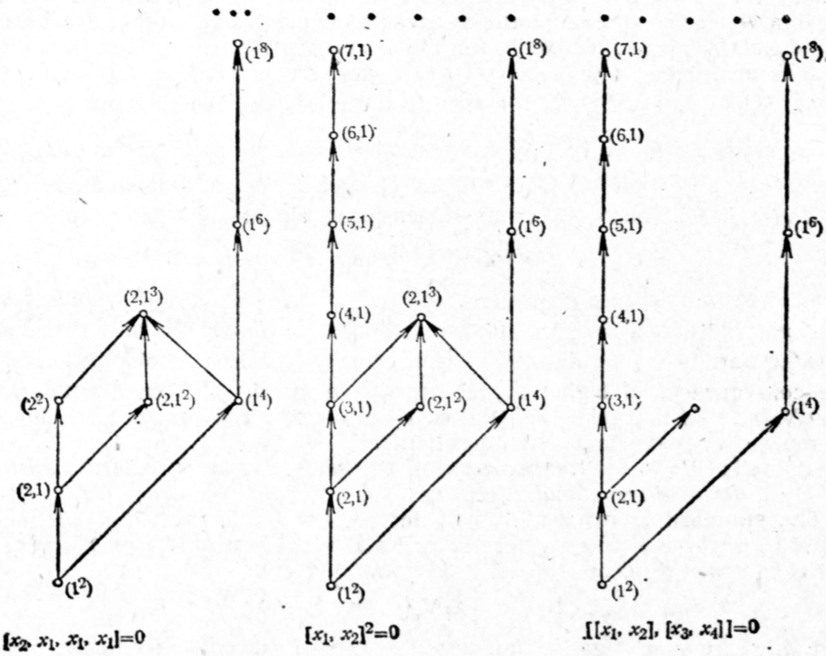

Fig. 1

The modules $N_m(1^{2n})$ are generated by the standard polynomials $S_{2n}(x_1, \ldots, x_{2n})$. The consequences of higher degree of the irreducible components of $B_m(\mathscr{V}_1)$ are marked by arrows at Fig. 1.

Proposition 1. *Let $\mathscr{U}$ be a subvariety of $\mathscr{V}_1$ and let $\mathscr{U}$ satisfy a standard identity. Then $\mathscr{U}$ is generated by a finite-dimensional algebra.*

Proof. Let $U=T(\mathscr{U})$ be the T-ideal of $\mathscr{U}$ in the relatively free algebra $F(\mathscr{V}_1)$ and let $S_{2n}(x_1, \ldots, x_{2n})\in U$. Clearly, there exists a canonical GL_m-module homomorphism $\nu: B_m(\mathscr{V}_1)\to B_m(\mathscr{U})$ and $N_m(1^{2k})\in \ker \nu$ if $k\geqq n$. In virtue of the complete reducibility of the GL_m-module $B_m(\mathscr{V}_1)$, the module isomorphism $B_m(\mathscr{V}_1)\cong B_m(\mathscr{U})\oplus \ker \nu$ holds. Therefore we can consider $B_m(\mathscr{U})$ as a submodule of $B_m(\mathscr{V}_1)$. Let $B_m(\mathscr{U})=\Sigma N_m(\lambda)$ and let m exceed the number of rows of the diagrams $[\lambda]$ (e. g. $m\geqq 2n-1$). Consider the finite-dimensional algebra $R=F_m(\mathscr{U})/F_m^{2n}(\mathscr{U})$. Obviously $R\in\mathscr{U}$, i. e. $B_m(\operatorname{var} R)\subset B_m(\mathscr{U})$ and the modules $N_m(\lambda)$ from $B_m(\mathscr{U})$ do not vanish on R. Hence $B_m(\mathscr{U})=B_m(\operatorname{var} R)$ and $\mathscr{U}=\operatorname{var} R$.

