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# Сердика

## Българско математическо списание

# B-COMBINATORY ALGEBRAS

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Particular algebraic systems are introduced with a view to generalize the recursion theory in both iterative operatory spaces (I v a n o v [3]) and $AC$-applicative systems from [10]. Some general conditions for such algebras called '$B$-combinatory algebras' are studied under which that generalization is possible. The main result gives an expression of the least solution of an arbitrary inequality in a $B$-combinatory algebra in the terms of that for a single inequality of special simple kind.

There is an elegant and natural abstract algebraic generalization of the usual theory of recursive functions based on the concept of operatory space, which was introduced and studied by I v a n o v in [1] – [4]. As sources of that generalization we should mention also the works [5] and [7] and especially the fundamental works of D. S k o r d e v ([6] and others) on the concept of combinatory space. On the other hand, there is an elegant algebraical treatment of the elementary theory of recursively enumerable sets based on the Plotkin's model $P\omega$ of the $\lambda$-calculus and considered by D. S c o t t in [8]. An abstract treatment of that kind in the spirit of Skordev's combinatory spaces was given in the thesis [9] (the main results concerned are shortly presented in [10] and [11] and another exposition is given in [12], which is more closely connected with the model of Plotkin). In that treatment, partially ordered algebras are used, called in [10] '$AC$-applicative systems'. In the present work we shall consider a possibility of uniform exposition of both above mentioned generalizations of the usual recursion theory, Ivanov's recursion theory in operatory spaces and recursion theory in $AC$-applicative systems. For that purpose we shall introduce partially ordered algebras, which are essentially a generalization of both operatory spaces and $AC$-applicative systems. Those algebras will be called '$B$-combinatory algebras'.

1. Let $\mathcal{F}$ be a set in which a binary operation $\varphi, \psi \to (\varphi\psi)$ (non-associative in general) is given, which we shall call multiplication. By letters like $\varphi, \psi, \chi, \varphi', \varphi_0$ etc. we shall denote arbitrary elements of $\mathcal{F}$ and we shall use the left grouping of brackets: $\varphi_0\varphi_1\varphi_2 \ldots \varphi_n$ means $(\ldots((\varphi_0\varphi_1)\varphi_2)\ldots\varphi_n)$. When in $\mathcal{F}$ elements $A, C$ and $D$ are given, such that the following identities are fulfilled in $\mathcal{F}$:

$$A\varphi\psi\chi = \varphi(\psi\chi) \tag{1}$$

$$C\varphi(D\psi\chi) = \varphi\psi\varkappa, \tag{2}$$

then we shall call $\mathcal{F}$ a 'preassociative combinatory algebra'. Special cases of preassociative combinatory algebras are semigroups with a left unit and $AC$-applicative systems from [10]. A preassociative combinatory algebra $\mathcal{F}$ will be called partially ordered if there are given a partial order $\leq$ in $\mathcal{F}$ and an element $O \in \mathcal{F}$ such that for all $\varphi, \psi, \varphi'$ and $\psi'$ in $\mathcal{F}$ $\varphi \leq \varphi' \& \psi \leq \psi' \Rightarrow \varphi\psi \leq \varphi'\psi'$ and $O \leq \varphi$.

The main definition in the present work is the following one: a $B$-c o m b i n a t o r y a l g e b r a is a partially ordered preassociative combinatory algebra $\mathcal{F}$ in which a binary operation $\varphi, \psi \to (\varphi, \psi)$, called 'branching operation', is given as well as elements $I, \check{D}, J, E, E_0, E_1, T, F$, such that the following conditions are fulfilled for all $\varphi, \psi, \chi, \varphi', \psi'$ in $\mathcal{F}$:

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with respect to that order for each $\varkappa$-sequence $\mathcal{J}_\iota \in \mathcal{F}^{(\varkappa)}$, and that is $\bigcap\{\mathcal{J} \in \mathcal{F}^{(\varkappa)}: (\forall \iota<\varkappa)(\mathcal{J}_\iota \subseteq \mathcal{J})\}$. Define in $\mathcal{F}^{(\varkappa)}$ a multiplication and a branching operation thus:

$$\mathcal{J}_0\mathcal{J}_1 = \{\vartheta \in \mathcal{F}: \exists\, \varphi_0 \exists\, \varphi_1(\varphi_0 \in \mathcal{J}_0 \;\&\; \varphi_1 \in \mathcal{J}_1 \;\&\; \vartheta \leqq \varphi_0\varphi_1)\};$$

$$(\mathcal{J}_0, \mathcal{J}_1) = \{\vartheta \in \mathcal{F}: \exists\, \varphi_0 \exists\, \varphi_1(\varphi_0 \in \mathcal{J}_0 \;\&\; \varphi_1 \in \mathcal{J}_1 \;\&\; \vartheta \leqq (\varphi_0, \varphi_1))\};$$

where $\mathcal{J}_0, \mathcal{J}_1 \in \mathcal{F}^{(\varkappa)}$. To see that $\mathcal{J}_0\mathcal{J}_1$ is closed under $\sup_{\iota<\varkappa}$ suppose $\vartheta_\iota \in \mathcal{J}_0\mathcal{J}_1$ for all $\iota<\varkappa$. Then $\vartheta_\iota \leqq \varphi^0_\iota\varphi^1_\iota$ for suitable $\varkappa$-sequences $\varphi^i_\iota \in \mathcal{J}_i\,(\iota<\varkappa, i<2)$, and therefore $\sup_{\iota<\varkappa}\vartheta_\iota \leqq (\sup_{\iota<\varkappa}\varphi^0_\iota)(\sup_{\iota<\varkappa}\varphi^1_\iota)$ and $\sup_{\iota<\varkappa}\vartheta_\iota \in \mathcal{J}_0\mathcal{J}_1$ since $\sup_{\iota<\varkappa}\varphi^i_\iota \in \mathcal{J}_i$ $(i<2)$. So $\mathcal{J}_0\mathcal{J}_1 \in \mathcal{F}^{(\varkappa)}$ and similarly $(\mathcal{J}_0, \mathcal{J}_1) \in \mathcal{F}^{(\varkappa)}$. Define for every $\varphi \in \mathcal{F}$ the element $\varphi^* \in \mathcal{F}^{(\varkappa)}$ as $\{\vartheta \in \mathcal{F}: \vartheta \leqq \varphi\}$.

