Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

ON THE MORSE-SMALE INDEX THEOREM AND THE PROBLEM OF MINIMAL SURFACES

THEMISTOCLES M. RASSIAS

The Morse-Smale index and nullity of some of the classical minimal surfaces are studied.

1. Let Γ be a Jordan curve in the ordinary Euclidean space R^3 represented by a continuous, one-to-one mapping $\alpha: \partial D_r^2 \to R^3$, where $D_r^2 = \{(u, v) \in R^2: u^2+v^2<r^2\}$ and ∂D_r^2 is the topological boundary of D_r^2. A minimal surface $q: D_r^2 \to R^3$ spanned by Γ is a map of class

$$C^0(\overline{D_r^2}, R^3) \cap C^2(D_r^2, R^3)$$

satisfying the differential equations: $\Delta q=0$, $|\partial q/\partial u|^2=|\partial q/\partial v|^2$, $\partial q/\partial u \cdot \partial q/\partial v=0$ in D_2^r, where q maps ∂D_r^2 onto Γ in a topological manner. For a surface in isometric representation, the Dirichlet integral equals twice the surface area. The Dirichlet integral is defined as

$$D[q]=\int\int_{D_r^2} |\nabla q|^2 \, dudv, \quad \nabla q=\left(\frac{\partial q}{\partial u}, \frac{\partial q}{\partial v}\right).$$

The *second variation* (derivative) of the area function for a variation whose de formation vector field is given by $g=fN$ is then

$$II(g, g)=\int\int_{D_r^2} f(-\Delta f+2KWf)\, dudv,$$

where $f: \overline{D_r^2} \to R$ is a smooth function such that $f\equiv 0$ on ∂D_r^2, and N is the unit normal field along $q(D_r^2)$. K is the Gaussian curvature of the minimal surface, W is the discriminant of the first fundamental form and Δ is the Laplacian of the surface

The operator $L=-\Delta+2KW$, used to define the second variation of the area function is a symmetric differential operator. This operator is also strongly elliptic. The operator L is known as the *Jacobi operator* for the minimal surface $q: \overline{D_r^2} \to R^3$ (given in a conformal parametric representation). L can be diagonalized with eigenvalues $\lambda_1<\lambda_2<\lambda_3<\cdots \to +\infty$, where each eigenspace V_{λ_i} is finite dimensional.

The quadratic form $II(g, g)$ is *the Hessian form* of second derivatives of the area function at the point q. At such a point the Hessian form has two *intrinsic invariants* that are used to define the *nullity* and the *index* of q.

Definition. *Let $q: D_r^2 \to R^3$ be a minimal immersion, given in parametric representation, $q(u, v)=(x(u, v), y(u, v), z(u, v))$ where $(u, v) \in D_r^2$ for $r \in R$*

a) *The "Morse-Smale Index of q" is* $I(q; D_r^2)$ *= the sum of the dimensions of those* V_λ *where* V_λ *is an eigenspace corresponding to the negative eigenvalue* λ *of the Jacobi operator for q. It is frequently written as*

$$I(q; D_r^2)=\dim(\bigoplus_{\lambda<0} V_\lambda)$$

b) *"The nullity of q" is*

$$N(q; D_r^2)=\dim V_0,$$

where V_0 *is the eigenspace corresponding to the zero eigenvalue of the Jacobi operator L for q (cf. [3], [4], [5], [10]).*

Definition. *A minimal immersion is defined to be "stable" if* $II(g, g)>0$ *for all* $(g\neq 0)$.

A minimal immersion is defined to be "unstable" if $II(g, g)<0$ *for some* $g(\neq 0)$.

A "Jacobi field" on $\overline{D_r^2}$ *is a normal field fN, where* $f: \overline{D_r^2}\to R$ *has the property of being a smooth function satisfying the "Jacobi equation"* $-\Delta f+2KWf=0$.

Remark: Clearly if $N(q; D_r^2)=0$, then

a) a minimal immersion q is stable $\Leftrightarrow I(q; D_r^2)=0$;

b) a minimal immersion q is unstable $\Leftrightarrow I(q; D_r^2)>0$. Consider the eigenvalue problem:

$$(E)\qquad \begin{cases} -\Delta f+2KWf=\lambda f \text{ where } f=f(u, v) \text{ on } \overline{D_{r_0}^2} \\ f=0 \text{ on } \partial D_{r_0}^2 \text{ (Boudary condition)} \end{cases}$$

Definition. *A conjugate boundary of a minimal immersion is a boundary* $r=r_0$ *such that zero is an eigenvalue of the Jacobi operator for the disc* $\overline{D_{r_0}^2}$ *in the eigenvalue problem (E). A first conjugate boundary of a minimal immersion is a boundary* $r=\tilde{r}$ *such that zero is the first eigenvalue of the Jacobi operator for the disc* $\overline{D_{\tilde{r}}^2}$ *in the eigenvalue problem (E), i. e.* $\lambda_1(\tilde{r})=0$.