Let $\mathscr{V}_2$ be the variety defined by the identity (2). By [2], $B_m^{(4)}(\mathscr{V}_2)=N_m(3,\ 1)+N_m(2,\ 1^2)+N_m(1^4)$, $B_m^{(5)}(\mathscr{U}_2)=N_m(4,\ 1)+N_m(2,\ 1^3)$, $B_m^{(2n)}(\mathscr{U}_2)=N_m(2n-1,\ 1)+N_m(1^{2n})$,

$B_m^{(2n+1)}(\mathscr{V}_2)=N_m(2n,\ 1)$, $n>2$ (Fig. 1). The irreducible GL_m-modules $N_m(n,1)$ and $N_m(1^{2n})$ are generated by $[x_2, x_1, \ldots, x_1]=x_2(ad\,x_1)^n$ and $S_{2n}(x_1,\ldots,x_{2n})$.

Proposition 2. *Let $\mathscr{U}$ be a subvariety of $\mathscr{V}_2$ and let $\mathscr{U}$ satisfy a standard identity. Then $\mathscr{U}$ is generated by a finite-dimensional algebra.*

Proof. Let $U=T(\mathscr{U})\subset F(\mathscr{V}_2)$ and $S_{2k}(x_1,\ldots,x_{2k})\in U$. First, assume $\Gamma_s(\mathscr{U})=0$ for s large enough, e. g. $s\geqq n$. Then, as in Proposition 1, $\mathscr{U}=\operatorname{var}R$ and $R=F_n(\mathscr{U})/F_n^n(\mathscr{U})$.

Now, let $\Gamma_n(\mathscr{U})\neq 0$ for any integer $n>1$. We shall establish $\mathscr{U}=\operatorname{var}(R_1\oplus R_2)$. The algebra R_1 takes into account the asymptotic behaviour of the identities for large degrees n and R_2 gives the details for the lower degrees. It is clear from Fig. 1 that there exists an integer n (e. g. $n=2k+1$), such that $B_m^{(s)}(\mathscr{U})=N_m(s-1,\ 1)$ for $s\geqq n$. Therefore, $N_m(s-1,\ 1)\subset B_m^{(s)}(\mathscr{U})$ for any $s>1$. Let R_1 be the algebra of 2×2 upper triangular matrices, $R_1=\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. Then R_1 satisfies the polynomial identities (2), (3) and (4), i. e. $R_1\in\mathscr{V}_2$ and $B_m^{(s)}(\operatorname{var}R_1)=N_m(s-1,1)$. Therefore $R_1\in\mathscr{U}$. Denote $R_2=F_m(\mathscr{U})/F_m^n(\mathscr{U})$ and $R=R_1\oplus R_2$. Obviously, the finite-dimensional algebra R belongs to $\mathscr{U}$. Moreover,

$$B_m^{(s)}(\operatorname{var}R)=B_m^{(s)}(\mathscr{U}) \tag{5}$$

for $s\geqq n$, since in this case $B_m^{(s)}(\operatorname{var}R_1)=B_m^{(s)}(\mathscr{U})$. The equality (5) holds for $s<n$ as well, because $B_m^{(s)}(\operatorname{var}R_2)=B_m^{(s)}(\mathscr{U})$ for $s<n$. Hence (5) is valid for any s, and $\mathscr{U}=\operatorname{var}R$.

Let the variety $\mathscr{V}_3$ be defined by the identity (3). The decomposition of $B_m^{(n)}(\mathscr{V}_3)$ and the consequences of higher degree are shown in Fig. 1. The generators of the modules $N_m(n,\ 1)$ and $N_m(1^{2n})$ are the same as for $\mathscr{V}_2$. The proof of the following assertion repeats verbatim that of Proposition 2.

Proposition 3. *Any subvariety of $\mathscr{V}_3$, satisfying a standard identity, is generated by a finite dimensional algebra.*

4. The standard identity of fourth degree. Let $\mathscr{V}_4$ be the variety defined by the identity (4). First, we shall consider the subvariety $\mathscr{M}=\operatorname{var}M_2(K)$ of $\mathscr{V}_4$, generated by the matrix algebra of second order. It is known [1] that

$$B_m(\mathscr{M})=K+\Sigma\, N_m(p+q+r,\ p+q,\ p),$$

where p, q, r are non-negative integers, $p+q>0$ and $(p+q+r,\ p+q,\ p)\neq(1^3)$. We shall establish that any subvariety of $\mathscr{M}$ is generated by a finite-dimensional algebra.