Proposition 2. *Under the presuppositions above $\mathcal{F}^{(\varkappa)}$ is a B-combinatory algebra with respect to $O^*, A^*, C^*, D^*, I^*, \bar{D}^*, \jmath^*, E^*, E^*_0, E^*_1, T^*, F^*$ as basic constants. The mapping $\varphi \to \varphi^*$ is a monomorphism of B-combinatory algebras $\mathcal{F} \to \mathcal{F}^{(\varkappa)}$ and $(\sup_{\iota<\varkappa}\varphi_\iota)^* = \sup_{\iota<\varkappa}\varphi^*_\iota$ for each $\varkappa$-sequence $\varphi_\iota \in \mathcal{F}$.*

The proof of the Proposition 2 is routine by the definitions. As an illustration we shall consider the checking of the condition B6 in $\mathcal{F}^{(\varkappa)}$: Let $\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2 \in \mathcal{F}^{(\varkappa)}$. We have to prove that

$$(E^*_0\mathcal{J}_2\mathcal{J}_0,\ E^*_1\mathcal{J}_2\mathcal{J}_1)E^* = \mathcal{J}_2(\mathcal{J}_0, \mathcal{J}_1).$$

Let $\mathcal{J}' = (E^*_0\mathcal{J}_2\mathcal{J}_0, E^*_1\mathcal{J}_2\mathcal{J}_1)$ and $\vartheta \in \mathcal{J}'E^*$. Then $\vartheta \leqq \vartheta'\varepsilon$ for suitable $\vartheta' \in \mathcal{J}'$ and $\varepsilon \leqq E$. Therefore $\vartheta' \leqq (\vartheta_0, \vartheta_1)$, where $\vartheta_i \in E^*_i\mathcal{J}_2\mathcal{J}_i\,(i<2)$, and similarly $\vartheta_i \leqq \psi_i\varphi_i$ for suitable $\psi_i \in E^*_i\mathcal{J}_2$ and $\varphi_i \in \mathcal{J}_i$. Again $\psi_i \leqq \varepsilon_i\chi_i$, where $\varepsilon_i \leqq E_i$ and $\chi_i \in \mathcal{J}_2$. Let $\chi = \sup\{\chi_0, \chi_1\}$. Since $\varkappa \geqq 2$ the element $\chi$ is correctly defined. Then

$$\begin{aligned}\vartheta \leqq \vartheta'\varepsilon \leqq (\vartheta_0, \vartheta_1)\varepsilon \leqq (\psi_0\varphi_0, \psi_1\varphi_1)\varepsilon \leqq (\varepsilon_0\chi_0\varphi_0, \varepsilon_1\chi_1\varphi_1)\varepsilon \\ \leqq (E_0\chi\varphi_0, E_1\chi\varphi_1)E = \chi(\varphi_0, \varphi_1) \in \mathcal{J}_2(\mathcal{J}_0, \mathcal{J}_1),\end{aligned}$$

and therefore $\vartheta \in \mathcal{J}_2(\mathcal{J}_0, \mathcal{J}_1)$. To prove the reverse inclusion let $\vartheta \in \mathcal{J}_2(\mathcal{J}_0, \mathcal{J}_1)$. Then $\vartheta \leqq \varphi_2(\varphi_0, \varphi_1)$, where $\varphi_i \in \mathcal{J}_i\,(i<3)$, and

$$\vartheta \leqq \varphi_2(\varphi_0, \varphi_1) = (E_0\varphi_2\varphi_0, E_1\varphi_2\varphi_1)E \in (E^*_0\mathcal{J}_2\mathcal{J}_0, E^*_1\mathcal{J}_2\mathcal{J}_1)E^*.$$

By the Proposition 2 we may consider $\mathcal{F}^{(\varkappa)}$ as an extension of $\mathcal{F}$, identifying $\varphi$ with $\varphi^*$. We shall adopt here this convention till the end of **2**.

Proposition 3. *In the above presuppositions let*

$$\alpha \sup_{\iota<\varkappa}\varphi_\iota = \sup_{\iota<\varkappa}(\alpha\varphi_\iota) \quad \text{and} \quad (\sup_{\iota<\varkappa}\varphi_\iota)\nu = \sup_{\iota<\varkappa}(\varphi_\iota\nu) \tag{8}$$

*for all $\alpha \in \mathcal{F}$, $\nu \in \mathcal{N}$ and each $\varkappa$-sequence $\varphi_\iota \in \mathcal{F}$. Then the conditions* I1, I2, *and* I3 *are fulfilled for the extension $\mathcal{F}^{(\varkappa)}$ of $\mathcal{F}$.*

Proof. For every $\mathcal{J} \in \mathcal{F}^{(\varkappa)}$ let $(\alpha \diagdown \mathcal{J}) = \{\vartheta \in \mathcal{F}: \alpha\vartheta \in \mathcal{J}\}$. The set $(\alpha \diagdown \mathcal{J})$ is non-empty since $\alpha O = O \in \mathcal{J}$. It follows easily by (8) that $(\alpha \diagdown \mathcal{J}) \in \mathcal{F}^{(\varkappa)}$. Obviously $\alpha(\alpha \diagdown \mathcal{J}) = \alpha^*(\alpha \diagdown \mathcal{J}) \subseteq \mathcal{J}$ and if $\mathcal{J}' \in \mathcal{F}^{(\varkappa)}$ is such that $\alpha\mathcal{J}' \subseteq \mathcal{J}$, then for each $\vartheta \in \mathcal{J}'$ $\alpha\vartheta \in \mathcal{J}$ and therefore $\mathcal{J}' \subseteq (\alpha \diagdown \mathcal{J})$. That proves I1. The proof of I2 is similar and I3 is obvious.

Corollary 1. *Let $\mathcal{F}_0$ be a B-combinatory algebra, let $\mathcal{F}$ be an extension of $\mathcal{F}_0$, and let $\varkappa$ be the first infinite ordinal. Suppose that the presuppositions of Proposition* 3 *are fulfilled,* (8) *holds for each $\nu \in \mathcal{F}$, and for every sequence $\varphi_i \in \mathcal{F}$ $(i<\varkappa)$*

$$\sup_{i<\varkappa}(I, \varphi_i) = (I, \sup_{i<\varkappa}\varphi_i). \tag{9}$$

*Then if for every monotonically increasing sequence $\varphi_i \in \mathcal{F}_0$ the supremum $\sup_{i<\varkappa} \varphi_i$ (in $\mathcal{F}$) again belongs to $\mathcal{F}_0$, then $\mathcal{F}^{(\varkappa)}$ is an iterative extension of $\mathcal{F}_0$.*

Proof. By the Propositions 2 and 3 $\mathcal{F}^{(\varkappa)}$ is an extension of $\mathcal{F}_0$ satisfying I1, I2, and I3. To check I4 define a mapping $\Gamma: \mathcal{F}^{(\varkappa)} \to \mathcal{F}^{(\varkappa)}$ by $\Gamma(\vartheta)=\varphi(I, \vartheta)\psi$. It follows from (9) and (8) that for each sequence $\vartheta_i \in \mathcal{F}\,(i<\varkappa)$

$$\sup_{i<\varkappa} \Gamma(\vartheta_i) = \Gamma(\sup_{i<\varkappa} \vartheta_i).$$

Then for every sequence $\mathcal{J}_i \in \mathcal{F}^{(\varkappa)}$ the set

$$\mathcal{J}' = \{\vartheta \in \mathcal{F}:\ \Gamma(\vartheta) \in \sup_{i<\varkappa} \Gamma(\mathcal{J}_i)\}$$

will be a $\varkappa$-ideal. But $\mathcal{J}_i \subseteq \mathcal{J}'$ for each $i<\varkappa$ and therefore

$$\sup_{i<\varkappa} \mathcal{J}_i \subseteq \mathcal{J}', \quad \text{whence} \quad \{\Gamma(\vartheta):\ \vartheta \in \sup_{i<\varkappa} \mathcal{J}_i\} \subseteq \sup_{i<\varkappa} \Gamma(\mathcal{J}_i).$$

Using here the definition of the multiplication and the branching operation in $\mathcal{F}^{(\varkappa)}$ we can see that $\Gamma(\sup_{i<\varkappa} \mathcal{J}_i) \subseteq \sup_{i<\varkappa} \mathcal{J}_i$ and the reverse inclusion is obvious, since the mapping $\Gamma$ is monotonically increasing. Then a usual argument (see, for instance, Proposition 1. 1. 3, p. 224 in [6]) shows that $\vartheta_0 = \sup_{i<\varkappa} \Gamma^i(O)$ is the least solution of $\Gamma(\vartheta) \leqq \vartheta$ with respect to $\vartheta$ in $\mathcal{F}^{(\varkappa)}$. By the Proposition 2 and the supposition that $\mathcal{F}_0$ is closed under $\sup_{i<\varkappa}$ of monotonically increasing sequences it follows that $\vartheta_0 \in \mathcal{F}_0$.