The multiplicity of a conjugate boundary $r=r_0$ *is the number of linearly independent Jacobi fields defined on* $\overline{D_{r_0}^2}$ *and vanishing on* $\partial D_{r_0}^2$*, i. e. the dimension of the eigenspace correšponding to the zero eigenvalue of the Jacobi operator for the disc* $\overline{D_{r_0}^2}$ *in the eigenvalue problem (E).*

Theorem 1.1. *The multiplicity of a first conjugate boundary of any minimal immersion* $q: \overline{D_{r_0}^2}\to R^3$ *is always one.*

Theorem 1. 2. *Let* $q: \overline{D_{r_0}^2}\to R^3$ *be a minimal immersion and* $r_0'<r_0$ *a first conjugate boundary of q. Then the following are true:*

1) $N(q; D_r^2)=1$ if $r=r_0'$ 4) $I(q; D_r^2)=0$ if $r<r_0'$

2) $N(q; D_r^2)=0$ if $r<r_0'$ 5) $I(q; D_r^2)\geq 1$ if $r>r_0'$

3) $I(q; D_r^2)=0$ if $r=r_0'$

Remark. If $q: \overline{D_r^2}\to R^3$ is a minimal immersion and ∂D_r^2 is a first conjugate boundary, then q is everywhere stable in the disc D_r^2.

2. An application of the Morse-Smale Index to the study of minimal surfaces in R^3. We are going to study some of the main topological and stability properties of some of the most important examples of complete minimal surfaces in R^3; by making use of the Morse-Smale Index theorem. We consider the eigenvalue formulation of the Index because this is the best way to get global results for complete minimal surfaces and in addition to being in a better situation to examine the eigenvalues of certain elliptic operators on manifolds. This leads to the study of the eigenvalues of the Laplacian which play a very significant role in mathematics (cf. [1]).

It is our purpose to analyze the following examples of complete minimal surfaces in R^3: 1. The Enneper's minimal surface; 2. The catenoid; 3. The (right) helicoid; 4. The Richmond's minimal surface.

2.1. The Enneper's minimal surface. A conformal parametric representation of a portion of the Enneper's minimal surface in R^3 is a mapping $q: D^2_{r_0} \to R^3$, for each $r_0>0$, given by $q(u, v)=(x(u, v), y(u, v), z(u, v))$, where $(u, v)\in D^2_{r_0}$ and

$$(1)\qquad \begin{cases} x(u, v)=u+uv^2-\frac{1}{3}u^3, \\ y(u, v)=-v-u^2v+\frac{1}{3}v^3, \\ z(u, v)=u^2-v^2. \end{cases}$$

This way we get a global expression of the Enneper's minimal surface in R^3. The boundary of the portion of Enneper's minimal surface as described above is a mapping $a: \partial D^2_{r_0} \to R^3$ given in a parametric representation as follows:

$$\Gamma_{r_0}=a(r_0, \theta)=(x(r_0, \theta), \ y(r_0, \theta), \ z(r_0, \theta)),$$

where

$$\begin{cases} x(r_0, \theta)=r_0\cos\theta-\frac{1}{3}r_0^3\cos 3\theta, \\ y(r_0, \theta)=-r_0\sin\theta-\frac{1}{3}r_0^3\sin 3\theta, \\ z(r_0, \theta)=r_0^2\cos 2\theta, \quad \theta\in[0, 2\pi]. \end{cases}$$

It follows from work of J. C. C. Nitsche that the portion of Enneper's minimal surface given by (1) in the disc $D^2_{r_0}$ can also be written (in an equivalent way) as

$$S_{r_0}=\{(x, y, z): \ x=r_0u+r_0^3uv^2-\frac{1}{3}r_0^3u^3, \ y=-r_0v-r_0^3u^2v+\frac{1}{3}r_0^3v^3, \\ z=r_0^2(u^2-v^2); \ u^2-v^2\leq 1\}.$$

It can be easily shown that its mean curvature H is zero; for this reason it is a minimal surface. From the topological point of view the Enneper's surface is simply connected. It is properly immersed in R^3, but this is not an embedding because the surface has self-intersections.