For a multihomogeneous polynomial $k(x_1,\ldots,x_m)$ in m variables, we use the notation $k(x_1;\ldots;x_i,u_1,\ldots,u_s;\ldots;x_m)$ for the partial linearization of k in x_i (i. e. the multilinear component in u_j of $k(x_1,\ldots,x_i+u_1+\cdots+u_s,\ldots,x_m)$).

Lemma 4. *Let $f(x_1,x_2,x_3)\in B_m(\mathscr{M})$ be a standard generator of the GL_m-module $N_m(p+q+r,\ p+q,\ p)$. Then the polynomial*

$$\begin{aligned} h(x_1,x_2,x_3)=(q-1)\{(q+1)f(x_1;x_2,[x_1,x_3];x_3)+f(x_1;x_2,x_3;x_3,[x_1,x_3])\}\\ -\{(q+1)f(x_1;x_2,x_3,[x_1,x_2];x_3)+f(x_1;x_2,x_3,x_3;x_3,[x_1,x_2])\}\end{aligned}$$

is a standard generator of $N_m(p+q+r+1,\ p+q-1,\ p+1)$.

Proof. Let S be the symmetric algebra of a linear space with a basis $S_3(x_i,x_j,x_k)$, $S_3(u,x_i,x_j)$, $[x_i,x_j]$, $[x_i,u]$, x_i, u, $1\leqq i<j<k\leqq m$, and let GL_m act on S in a natural way, fixing the variable u. Then the element

$$g(x_1,x_2,x_3)=S_3^p(x_1,x_2,x_3)\,[x_1,x_2]^q x_1^r$$

is a standard generator of $N_m(p+q+r,\ p+q,\ p)$. The polynomial from S:

$$g_1(x_1,x_2,x_3,u)=(q+1)g(x_1;x_2,u;x_3)+g(x_1;x_2,x_3;x_3,u)$$

$$=qS_3^{p-1}(x_1, x_2, x_3)\ [x_1, x_2]^{q-1}\ x_1^r\ \{(q+1)S_3(x_1, x_2, x_3)\ [x_1, u]$$

$$+p(S_3(u, x_1, x_2)\ [x_1, x_3]+S_3(u, x_2, x_3)\ [x_1, x_1]+S_3(u, x_3, x_1)\ [x_1, x_2])\}$$

generates $N_m(p+q+r, p+q-1, p)$. We have indicated by braces the positions of the skew symmetric triples and pairs in g_1. Analogously the polynomial

$$f_1(x_1, x_2, x_3, u)=(q+1)f(x_1;\ x_2, u;\ x_3)+f(x_1;\ x_2, x_3;\ x_3, u)$$

generates $N_m(p+q+r, p+q-1, p)$, when GL_m does not act on the variable u. Now, let $g_2(x_1, x_2, x_3)=S_3^p(x_1, x_2, x_3)\ [x_1, x_2]^{q-1}\ x_1^{r+1}\in S$ generate $N_m(p+q+r, p+q-1, p)$. Consider the tensor product $S\otimes_K N_m(1^2)$, where $N_m(1^2)\subset S$ is spanned by $[x_i, x_j]$. The element

$$g_3(x_1, x_2, x_3)=(q-1)\,g_2(x_1, x_2, x_3)\otimes[x_1, x_3]-g_2(x_1;\ x_2, x_3;\ x_3)\otimes[x_1, x_2]$$

$$=(q-1)S_3^p(x_1, x_2, x_3)\ [x_1, x_2]^{q-2}x_1^{r+1}\{[x_1, x_2]\otimes[x_1, x_3]$$