Corollary 2. *Let $\mathcal{F}_0$ be a B-combinatory algebra with an extension $\mathcal{F}$ and let $\varkappa$ be a cardinal number, greater than the power of $\mathcal{F}_0$. Suppose that $\mathcal{F}$ and $\varkappa$ satisfy the conditions of Proposition 3 and $\mathcal{F}_0$ is closed under supremum $\sup_{\iota<\varkappa} \varphi_i$ of monotonically increasing $\varkappa$-sequences $\varphi_\iota \in \mathcal{F}$. Then $\mathcal{F}^{(\varkappa)}$ is an iterative extension of $\mathcal{F}_0$.*

Proof. As in the previous proof, $\mathcal{F}^{(\varkappa)}$ is an extension of $\mathcal{F}_0$ satisfying I1, I2, and I3, and define $\Gamma$ in the same way. Then by induction on $\iota$ define for everv ordinal $\iota$ an element $\varphi_i \in \mathcal{F}^{(\varkappa)}$ thus:

$$\varphi_i = \sup_{\mu<\iota} \Gamma(\varphi_\mu). \tag{10}$$

A usual argument (see, for instance, the proof of Proposition 1. 1. 5, p. 226 in [6]) shows that the sequence $\varphi_i$ monotonically increases and if $\lambda$ is the least ordinal for which $\varphi_\lambda = \varphi_{\lambda+1}$, then $\varphi_\lambda$ is the least solution of $\Gamma(\vartheta) \leqq \vartheta$ in $\mathcal{F}^{(\varkappa)}$. By (10), using Proposition 2 and an induction on $\iota$ we may see that $\varphi_i \in \mathcal{F}$ for each $\iota<\varkappa$. Since the sequence $\varphi_\iota\,(\iota \leqq \varkappa)$ monotonically increases, the same induction shows that $\varphi_\iota \in \mathcal{F}_0$ for each $\iota \leqq \varkappa$. But $\lambda<\varkappa$ because $\varkappa$ is greater than the power of $\mathcal{F}_0$ and therefore $\varphi_\lambda \in \mathcal{F}_0$.

The corollaries 1 and 2 are analogous to the properties (**) and (*) from [1] and [3]. We can easily extend them so that they may comprise the case with a non-trivial rank function in the sense of [3]. The conditions of the corollaries 1 and 2 are usually satisfed with $\mathcal{F}_0$ as $\mathcal{F}$ when $\sup_{\iota<\varkappa} \varphi_\iota$ exists in $\mathcal{F}_0$ for every $\varkappa$-sequence $\varphi_\iota \in \mathcal{F}_0$. Such is the case with all special instances of the example 1 from [10] and [12]. Otherwise, to apply those corollaries we should firstly find an extension $\mathcal{F}$ of $\mathcal{F}_0$ in which that supremum always exists. As an illustration we shall consider the following example:

Example 3. Let $\mathcal{F}_0$ be the operatory space in [2], example 1. Then $\mathcal{F}_0$ is the set of all partial functions $\varphi: M \to M$, where $M$ is a set in which two disjoint subsets $M_0, M_1$, two mappings $f_i: M \to M_i\,(i<2)$, and a partial mapping $f: M \to M$ are given such that the domain of $f$ contains $M_0 \cup M_1$ and $f(f_i(x))=x$ for all $x \in M_i$ and $i<2$. According to example 2, $\mathcal{F}_0$ is a $B$-combinatory algebra in which $\leqq$ is the relation of the inclusion of graphs, the multiplication is the composition of partial functions, and the branching operation is defined by

$$(\varphi, \psi)(x) = \begin{cases} \varphi(f(x)), & \text{if } x \in M_0 \\ \psi(f(x)), & \text{if } x \in M_1 \\ \text{undefined otherwise.} \end{cases}$$

The mapping $x \longmapsto x$ is the value of all basic constants of $\mathscr{F}_0$ except $T, F$ and $O$ whose values are $f_0, f_1$ and the least element of $\mathscr{F}_0$ respectively. Let $\mathscr{F}$ be the set of all many-valued partial mappings $\varphi: M \to M$. Define in $\mathscr{F} \leq$ as the inclusion of graphs, and the multiplication as the composition of many-valued mappings: $y \in (\varphi, \psi)(x) \Leftrightarrow \exists z(z \in \psi(x) \,\&\, y \in \varphi(z))$.

Define branching operation by

$$y \in (\varphi, \psi)(x) \Leftrightarrow (x \in M_0 \,\&\, y \in \varphi(f(x))) \vee (x \in M_1 \,\&\, y \in \psi(f(x))).$$

Then (as is shown in [3]) $\mathscr{F}$ is a $B$-combinatory algebra, which is an extension of $\mathscr{F}_0$ Obviously $\sup_{\iota<\varkappa} \varphi_\iota$ exists in $\mathscr{F}$ for every $\varkappa$-sequence $\varphi_\iota \in \mathscr{F}$ and every ordinal $\varkappa$, and $\mathscr{F}_0$ is closed under $\sup_{\iota<\varkappa}$ of monotonically increasing $\varkappa$-sequences. It is easy to see that the conditions of Proposition 3 are fulfilled and we may apply corollary 1, according to which $\mathscr{F}^{(\omega)}$ is an iterative extension of $\mathscr{F}_0$.

**3.** From now on, the letters like $t, s, r, t_0, t'$ etc. will denote **terms**, defined inductively as follows: the variables $(x, x_0, x_1, \ldots)$ are terms; the symbols for the basic constants $O, A, \ldots$ (denoted as the constants themselves) are terms; if $t$ and $s$ are terms, then $(ts)$ and $\langle t, s\rangle$ are terms. Let $\mathscr{F}$ be a $B$-combinatory algebra and let $\Theta$ be an evaluation in $\mathscr{F}$, i. e. $\Theta$ is a function $\Theta: X \to \mathscr{F}$ where $X$ is a finite set of variables. Suppose that the term $t$ does not contain variables out of $X$. Then the value $\widetilde{\Theta}(t)$ of $t$ by $\Theta$ is defined inductively thus: $\widetilde{\Theta}(t) = \Theta(t)$, if $t$ is a variable; $\widetilde{\Theta}(t) = \gamma$, if $t$ is a symbol for some basic constant $\gamma \in \mathscr{F}$; $\widetilde{\Theta}((ts)) = \widetilde{\Theta}(t)\widetilde{\Theta}(s)$ and $\widetilde{\Theta}(\langle t, s\rangle) = (\widetilde{\Theta}(t), \widetilde{\Theta}(s))$. The left grouping of brackets will be adopted for terms $t_0t_1t_2 \ldots t_n = (\ldots((t_0t_1)t_2)\ldots t_n)$, where $=$ will denote the identity of terms.