Theorem 2.1.1. *The boundary curve Γ_{r_0} for the portion of Enneper's minimal surface S_{r_0} is a Jordan curve for $0<r_0<\sqrt{3}$. The multiple points of Γ_{r_0} appear for $r_0>0$ and $0\leq\theta_1<\theta_2<2\pi$ in the following places:*

For $r_0=\sqrt{3}$: $\theta_1=0$, $\theta_2=\pi$; or $\theta_1=\frac{\pi}{2}$, $\theta_2=\frac{3\pi}{2}$.

For $r_0>\sqrt{3}$: $r_0=\dfrac{1}{\sqrt{\frac{1}{3}\cos^2\theta_1-\sin^2\theta_1}}$ $\quad 0<\theta_1<\frac{\pi}{6}$ *and* $\theta_2=\pi-\theta_1$

or $r_0=\dfrac{1}{\sqrt{\frac{1}{3}\sin^2\theta_1-\cos^2\theta_1}}$ $\quad \frac{\pi}{3}<\theta_1<\frac{\pi}{2}$ *and* $\theta_2=2\pi-\theta_1$

or $r_0=\dfrac{1}{\sqrt{\frac{1}{3}\sin^2\theta_1-\cos^2\theta_1}}$ $\quad \frac{\pi}{2}<\theta_1<\frac{2\pi}{3}$ *and* $\theta_2=2\pi-\theta_1$

or $r_0=\dfrac{1}{\sqrt{\frac{1}{3}\cos^2\theta_1-\sin^2\theta_1}}$ $\quad \pi<\theta_1<\frac{7\pi}{6}$ *and* $\theta_2=3\pi-\theta_1$.

Theorem 2.1.2. *The Morse-Smale Index* $I(q;D^2_{r_0})$ *of the Enneper's minimal surface* $q: D^2_{r_0}\to R^3$ *is bounded above by the sum of the dimensions of the eigenspaces corresponding to those eigenvalues* λ *of the Bessel equation which are less than* $8r_0^2$ *for all* r_0 *in* R.

Note: The Bessel equation is an ordinary linear differential equation given by $\frac{d^2y}{d\rho^2}+\frac{1}{\rho}\frac{dy}{d\rho}+(1-\frac{n^2}{\rho^2})y=0$ for all $n=0, 1, 2, \ldots$, where

$$\begin{cases} y=y(\rho), \quad \rho\in[0,\infty) \text{ and} \\ \rho=rr_0\sqrt{\lambda+8} \text{ for } 0\leq r\leq 1 \text{ and } r_0 \text{ given real number} \\ \text{with } r_0\in[0,\infty). \end{cases}$$

Corollary. *The Morse-Smale Index* $I(q;D^2_{r_0})$ *of the Enneper's minimal surface* $q: D^2_{r_0}\to R^3$ *varies as follows:*

1) $I(q;D^2_{r_0})=0$ for $r_0=0$;
2) $I(q;D^2_1)\leq 1$;
3) $I(q;D^2_{\sqrt{3}})\leq 2$;
4) $I(q;D^2_2)\leq 4$;
5) $I(q;D^2_{r_0})\to\infty$ *as* $r_0\to\infty$.

Theorem 2.1.3 *The portion of Enneper's minimal surface* $q: D^2_{r_0}\to R^3$ *for each* $r_0>0$ *given by (1), is an unstable extremal (i. e. a non-relative minimum of the area functional) for all* r_0 *with* $\bar{r}<r_0<\sqrt{3}$ *and* $\bar{r}=1.681475$.

2.2. *The Catenoid.* We define as catenoid the surface obtained by revolving the curve $y=\cosh z$ about the z-axis. This way we get a global expression of the surface. It can be easily shown that its mean curvature H is zero. It is easy to see that it is properly embedded in R^3, and that it is not an algebraic minimal surface. The catenoid and Enneper's minimal surface are the only surfaces whose normal map is one-to-one (K. Voss).

A conformal parametric representation of a portion of the catenoid in R^3 is a mapping $q: D^2_{r_0}\to R^3$, for each $r_0>0$ given by $q(u,v)=(x(u,v), y(u,v), z(u,v))$ where $(u,v)\in D^2_{r_0}$ and

$$\begin{cases} x(u,v)=\cosh u\cos v, \\ y(u,v)=\cosh u\sin v, \\ z(u,v)=u. \end{cases}$$

Theorem 2.2.1 *The Morse-Smale Index* $I(q;\ D^2_{r_0})$ *of the catenoid* $q: D^2_{r_0} \to R^3$ *is bounded above by the sum of the dimensions of the eigenspaces corresponding to those eigenvalues* λ *of the Bessel equation which are less than* $2r_0^2$ *for all* r_0 *in* R.