$$+[x_3, x_1]\otimes[x_1, x_2]+[x_2, x_3]\otimes[x_1, x_1]\}$$

generates $N_m(p+q+r+1, p+q-1, p+1)$. The same holds for

$$(q-1)g_1(x_1, x_2, x_3, [x_1, x_3])-g_1(x_1;\ x_2, x_3;\ x_3;\ [x_1, x_2])$$

and

$$h(x_1, x_2, x_3)=(q-1)f_1(x_1, x_2, x_3, [x_1, x_3])-f_1(x_1;\ x_2, x_3;\ x_3;\ [x_1, x_2])$$

$$=(q-1)\{(q+1)f(x_1;\ x_2, [x_1, x_3];\ x_3)+f(x_1;\ x_2, x_3;\ x_3, [x_1, x_3])\}$$

$$-\{(q+1)f(x_1;\ x_2, x_3, [x_1, x_2];\ x_3)+f(x_1;\ x_2, x_3, x_3;\ x_3\ [x_1, x_2])\}.$$

Later we shall verify that $h(x_1, x_2, x_3)\neq 0$ in $B_m(\mathcal{M})$ and this will complete the proof.

Lemma 5. *Let* $0\neq f(x_1, \ldots, x_m)\in N_m(\lambda_0)\subset B_m(\mathcal{M})$, *where* $\lambda_0=(p_0+q_0+r_0, p_0+q_0, p_0)$. *Then the polynomial identities from* $N_m(\mu)\subset B_m(\mathcal{M})$ *are consequences of* $f(x_1, \ldots, x_m)$ *for* $\mu=(p+q+r, p+q, p)$, *if* $2p+q\geq 2p_0+q_0$ *and* $n=3p+2q+r$ *is large enough.*

Proof. We use arguments from [8]. Without loss of generality we assume that $f=f(x_1, x_2, x_3)$ is a standard generator of $N_m(\lambda_0)\subset B_m(\mathcal{M})$. In order to verify that $f(x_1, x_2, x_3)\neq 0$ in $B_m(\mathcal{M})$, it sufficies to establish $f(a, b, c)\neq 0$, where a, b, c are elements of $M_2(K)$ such that [1]

$$\begin{aligned} &ab=-ba=c/2, \quad bc=-cb=a/2, \quad ca=-ac=b/2,\\ &[a, b]=c, \quad [b, c]=a, \quad [c, a]=b,\\ &a^2=b^2=c^2=-e/4, \quad [a, b, b]=[a, c, c]=-a,\\ &aS_3(\mathrm{ad}a, \mathrm{ad}b, \mathrm{ad}c)=-2a. \end{aligned} \tag{6}$$

The polynomial $f(x_1, x_2, [x_1, x_2])$ is a standard generator of $N_m(2p_0+q_0+r_0, 2p_0+q_0)$ and $f(a, b, [a, b])=f(a, b, c)\neq 0$. Therefore, $N_m(2p_0+q_0+r_0, 2p_0+q_0)\subset B_m(\mathcal{M})$ follows from $f(x_1, x_2, x_3)$. Assume that $2p+q=2p_0+q_0$. We shall find an integer r such that $N_m(\mu)\subset B_m(\mathcal{M})$ is a consequence of $f(x_1, x_2, x_3)$ for $\mu=(p+q+r, p+q, p)$. We use descending induction on q. If $q>0$ we fix a standard generator of $N_m(\mu)$

$$f_\mu(x_1, x_2, x_3)=[x_2, x_1]\ (\mathrm{ad}\,x_1)^r S_3^p(\mathrm{ad}\,x_1, \mathrm{ad}\,x_2, \mathrm{ad}\,x_3)\ [x_2, x_1]^{q-1}\in B_m(\mathcal{M}). \tag{7}$$

For $q=0$ the generator is

(8) $f_\mu(x_1, x_2, x_3)=\Sigma(\operatorname{sign}\sigma)x_{\sigma(1)}(\operatorname{ad}x_1)^r S_3^{p-1}(\operatorname{ad}x_1, \operatorname{ad}x_2, \operatorname{ad}x_3)[x_{\sigma(2)}, x_{\sigma(3)}]$, $\sigma\in\operatorname{Sym}(3)$.