The expressions having one of the forms

C1. $It \longmapsto t$;

C2. $At_0t_1t_2 \longmapsto t_0(t_1t_2)$;

C3. $\langle E_0tt_0, E_1tt_1\rangle E \longmapsto t\langle t_0, t_1\rangle$;

will be called contractions. Till the end of the paper let $Y(x)$, $Y_0(x)$, $Y'(x)$ etc. denote terms having at most one occurrence of the variable $x$, and let $Y(t)$ be the result of substituting $t$ for $x$ in $Y(x)$. Define some notations by the equivalences: $t \underset{1}{\Vdash\!\!\to} s \Leftrightarrow$ "there is a contraction $p \longmapsto q$, such that $t = Y(p)$ and $s = Y(q)$ for suitable $Y(x)$, in which $x$ occurs"; $t \underset{n}{\Vdash\!\!\to} s \Leftrightarrow$ "there are $n+1$ terms $s_0, s_1, \ldots, s_n$, such that $s_0 \underset{1}{\Vdash\!\!\to} s_1, \ldots, s_{n-1} \underset{1}{\Vdash\!\!\to} s_n$ and $s_0 = t$ and $s_n = s$"; $t \Vdash\!\!\to s \Leftrightarrow \exists n(t \underset{n}{\Vdash\!\!\to} s)$; $t$ is normal' $\Leftrightarrow \forall s(t \Vdash\!\!\to s \Rightarrow t = s)$; and $t \cong s \Leftrightarrow$ "for every $B$-combinatory algebra $\mathscr{F}$ and every evaluation $\Theta: X \to \mathscr{F}$ such that $X$ contains all variables in $t, s$, $\widetilde{\Theta}(t) = \widetilde{\Theta}(s)$". If $p \longmapsto q$ is a contraction, then obviously $p \cong q$. Therefore

(11) $$t \Vdash\!\!\to s \Rightarrow t \cong s.$$

Denoting by $\underline{m}(t)$ the number of brackets (in $t$ we see from C1—C3 that $\underline{m}(q) < \underline{m}(p)$) for every contraction $p \longmapsto q$. Then for each $n$

(12) $$t \underset{n+1}{\Vdash\!\!\to} s \Rightarrow \underline{m}(s) < \underline{m}(t).$$

Therefore for every term $t$ there is a natural number $n \leqq \underline{m}(t)$ and a normal $s$ such that $t \mathrel{\|\!\!\underset{n}{\to}} s$. That term $s$ will be denoted by $t^N$ and we shall call $t^N$ 'the normal form of $t$'. The last definition is correct by the Church--Rosser property for $\longrightarrow$:

Proposition 4. *If* $t \mathrel{\|\!\!\to} t_0$ *and* $t \mathrel{\|\!\!\to} t_1$, *then there is a term* $s$ *such that* $t_0 \mathrel{\|\!\!\to} s$ *and* $t_1 \mathrel{\|\!\!\to} s$.

Proof. At first we shall show that

$$(13) \qquad t \mathrel{\|\!\!\underset{1}{\to}} t_0 \;\&\; t \mathrel{\|\!\!\underset{1}{\to}} t_1 \Rightarrow \exists\, s(t_0 \mathrel{\|\!\!\to} s \;\&\; t_1 \mathrel{\|\!\!\to} s).$$

It is sufficient to consider the case when $t \longrightarrow t_0$ is a contraction. The general case is obtained then easily by an induction on the length of the term $t$. Let $p \longrightarrow q$ be a contraction such that $t = Y(p)$ and $t_1 = Y(q)$ for suitable $Y(x)$, in which x occurs. If $Y(x) = x$ then $t \longrightarrow t_1$ is a contraction, whence by C1—C3 $t_0 = t_1$ and (13) is trivial. Let $Y(x) \neq x$ and consider the possible cases for the contraction $t \longrightarrow t_0$:

1) $t \longrightarrow t_0$ has the form of C1. Then $Y(x) = IY_0(x)$ for suitable $Y_0(x)$ such that $t_0 = Y_0(p)$. If $s = Y_0(q)$ then $t_0 \mathrel{\|\!\!\underset{1}{\to}} s$ and $t_1 = IY_0(q) \mathrel{\|\!\!\underset{1}{\to}} s$.

2) $t \longrightarrow t_0$ has the form of C2. Then $Y(x)$ should be of the form $AY_0(x)Y_1(x)Y_2(x)$ and $t_0 = Y_0(p)\,(Y_1(p)Y_2(p))$. If $s$ is the term $Y_0(q)\,(Y_1(q)Y_2(q))$ then $t_0 \mathrel{\|\!\!\underset{1}{\to}} s$ and $t_1 = Y(q) \mathrel{\|\!\!\underset{1}{\to}} s$.

3) $t \longrightarrow t_0$ has the form of C3. Then there are two possible cases for the term $Y(x)$:

3.1) $Y(x)$ has the form $\langle E_0 r Y_0(x), E_1 r Y_1(x)\rangle E$, where $r$ does not contain $x$. Then defining $s = r\,\langle Y_0(q), Y_1(q)\rangle$ we have $t_1 = Y(q) \mathrel{\|\!\!\underset{1}{\to}} s$ and $t_0 = r\langle Y_0(p), Y_1(p)\rangle \mathrel{\|\!\!\underset{1}{\to}} s$.

3.2) $Y(x)$ has the form $\langle E_0 Y'(x) r_0, E_1 Y'(p) r_1\rangle E$ or $\langle E_0 Y'(p) r_0, E_1 Y'(x) r_1\rangle E$, where $r_0 r_1$ does not contain $x$. Then $t_0 = Y'(p)\,\langle r_0, r_1\rangle \mathrel{\|\!\!\underset{1}{\to}} s$ and $t_1 = Y(q) \mathrel{\|\!\!\underset{1}{\to}} \langle E_0 Y'(q) r_0, E_1 Y'(q)\, r_1\rangle E \mathrel{\|\!\!\underset{1}{\to}} s$, where $s = Y'(q)\langle r_0, r\rangle$.