Corollary. *The Morse-Smale Index* $I(q;\ D^2_{r_0})$ *of the catenoid* $q: D^2_{r_0} \to R^3$ *varies as follows:*

1) $I(q;\ D^2_{r_0})=0$ *for* $r_0=0$;
2) $I(q;\ D^2_1)=0$;
3) $I(q;\ D^2_{\sqrt{3}})\leqq 1$;
4) $I(q;\ D^2_2)\leqq 1$;
5) $I(q;\ D^2_3)\leqq 2$;
6) $I(q;\ D^2_4)\leqq 4$;
7) $I(q;\ D^2_5)\leqq 6$;
8) $I(q;\ D^2_{r_0})\to\infty$ *as* $r_0\to\infty$.

Theorem 2.2.2. *The first conjugate boundary of the catenoid* $q: D^2_{r_0} \to R^3$ *occurs for* $r_0=\mu_1^{(0)}/\sqrt{2}\simeq 1.7$, *where* $\mu_1^{(0)}$ *is the first zero of the Bessel equation for the unit disc.*

Note: $\mu_1^{(0)}\simeq 2.4048256$ (cf. [11])

2.3. The (right) helicoid. A (right) helicoid is the surface generated by a line which moves along the z-axis in such a way that it remains parallel to the (x, y)-plane and passes through the points of a circular helix; in other words, the surface is generated by a line perpendicular to the z-axis under a "screw" motion. A parametric representation of a portion of the (right) helicoid in R^3 is a mapping $q: D^2_{r_0}\to R^3$ for each $r_0>0$, given by $q(v, u)=(x(v, u),\ y(v, u),\ z(v, u))$ where $(v, u)\in D^2_{r_0}$ and

$$\begin{cases} x(v, u)=u\cos v, \\ y(v, u)=u\sin v, \\ z(v, u)=v. \end{cases}$$

This way we obtain a global expression of the surface. It can be easily verified that its mean curvature H is zero. From the topological viewpoint it is simply connected. The (right) helicoid is properly embedded in R^3 and it is not an algebraic minimal surface. Monge and Sophus Lie considered this surface as an important example for their study on the theory of minimal surfaces. They tried to solve the classification problem of minimal surfaces, but without success. Their methods are completely different from ours.

Theorem 2.3.1. *The Morse-Smale Index* $I(q;\ D^2_{r_0})$ *of the (right) helicoid* $q: D^2_{r_0}\to R^3$ *is bounded above by the sum of the dimensions of the eigenspaces corresponding to those eigenvalues* λ *of the Bessel equation which are less than* $2r_0^2\sqrt{1+r_0^2}$ *for all* r_0 *in* R.

Corollary. *The Morse-Smale Index* $I(q;\ D^2_{r_0})$ *of the (right) helicoid* $q: D^2_{r_0}\to R^3$ *varies as follows:*

1) $I(q;\ D^2_{r_0})=0$ *for* $r_0=0$;
2) $I(q;\ D^2_1)=0$;
3) $I(q;\ D^2_{\sqrt{3}})\leq 1$;
4) $I(q;\ D^2_2)\leq 2$;
5) $I(q;\ D^2_3)\leq 6$;
6) $I(q;\ D^2_{r_0})\to\infty$ *as* $r_0\to\infty$.

2.4. The H. W. Richmond's minimal surface. A conformal parametric representation of a portion of the H. W. Richmond's minimal surface in R^3 is a mapping $q: D^2_{r_0} \to R^3$ for each $r_0>0$, given by $q(u, v)=(x(u, v), y(u, v), z(u, v))$, where $(u, v) \in D^2_{r_0}$ and

$$\begin{cases} x(u, v)=\frac{1}{3}u^3-uv^2+u/(u^2+v^2) \\ y(u, v)=\frac{1}{3}v^3-u^2v-v/(u^2+v^2) \\ z(u, v)=2u. \end{cases}$$

This way we get a global expression of Richmond's surface. It can be shown that its mean curvature is zero. Richmond's minimal surface is an immersion everywhere except at the origin (0, 0).

This surface was invented by H. W. Richmond in 1904.