Having in mind (6), we verify that $f_\mu(a, b, c)\neq 0$. Let $f_\mu(x_1, x_2, x_3)$ follow from $f(x_1, x_2, x_3)$. By Lemma 4, the polynomial

$$h(x_1, x_2, x_3)=(q-1)\{(q+1)f_\mu(x_1; x_2, [x_1, x_3]; x_3)+f_\mu(x_1; x_2, x_3; x_3, [x_1, x_3])\}$$
$$-\{(q+1)f_\mu(x_1; x_2, x_3, [x_1, x_2]; x_3)+f_\mu(x_1; x_2, x_3, x_3; x_3, [x_1, x_2])\}$$

belongs to $N_m(p+q+r+1, p+q-1, p+1)$ and is a consequence of $f(x_1, x_2, x_3)$ as well. Again $h(a, b, c)\neq 0$. Similar considerations carried out in [8], show that all modules $N_m(\mu)\subset B_m(\mathcal{M})$ belong to the T-ideal $T(f)$ of $F(\mathcal{M})$ for $\mu=(p+q+r, p+q, p)$ if $2p+q\geq 2p_0+q_0$ and n is large enough.

Lemma 6. *Let $E_s=K[t]/(t^{s+1})$ be the factor algebra of the polynomials in one variable modulo the principal ideal generated by t^{s+1}. Let us denote by R_s the subalgebra of $M_2(E_s)$ generated by $e_{11}, e_{22}, te_{12}, te_{21}$. Then $R_s\in\mathcal{M}$ and*

$$B_m(\operatorname{var}R_s)=\Sigma N_m(\mu)\subset B_m(\mathcal{M}),$$

where $\mu=(p+q+r, p+q, p)$ and $2p+q\leq s$.

Proof. The elements

$$a=-1/2(e_{11}-e_{22})\sqrt{-1},\quad b=1/2(e_{12}+e_{21})\sqrt{-1},\quad c=1/2(e_{12}-e_{21})$$

satisfy (6) and $f_\mu(a, b, c)\neq 0$ for any standard generator $f_\mu(x_1, x_2, x_3)$ of $N_m(\lambda)\subset B_m(\mathcal{M})$ [1]. It is clear that $R_s\in\mathcal{M}$. Moreover, $a, tb, tc\in M_2(E_s)$ and $f_\mu(a, tb, tc)=t^{2p+q}f(a, b, c)\neq 0$ if $2p+q\leq s$. Now, assume $2p+q>s$. It suffices to substitute for x_1, x_2, x_3 the elements $\bar{x}_i=\zeta_i(t)a+t\eta_i(t)b+t\theta_i(t)c$, $\zeta_i, \eta_i, \theta_i\in E_s$.
Then in (7) and (8)

$$[\bar{x}_2, \bar{x}_1]=t(t(\theta_1\eta_2-\theta_2\eta_1)a+(\zeta_1\theta_2-\zeta_2\theta_1)b+(\eta_1\zeta_2-\eta_2\zeta_1)c,$$

$$S_3(\operatorname{ad}\bar{x}_1, \operatorname{ad}\bar{x}_2, \operatorname{ad}\bar{x}_3)=t^2\begin{vmatrix}\zeta_1 & \eta_1 & \theta_1\\ \zeta_2 & \eta_2 & \theta_2\\ \zeta_3 & \eta_3 & \theta_3\end{vmatrix}S_3(\operatorname{ad}a, \operatorname{ad}b, \operatorname{ad}c),$$

$$f_\mu(\bar{x}_1, \bar{x}_2, \bar{x}_3)=t^{2p+q}k_\mu=0,$$

where $k_\mu\in M_2(E_s)$. Hence, the modules $N_m(\mu)\subset B_m(\mathcal{M})$ will vanish for R_s if $2q+q>s$.