The Church—Rosser property for $\|\!\!\to$ can be obtained easily from (13) since by (12) we may use induction on $\underline{m}(t)$ to prove

$$(14) \qquad t \mathrel{\|\!\!\to} t_0 \;\&\; t \mathrel{\|\!\!\underset{1}{\to}} t_1 \Rightarrow \exists\, s(t_0 \mathrel{\|\!\!\to} s \;\&\; t_1 \mathrel{\|\!\!\to} s).$$

The basis of the induction is trivial since if $\underline{m}(t) = 0$ then $t \mathrel{\|\!\!\underset{1}{\to}} t_1$ is impossible by (12). In the induction step we shall use a second induction on $n$ to prove that for all $t_0, t_1$ such that if $t \mathrel{\|\!\!\underset{n}{\to}} t_0$ and $t \mathrel{\|\!\!\underset{1}{\to}} t_1$ then there is a term $s$ such that $t_0 \mathrel{\|\!\!\to} s$ and $t_1 \mathrel{\|\!\!\to} s$. If $n = 0$ then $t = t_0$ and we may take $t_1$ for $s$. The case $n = 1$ is immediate by (13). Let $n > 1$ and assume the hypothesis of the induction for $n-1$. Then $t \mathrel{\|\!\!\underset{n-1}{\to}} t'$ and $t' \mathrel{\|\!\!\underset{1}{\to}} t_0$ for suitable $t'$ and by the hypothesis of the induction there is a term $s'$ such that $t' \mathrel{\|\!\!\to} s'$ and $t_1 \mathrel{\|\!\!\to} s'$. By (12) $m(t') < m(t)$ and from $t' \mathrel{\|\!\!\to} s'$ and $t' \mathrel{\|\!\!\underset{1}{\to}} t_0$ by the hypothesis of the former induction there is a term $s$ such that $s' \mathrel{\|\!\!\to} s$ and $t_0 \mathrel{\|\!\!\to} s$. Then from $t_1 \mathrel{\|\!\!\to} s$ it follows also $t_1 \mathrel{\|\!\!\to} s$. That finishes both induction steps and proves (14). Finally an easy induction on $n$ shows that

$$t \mathrel{\|\!\!\to} t_0 \;\&\; t \mathrel{\|\!\!\underset{n}{\to}} t_1 \Rightarrow \exists\, s(t_0 \mathrel{\|\!\!\to} s \;\&\; t_1 \mathrel{\|\!\!\to} s)$$

and proves the proposition. We shall omit the details.

**4.** From now on, a term $t$ will be called an $A$-term if only the (symbol of the) constant $A$ and the brackets ( , ) occur in $t$. Then for each $t$ we shall define two terms $t^*$ and $t^0$ inductively thus:

1*) if $t$ is a (symbol of a) constant or a variable or an $A$-term or there are $t_0$ and $t_1$ such that $t=\langle t_0, t_1\rangle$, then $t^*=I$ and $t^0=t$;

2*) if $t=t's$ where $s$ is a constant or a variable or an $A$-term, but $t$ is not an $A$-term then $t^*=t'$ and $t^0=s$;

3*) if $t=t's$ where $s=\langle s_0, s_1\rangle$ for suitable $s_0$ and $s_1$ then $t^*=\langle E_0 t's_0, E_1 t's_1\rangle$ and $t^0=E$;

4*) if $t=t's$ and no one of the previous three cases holds then $t^*=At's^*$ and $t^0=s^0$.

Using an induction corresponding to the definition above we easily see that

$$t^*t^0 \,||\!\rightarrow t \quad \text{and} \tag{15}$$

$$\text{if } t \text{ is normal then such are also } t^* \text{ and } t^0. \tag{16}$$

Till the end of the paper we shall fix a $B$-combinatory algebra $\mathcal{F}$, an evaluation $\Theta_0\colon \mathrm{X}\rightarrow\mathcal{F}$, and a variable $x$ such that $x\notin \mathrm{X}$. All terms below will be supposed not containing variables out of $\mathrm{X}\cup\{x\}$. The set of all such terms will be denoted by $\mathcal{T}$ and let $\mathcal{T}_0=\{t\in\mathcal{T}: x \text{ does not occur in } t\}$; $\mathcal{C}=\{t\colon t\in \mathrm{X} \text{ or } t \text{ is a basic constant symbol}\}$; $\mathcal{A}=\{t\colon t\in\mathcal{C} \text{ or } t \text{ is an } A\text{-term}\}$. We shall write $\tilde{t}$ for $\tilde{\Theta}_0(t)$ if $t\in\mathcal{T}$, and if $t\in\mathcal{T}$ we shall write $\tilde{t}(\varphi)$ for $\tilde{\Theta}(t)$ where $\Theta\colon \mathrm{X}\cup\{x\}\rightarrow\mathcal{F}$ is the evaluation $\Theta$ such that $\Theta(y)=\Theta_0(y)$ for $y\in \mathrm{X}$ and $\Theta(x)=\varphi$.

Definition. *A function $k\colon\mathcal{T}\rightarrow\mathcal{N}$ is a code function for a term $r$, and an element $\sigma\in\mathcal{F}$ is a reading element of $k$ iff the following five conditions are satisfied for every $t\in\mathcal{T}$:*

(K0) $k(t)=k(t^N)$;

(K1) *if* $t=I$ *then* $\sigma k(t)=DTI$;

(K2) *if* $t\neq I$, *t is normal and* $t^0\in\mathcal{A}$ *then* $\sigma k(t)=DF(Dk(t^*)\widetilde{t^0})$;

(K3) *if t is normal and* $t^0=\langle t_0, t_1\rangle$ *then* $\sigma k(t)=DF(J(k(t_0), k(t_1)))$;

(K4) *if t is normal and* $t^0=x$ *then* $\sigma k(t)=DF(\bar{D}k(t^*r))$.

Now the main result of the paper is the following theorem:

Theorem 1. *Let $\mathcal{F}_1$ be an extension of $\mathcal{F}$ satisfying* I1, I2 *and* I3. *Let $k$ be a code function for $r$ and $\sigma$ be a reading element of $k$. Suppose that $\omega\in\mathcal{F}$ and $\omega$ is the least solution in $\mathcal{F}_1$ of*

$$AAC(I, C\xi)\sigma\leq\xi \tag{17}$$

*with respect to $\xi$. Then the element $\chi=\omega k(x)$ is the least solution of $\tilde{r}(\xi)\leq\xi$ with respect to $\xi$ in $\mathcal{F}_1$.*

Proof. Denote $AAC(I, C\xi)\sigma$ by $\Omega(\xi)$. Then for every $\varphi\in\mathcal{F}_1$ and for every normal $t\in\mathcal{T}$ we have

$$\text{if } t=I \text{ then } \Omega(\varphi)k(t)=I; \tag{18.1}$$

$$\text{if } t\neq I \text{ and } t^0\in\mathcal{A} \text{ then } \Omega(\varphi)k(t)=\varphi k(t^*)\tilde{t}^0; \tag{18.2}$$

$$\text{if } t^0=\langle t_0, t_1\rangle \text{ then } \Omega(\varphi)k(t)=(\varphi k(t_0), \varphi k(t_1)); \tag{18.3}$$

$$\text{if } t^0=x \text{ then } \Omega(\varphi)k(t)=\varphi k(t^*r). \tag{18.4}$$