Proposition 7. *Any subvariety of $\mathcal{M}=\operatorname{var}M_2(K)$ is generated by a finite-dimensional algebra.*

Proof. Let $\mathcal{U}\subset\mathcal{M}$ and $U=T(\mathcal{U})$ be the T-ideal of $\mathcal{U}$ in $F(\mathcal{M})$. If $\mathcal{U}=\mathcal{M}$, then $\mathcal{U}$ is generated by the four-dimensional algebra $M_2(K)$. Now, assume that $\mathcal{U}\neq\mathcal{M}$ and $B_m(\mathcal{M})\cap U=\Sigma N_m(\lambda)$, where $\lambda=(p+q+r, p+q, p)\in I$. Denote by $\lambda_0=(p_0+q_0+r_0, p_0+q_0, p_0)$ the partition from I with a minimal value of $2p+q$, and let $s+1=2p_0+q_0$. In virtue of Lemma 5, there exists an integer n_0, such that $N_m(\mu)\subset B_m(\mathcal{M})\cap U$ if $n=3p+2q+r\geq n_0$ and $2p+q\geq s+1$ (Fig. 2). Let $R=R_s\oplus F_3(\mathcal{U})/F_3^{n_0}(\mathcal{U})$. By Lemma 6, $R\in\mathcal{U}$. As in the preceding propositions we conclude that $\mathcal{U}=\operatorname{var}R$.

Now we shall consider the variety $\mathcal{V}_4$ defined by the standard identity (4). Kemer [4] proved, that $T(M_2(K))/T(S_4)$ is spanned by $x_{i_1}\ldots x_{i_n}g(x_{i_{n+1}}, \ldots, x_{i_{n+5}})$, $x_{i_1}\ldots x_{i_n}h(x_{i_{n+1}}, \ldots, x_{i_{n+6}})$, where $i_1\leq\cdots\leq i_n$, $n\geq 0$. Here g and h are the linearizations of the polynomials $[[x, y]^2, x]$ and $2[x, y]^3+[[x, y][x, y, y], x]$, respectively. Combining this description with the arguments from [6] we easily establish that

$$T(M_2(K))\cap B_m^{(n)})/(T(S_4)\cap B_m^{(n)})=0 \quad \text{for } n>6.$$

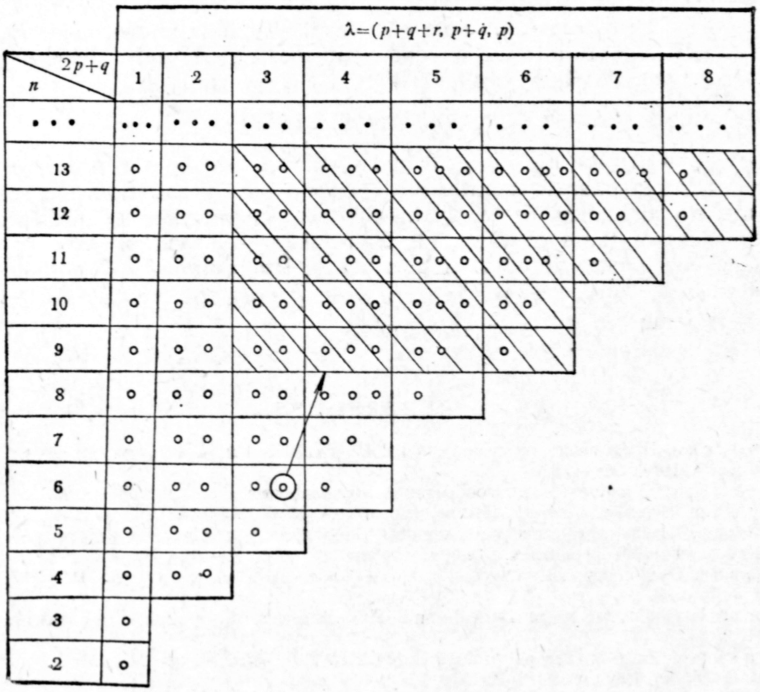

Fig. 2

The proof of the next lemma follows immediately.