Indeed,

$$\Omega(\varphi)k(t)=A(C(I, C\varphi))\sigma k(t)=C(I, C\varphi)(\sigma k(t)),$$

and by (K1)—(K4) we have: if $t=I$ then

$$\Omega(\varphi)k(t)=C(I, C\varphi)(DTI)=(I, C\varphi)TI=II=I;$$

if $t \neq I$ and $t^0 \in \mathscr{A}$ then

$$\Omega(\varphi)k(t) = C(I, C\varphi)(DF(Dk(t^*)\tilde{t}^0)) = (I, C\varphi)F(Dk(t^*)\tilde{t}^0) = C\varphi(Dk(t^*)\tilde{t}^0) = \varphi k(t^*)\tilde{t}^0;$$

if $t^0 = \langle t_0, t_1 \rangle$ then

$$\Omega(\varphi)k(t) = C(I, C\varphi)(DF(J(k(t_0), k(t_1)))).$$
$$= (I, C\varphi)F(J(k(t_0), k(t_1))) = C\varphi(J(k(t_0), k(t_1))) = (\varphi k(t_0), \varphi k(t_1));$$

and if $t^0 = x$ then

$$\Omega(\varphi)k(t) = C(I, C\varphi)(DF(\check{D}k(t^*r))) = (I, C\varphi)F(\check{D}k(t^*r))$$
$$= C\varphi(\check{D}k(t^*r)) = \varphi k(t^*r).$$

Since the mapping $\Omega: \mathscr{F}_1 \to \mathscr{F}_1$ monotonically increases and $\Omega(\omega) \leqq \omega$ then $\Omega(\omega)$ again is a solution of (17) and since $\omega$ is the least solution of (17) in $\mathscr{F}_1$, then $\omega \leqq \Omega(\omega)$ and therefore $\Omega(\omega) = \omega$. Thence by (18.1) — (18.4) we obtain that for each normal $t \in \mathscr{T}$:

(19.1) if $t = I$ then $\omega k(t) = I$;

(19.2) if $t \neq I$ and $t^0 \in \mathscr{A}$ then $\omega k(t) = \omega k(t^*)\tilde{t}^0$;

19.3) if $t^0 = \langle t_0, t_1 \rangle$ then $\omega k(t) = (\omega k(t_0), \omega k(t_1))$;

(19.4) if $t^0 = x$ then $\omega k(t) = \omega k(t^*r)$.

(Now we shall show that for every $t \in \mathscr{T}_0$

(20) $$\omega k(t) = \tilde{t}.$$

Every term $t \in \mathscr{T}_0$ regarded as a word has the form $w_0 w_1$ where the word $w_1$ contains the letters $A$,(,) only, and $w_0$ im empty or the last letter in $w_0$ is different from $A$, (and). Then by $a(t)$ we shall denote the number of letters in $w_1$. Let $b(t)$ be the number of $\langle$ in $t$ and denote by $c(t)$ the number of occurences in $t$ of terms $q \in \mathscr{C} \setminus \{A\}$, which are out of the scope of every pair corresponding brackets $\langle , \rangle$ in $t$. Denoting by $\omega_0$ the first infinite ordinal let $\mu(t) = b(t)\omega_0^2 + c(t)\omega_0 + a(t)$. Then by induction on $\mu(t)$ we shall show that (20) holds for every normal $t \in \mathscr{T}_0$. Suppose that $t \in \mathscr{T}_0$ is normal and consider the cases for $t$:

1. $t \in \mathscr{A}$. Then if $t = I$ (20) is the same as (19.1) and if $t \neq I$ then by (19.1) and (19.2)

$$\omega k(t) = \omega k(t^*)\tilde{t}^0 = \omega k(I)\tilde{t} = I\tilde{t} = \tilde{t}.$$

2. $t = \langle t_0, t_1 \rangle$. Then $t_i \in \mathscr{T}_0$, $t_i$ is normal and $\mu(t_i) < = \mu(t)$ since $b(t_i) < b(t)$ for each $i < 2$. Thence by (19.3) and the hypothesis of the induction $\omega k(t) = (\omega k(t_0), \omega k(t_1)) = (\tilde{t}_0, \tilde{t}_1) = \tilde{t}$.

3. No one of the previous two cases holds. Then $t$ should be of the form

(21) $$t = (t_0(t_1 \ldots (t_m s) \ldots))$$

where $s \in \mathscr{A}$ or $s$ is of the form $\langle s_0, s_1 \rangle$ but $t_m s$ isn't an $A$-term. To see that $\mu(t^*) < \mu(t)$ when $s \in \mathscr{A}$, consider the following two cases:

3.1. $s \in \mathscr{C} \setminus \{A\}$. Then by the definition of $t^*$

(22) $$t^* = (At_0(At_1 \ldots (At_{m-1}t_m) \ldots)).$$

By (21) and (22) we see that $c(t^*) < c(t)$ and $b(t^*) = b(t)$ whence $\mu(t^*) < \mu(t)$.

3.2. $s$ is an $A$-term. Then $t_m$ must contain a letter different from $A$, (and) since otherwise $t_m s$ would be an $A$-term, and (22) holds again. Then $a(t^*)<a(t)$, $b(t^*)=b(t)$ and $c(t^*)=c(t)$ whence $\mu(t^*)<\mu(t)$.

Therefore when $s\in\mathscr{A}$ we have $\omega k(t^*)=\tilde{t}^*$ and by (19.2), (11) and (15) $\omega k(t)=\omega k(t^*)\tilde{t}^0=\tilde{t}^*\tilde{t}^0=\widetilde{t^*t^0}=\tilde{t}$. It remains to consider the case.

3.3. $s$ is of the form $\langle s_0, s_1\rangle$. Then defining inductively $t^{*^1}=t^*$ ane $t^{*^{i+1}}=(t^{*^i})^*$ we have $t^{*^m}=\langle t', t''\rangle$ where

$$t'=(t_0'(t_1'\ldots(t_{m-1}'s_0)\ldots)),\quad t''=(t_0''(t_1''\ldots(t_{m-1}''s_1)\ldots));$$

$$t_{m-1}'=E_0t_{m-1},\ t_{m-2}'=E_0(At_{m-2}),\quad t_{m-3}'=E_0(A(At_{m-3})),\ldots;$$

$$t_{m-1}''=E_1t_{m-1},\ t_{m-2}''=E_1(At_{m-2}),\ t_{m-3}''=E_1(A(At_{m-3})),\ldots.$$

Obviously $b(t')<b(t)$ and $b(t'')<b(t)$ whence $\mu(t')<\mu(t)$ and $\mu(t'')<\mu(t)$. Therefore $\omega k(t')=\tilde{t}'$ and $\omega k(t'')=\tilde{t}''$, and by 1*) and (19.3)

$$\omega k(t^{*^k})=(\omega k(t'),\ \omega k(t''))=(\tilde{t}',\ \tilde{t}'')=\widetilde{t^{*m}}.$$

But

$$\omega k(t)=\omega k(t^*)\tilde{t}^0=\omega k(t^{*^2})\widetilde{t^{*0}}\,\tilde{t}^0=\cdots=\omega k(t^{*^k})\overline{\widetilde{t^{*m-1}0}}\ldots\tilde{t}^0$$

$$=\widetilde{t^{*^m}}\widetilde{t^{*m-1}0}\ldots\tilde{t}^0=\widetilde{t^{*m-1}}\widetilde{\widetilde{t^{*m-2}0}}\ldots\tilde{t}^0=\cdots=\tilde{t}^*\,\widetilde{t^0}=\tilde{t}.$$