Lemma 8. *Let* $f_1(x_1, \ldots, x_m)=f_2(x_1, \ldots, x_m)$ *be an identity for* $M_2(K)$. *Then* f_1 *and* f_2 *have the same consequences in* $B_m^{(n)}(\mathscr{V}_4)$ *modulo* $T(\mathscr{M})$ *for* $n\leq 6$ *and the same consequences in* $B_m^{(n)}(\mathscr{V}_4)$ *for* $n>6$.

Proposition 9. *Any variety satisfying* $S_4(x_1, x_2, x_3, x_4)=0$ *is generated by a finite dimensional algebra.*

Proof. Let $\mathscr{U}\subset\mathscr{V}_4$. If $\mathscr{M}\subset\mathscr{U}$, then the arguments from the proof of Proposition 2 show, that $\mathscr{U}=\operatorname{var}(M_2(K)\oplus F_m(\mathscr{U})/F_m^7(\mathscr{U}))$ for some integer m (e. g. $m>6$). In the other case, by Proposition 7, $\mathscr{U}\cap\mathscr{M}$ is generated by a finite dimensional algebra R. It follows from Lemma 8 that $\mathscr{U}=\operatorname{var}(R\oplus F_7(\mathscr{U})/F_7^7(\mathscr{U}))$.

5. Proofs of the main results. Theorem A is an immediate corollary of Propositions 1, 2, 3 and 9.

Proof of Theorem B. Let $\mathscr{U}\subset\mathscr{L}_d$ satisfy a standard identity $S_k(x_1, \ldots, x_k)=0$. By [3], the Capelli identity

$$d_r(x_1, \ldots, x_r; y_1, \ldots, y_{r-1})=\Sigma(\operatorname{sign}\sigma)x_{\sigma(1)}\, y_1 x_{\sigma(2)}, \ldots, y_{r-1}x_{\sigma(r)}=0$$

follows from $S_k(x_1, \dots, x_k)=0$ for some r. The Capelli identity shows that in the decomposition $F_m(\mathcal{U})=\Sigma N_m(\lambda)$ the diagrams $[\lambda]$ have less than r rows. Hence, $\mathcal{U}$ is generated by its relatively free algebra $F_{r-1}(\mathcal{U})$. It was proved in [6] that $B^{(n)}_{r-1}(\mathcal{L}_d)=0$, if n is large enough. Therefore, there exists an integer n_0 such that $B^{(n)}_{r-1}(\mathcal{U})=0$ for $n\geqq n_0$. Consequently, as in Proposition 1 we obtain that $\mathcal{U}=\operatorname{var}F_{r-1}(\mathcal{U})/F^{n_0}_{r-1}(\mathcal{U})$.

P r o o f o f T h e o r e m C. Denote $\mathcal{V}=\operatorname{var}sl_2(K)$. It is known that $F_m(\mathcal{V})$ can be embedded as a Lie algebra into the adjoint algebra of $F_m(\mathcal{M})$ and $F_m(\mathcal{V})=N_m(1)+F_m(\mathcal{V})\cap B_m(\mathcal{M})$. Moreover, it follows from [1] that

$$F_m(\mathcal{V})\cap B_m(\mathcal{M})=\Sigma N_m(\mu),$$

where $\mu=(p+q+r,\ p+q,\ p)$, $p+q>0$ and q or r is odd. In the proof of Lemma 4 we found out the standard generator $h(x_1, x_2, x_3)$ by linearization and substitution of Lie elements. As in Lemma 5 we obtain that the polynomials from $N_m(\mu)\subset F_m(\mathcal{V})$ are Lie consequences of those of $N_m(\lambda_0)$ if $2p+q\geqq 2p_0+q_0$ and $n=3p+2q+r$ is large enough. Then we repeat verbatim the proof of Proposition 7. The finite-dimensional algebra R with $\mathcal{U}=\operatorname{var}R\subset\mathcal{V}$ is $R=L_s+F_m(\mathcal{U})/F^{n_0}_m(\mathcal{U})$ and L_s is the subalgebra of $sl_2(K[t]/(t^{s+1}))$ generated by $e_{11}-e_{22}$, te_{12}, te_{21}.

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