That finishes the case 3. and proves (20) for normal $t$. Then for an arbitrary $t\in\mathscr{T}_0$ by (KO) and (11) we have

$$\omega k(t)=\omega k(t^N)=\widetilde{t^N}=\tilde{t}.$$

The next step will be to prove that if $N\neq E$, $t\in\mathscr{T}_0$ and $ts$ is normal then

$$\tilde{t}(\omega k(s))\leq\omega k(ts). \tag{23}$$

Suppose that $N\neq E$ and using I1 and I2 define for all $t, s\in\mathscr{T}$ an element $\vartheta_{t,s}\in\mathscr{F}$ by $\vartheta_{t,s}=((\tilde{t}\setminus\omega k(ts))/k(s))$. Then using I3 define $\vartheta_0=\inf\{\vartheta_{t,s}:\ t\in\mathscr{T}_0$ and $ts$ is normal$\}$ and $\vartheta=\inf\{\vartheta_0,\omega\}$. $\vartheta$ is the greatest element in $\mathscr{F}_1$ such that $\vartheta\leq\omega$ and for all $t, s\in\mathscr{T}$

$$t\in\mathscr{T}_0\ \&\ ts \text{ is normal}\ \Rightarrow\tilde{t}(\vartheta k(s))\leq\omega k(ts). \tag{24}$$

Therefore to prove (23) it is enough to show that $\omega\leq\vartheta$ and since $\omega$ is the least so lution of (17) in $\mathscr{F}_1$, it will be enough to show that $\Omega(\vartheta)\leq\vartheta$. But $\Omega(\vartheta)\leq\Omega(\omega)\leq\omega$, and therefore by the definition of $\vartheta$ it will be enough to prove the implication

$$t\in\mathscr{T}_0\ \&\ ts\ \text{is normal}\ \Rightarrow\tilde{t}(\Omega(\vartheta)k(s))\leq\omega k(ts).$$

To do that let $\Psi=\tilde{t}(\Omega(\vartheta)k(s))$, assume the hypothesis of the last implication, and consider the cases:

1. $s\in\mathscr{A}$. Then if $s=I$ then by (18.1) $\Omega(\vartheta)k(s)\leq\tilde{s}$, and if $s\neq I$ then by (18.2), 1*) and (19.1)

$$\Omega(\vartheta)k(s)=\vartheta k(s^*)\tilde{s}^0=\vartheta k(I)\tilde{s}\leq\omega k(I)\tilde{s}=I\tilde{s}=\tilde{s}.$$

Therefore according to (20) $\Psi\leq\tilde{t}\,\tilde{s}=\widetilde{ts}=\omega k(ts)$.

2. $s^0 \in \mathscr{A}$ and $s \notin \mathscr{A}$. Then $(ts)^* = Ats^*$ and $(ts)^0 = s^0$, and by (16) $Ats^*$ is normal. Besides that according to (18.2) $\Omega(\vartheta)k(s) = \vartheta k(s^*)\widetilde{s^0}$ whence $\Psi = \tilde{t}(\vartheta k(s^*)\widetilde{s^0}) = A\tilde{t}(\vartheta k(s^*))\widetilde{s^0}$ and by (24) and (19.2) $\Psi \leqq \omega k(Ats^*)s^0 = \omega k(ts)$.

3. $s^0 = \langle s_0, s_1\rangle$. Then according the definition of $s^0$ $s = \langle s_0, s_1\rangle$, and $(ts)^* = \langle E_0 ts_0, E_1 ts_1\rangle$, and $(ts)^0 = E$ whence by (18.3) $\Omega(\vartheta)k(s) = (\vartheta k(s_0), \vartheta k(s_1))$ and therefore

$$\Psi = \tilde{t}(\vartheta k(s_0), \vartheta k(s_1)) = (E_0 \tilde{t}(\vartheta k(s_0)), E_1 \tilde{t}(\vartheta k(s_1)))E.$$

But $E_i ts_i$ is normal for each $i < 2$ since $ts$ is normal, and thence with the help of (24) (19.3) and (19.2)

$$\Psi \leqq (\omega k(E_0 ts_0), \omega k(E_1 ts_1))E = \omega k(\langle E_0 ts_0, E_1 ts_1\rangle)E = \omega k((ts)^*)(\widetilde{ts})^0 = \omega k(ts).$$

4. $s^0 = x$. Then by (18.4) and (KO) $\Psi = \tilde{t}(\vartheta k(s^* r)) = \tilde{t}(\vartheta k((s^* r)^N))$. To be able to apply here (24) we should know that $t(s^* r)^N$ is normal. But otherwise a contraction of the form $t(s^* r)^N \to q$ must exist since $t$ and $(s^* r)^N$ are normal. By the normality of $ts$ that contraction can be only of the form C3 and then $(s^* r)^N = E$, which is possible only when $s^* = I$ and $r^N = E$. That can be seen easily by the normality of $s^*$. The contradiction with the assumption $r^N \neq E$ shows that $t(s^* r)^N$ is normal, and applying (24) we have $\Psi \leqq \omega k(t(s^* r)^N)$. Now if $s = x$ then $(ts)^* = t$ and

$$(t(s^* r)^N)^N = (t(s^* r))^N = (t(Ir))^N = (tr)^N = ((ts)^* r)^N$$

and if $s \neq x$ then $(ts)^* = Ats^*$ and again

$$(t(s^* r)^N)^N = (t(s^* r))^N = (Ats^* r)^N = ((ts)^* r)^N.$$

But $s^0 = x$ and therefore by (KO) and (19.4)

$$\Psi \leqq \omega k((t(s^* r)^N)^N = \omega k((ts)^* r) = \omega k(ts).$$

The proof of (23) is finished. Now let $\mathscr{T}_1$ be the set of all terms of the form $t_0(t_1 \ldots (t_{m-1}x) \ldots)$ where $t_0, \ldots, t_{m-1}$ are in $\mathscr{T}_0$, and denote by $p(t)$ the result of substituting $t$ for $x$ in $p \in \mathscr{T}_1$. We shall show that if $r^N \neq E$, $p \in \mathscr{T}_1$ and $p(ts)$ is normal then

$$\tilde{p}(\omega k(t))(\omega k(s)) \leqq \omega k(p(t)s). \tag{25}$$

The proof of (25) is analogous to that of (23). Suppose $r \neq E$, fix $t$ and denote $\omega k(t)$ shortly by $\varphi$. Then consider the greatest element $\vartheta \in \mathscr{T}_1$ such that $\vartheta \leqq \omega$ and for all $p \in \mathscr{T}_1$ and $t, s \in \mathscr{T}$

$$p(t)s \text{ is normal} \Rightarrow \tilde{p}(\varphi)(\vartheta k(s)) \leqq \omega k(p(t)s). \tag{26}$$

That element $\vartheta$ exists by I1, I2 and I3 in **2**. To prove (25) it is enough to show that for all $p \in \mathscr{T}_1$ and $t, s \in \mathscr{T}$

$$p(t)s \text{ is normal} \Rightarrow \tilde{p}(\varphi)(\Omega(\vartheta)k(s)) \leqq \omega k(p(t)s), \tag{27}$$

since then by $\Omega(\vartheta) \leqq \Omega(\omega) \leqq \omega$ it follows that $\Omega(\vartheta) \leqq \vartheta$ whence $\omega \leqq \vartheta$ and then by (26) we have (25). To prove (27) let $\Psi = \tilde{p}(\varphi)(\Omega(\vartheta)k(s))$, where $p \in \mathscr{T}_1$, assume the hypothesis in (27) and consider the cases for $s$:

1. $s \in \mathscr{T}_0$. Then as in the proof of (23) $\Omega(\vartheta)k(s) \leqq \tilde{s}$ and $\Psi \leqq \tilde{p}(\varphi)\tilde{s} = \tilde{p}(\omega k(t))\tilde{s}$. But $p(t)$ is normal since $p(t)s$ is such. Then applying (23) several times (or by induction on $m$ where $p = (t_0 \ldots (t_{m-1}x) \ldots)$) we have $\tilde{p}(\omega k(t)) \leqq \omega k(p(t))$ whence by (19.2)

$$\Psi \leqq \omega k(p(t))\tilde{s} = \omega k((p(t)s)^*)(\widetilde{p(t)s})^0 = \omega k(p(t)s).$$

2. $s \notin \mathscr{A}$ and $s^0 \in \mathscr{A}$. Then using (19.2) and (26) with $Ap$ as $p$ we have

$$\Psi = \tilde{p}(\varphi)\,(\vartheta k(s^*)\tilde{}s^0) = \tilde{A}p(\varphi)\,(\vartheta k(s^*))\tilde{}s^0 \leq \omega k(Ap(t)s^*)\tilde{}s^0 = \omega k((p(t)s)^*)\,(p\tilde{(t)}\tilde{s})^0 = \omega k(p(t)s).$$

3. $s = \langle s_0, s_1\rangle$. Then $(p(t)s)^* = \langle E_0 p(t)s_0,\ E_1 p(t)s_1\rangle$, $(p(t)s)^0 = E$ and by the help of (26), (19.3) and (19.2) we have

$$\Psi = \tilde{p}(\varphi)\,(\vartheta k(s_0),\ \vartheta k(s_1)) = (E_0\tilde{p}(\varphi)\,(\vartheta k(s_0)),\ E_1\tilde{p}(\varphi)\,(\vartheta k(s_1)))E$$

$$\leq (\omega k(E_0 p(t)s_0),\ \omega k(E_1 p(t)s_1))E = \omega k(\langle E_0 p(t)s_0,\ E_1 p(t)s_1\rangle)E = \omega k(p(t)s).$$

4. $s^0 = x$. Then $\Psi = \tilde{p}(\varphi)\,(\vartheta k(s^* r)) = \tilde{p}(\varphi)\,(\vartheta k((s^* r)^N))$ and as in the proof of (23) we see that $p(t)(s^* r)^N$ is normal. Thence by (26)

$$\Psi \leq \omega k(p(t)\,(s^* r)^N) = \omega k(p(t)\,(s^* r)).$$

But if $s = x$ then

$$(p(t)\,(s^* r))^N = (p(t)\,(Ir))^N = (p(t) r)^N = ((p(t)s)^* r)^N,$$

and if $s \neq x$ then again

$$(p(t)\,(s^* r))^N = (Ap(t)s^* r)^N = ((p(t)s)^* r)^N$$

whence by (19.4) $\Psi \leq \omega k((p(t)s)^* r) = \omega k(p(t)s$.

Thus the proof of (25) is finished. Applying (25) with $x$ as $p$ we see that if $r^N \neq E$ and $ts$ is normal then

(28) $$\omega k(t)\,(\omega k(s)) \leq \omega k(ts).$$

Then for $\chi = \omega k(x)$ we may easily prove that

(29) $$r^N \neq E\ \&\ t \text{ is normal } \Rightarrow \tilde{t}(\chi) \leq \omega k(t).$$

The proof is by induction on $t$ (i. e. by an induction corresponding to the inductive definition of 'term'): for $t = x$ (29) is trivial; for $t \in \mathscr{C}$ it is obvious from (20); and the inductive step is immediate by (28) and (19.3). By (11) and (29) with $r^N$ as $t$ it follows that if $r^N \neq E$ then $\tilde{r}(\chi) = \tilde{r}^N(\chi) \leq \omega k(r^N)$. But if $r^N = E$ then by (20) $\tilde{r}(\chi) = \widetilde{r^N}(\chi) = E = \omega k(E) = \omega k(r^N)$. Therefore according to (KO) and (19.4)

$$\tilde{r}(\chi) \leq \omega k(r) = \omega k(Ir) = \omega k((x)^* r) = \omega k(x) = \chi,$$

i. e. $\chi$ is a solution of $\tilde{r}(\xi) \leq \xi$.

Now let $\chi' \in \mathscr{F}_1$ and $\tilde{r}(\chi') \leq \chi'$. Then we shall show that

(30) $$\omega k(t) \leq \tilde{t}(\chi')$$

for all normal $t$. Define $\eta$ as the greatest element of $\mathscr{F}_1$ such that

(31) $$\eta k(t) \leq \tilde{t}(\chi')$$

for all normal $t$. That definition is possible by I2 and I3 (2). It is enough to show that for every normal $t$

(32) $$\Omega(\eta) k(t) \leq \tilde{t}(\chi')$$

since then $\Omega(\eta) \leq \eta$, $\omega \leq \eta$ and by (31) it follows (30). To do that suppose $t$ is normal and consider the cases for $t$:

1. $t = I$. Then by (18.1) $\Omega(\eta) k(t) = I = \tilde{t}(\chi')$.

2. $t \neq I$ and $t^0 \in \mathscr{A}$. Then using (18.2), (31), (15), (16) and (11)

$$\Omega(\eta)k(t)=\eta k(t^*)\widetilde{t^0} \leqq \widetilde{t^*}(\chi')\widetilde{t^0}=\widetilde{t^* t^0}(\chi')=\widetilde{t}(\chi').$$

3. $t^0=\langle t_0, t_1\rangle$. Then by (18.3) and (31)

$$\Omega(\eta)k(t)=(\eta k(t_0),\ \eta k(t_1)) \leqq (\widetilde{t^0}(\chi'),\ \widetilde{t_1}(\chi'))=\widetilde{t}(\chi').$$

4. $t^0=x$. Then using (18.4), (31), (11), $\widetilde{r}(\chi') \leqq \chi'$, and (15) we have

$$\Omega(\eta)k(t)=\eta k(t^* r)=\eta k((t^* r)^N)=\overline{(t^* r)^N}(\chi')$$

$$=\widetilde{t^*}(\chi')\widetilde{r}(\chi') \leqq \widetilde{t^*}(\chi')\chi'=\widetilde{t^* t^0}(\chi')=\widetilde{t}(\chi').$$

Thus (30) is proved and thence we have $\chi=\omega k(x) \leqq \widetilde{x}(\chi')=\chi'$, i. e. $\chi$ is the least solution of $\widetilde{r}(\xi) \leqq \xi$ in $\mathscr{F}_1$. Theorem 1 is proved.

Theorem 1 may be used as a fundamental instrument in the exposition of a recursion theory in $B$-combinatory algebras, which is similar to the theories of D. Skordev [6] and L. Ivanov [3]. By suitable application of theorem 1 we may obtain a first recursion theorem, a parametrization and second recursion theorem, a normal form theorem for recursive operators and other corollaries. We are intended to do that in another paper.

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