Theorem 2.4.1. *The nullity of Richmond's minimal surface (in the sense of the eigenvalue problem (E)) is zero.*

Theorem 2.4.2. *I) The negative eigenvalues of:*

$$(A)\qquad \begin{cases} r^2R''+rR'+\lambda\cdot\frac{(r^4+1)^2}{r^2}\cdot R=(n^2-32)R \text{ where } R=R(r),\ r\in[0, r_0] \\ R(r_0)=0 \text{ (Boundary condition) and } R\in C^2_p\,[0, r_0] \end{cases}$$

corresponding to Richmond's minimal surface occur only for $n=0, 1, 2, 3, 4, 5$.

Note: C^2_p stands for the space of piecewise C^2 functions.

II) *No negative eigenvalues occur in* (A) *if*

i)	$\lambda<-32\,r_0^2$ *and* $n=0$;	iv)	$\lambda<-23\,r_0^2$ *and* $n=3$;
ii)	$\lambda<-31\,r_0^2$ *and* $n=1$;	v)	$\lambda<-16\,r_0^2$ *and* $n=4$;
iii)	$\lambda<-28\,r_0^2$ *and* $n=2$;	vi)	$\lambda<-7\,r_0^2$ *and* $n=5$.

Research Problem: ([7]). Find the sum (or an upper bound of the sum) of the dimensions of the eigenspaces corresponding to the negative eigenvalues of the eigenvalue problem (A), as a function of r_0.

In the following we pose some research problems whose solution we hope to help towards a further understanding of the theory of minimal surfaces and the problem of Plateau.

Problem: Does there exist a harmonic homeomorphism of the open unit ball B in R^3 onto R^3?, i. e. do there exist harmonic functions f_1, f_2, f_3 defined in $B=\{z=(z_1, z_2, z_3) : |z|<1\}$, such that the mapping $z\to(f_1, f_2, f_3)$ is a homeomorphism of B onto all of R^3?

Remark: It is a theorem of T. Rado that there exists no harmonic homeomorphism of the open unit disc in R^2 onto R^2.

Problem: Does there exist a harmonic mapping $q: D\to R^3$, such that $\frac{\partial q}{\partial u}\cdot\frac{\partial q}{\partial u}=\frac{\partial q}{\partial v}\cdot\frac{\partial q}{\partial v}$, $\frac{\partial q}{\partial u}\cdot\frac{\partial q}{\partial v}=0$ and $q|_{\partial D}: \partial D\to\Gamma$ is not a homeomorphism but it is homotopic to a homeomorphism? Here Γ stands for a C^1 Jordan curve in R^3 and D for the open unit disc in R^2, i. e. $D=\{(u, v)\in R^2: u^2+v^2<1\}$.

Problem: ([6]). Give an example of a Jordan curve Γ in R^3 spanning five different minimal surfaces of the type of the disc, all of which are explicitly known. Is there any characterization of this phenomenon in the sense of Morse theory on Hilbert (or Banach) manifolds as developed by R. Palais and S. Smale (cf. [2, **8**, 9]).

REFERENCES

1. M. F. Atiyah. Eigenvalues and Riemannian Geometry. — Manifolds, Tokyo, 1973, 5-9.
2. R. Palais, S. Smale. A generalized Morse theory. *Bull. Amer. Math. Soc.*, **70**, 1964, 165-172.
3. Th. M. Rassias. Sur la multiplicité du premier bord conjuguè d'une hypersurface minimale de R^n, $n \geqq 3$. *C. R. Acad. Sci. Paris*, **284**, 1977, 497-499.
4. Th. M. Rassias. On certain properties of eigenvalues and the Poincarè inequality. — In: Global Analysis — Analysis on Manifolds (ed. Th. M. Rassias). Leipzig, 1983, 282-300.
5. Th. M. Rassias. Foundations of Global Nonlinear Analysis. Leipzig (To Appear).
6. Th. M. Rassias. Query No. 118. *Notices Amer. Math. Soc.*, **24**, 1977, No 2, p. 136.
7. Th. M. Rassias. Query No. 257. *Notices Amer. Math. Soc.*, **29**, 1982, No. 4, p. 326.
8. Th. M. Rassias. Morse theory and Plateau's problem. — In: Selected Studies: Physics-Astrophysics, Mathematics, History of Science. A volume dedicated to the memory of Albert Einstein (eds: Th. M. Rassias and G. M. Rassias). Amsterdam, 1982, 261-292.
9. S. Smale. Morse theory and a non-linear generalization of the Dirichlet problem. *Ann. Math.*, **2** (80), 1964, 382-396.
10. S. Smale. On the Morse Index theorem. *J. Math. Mech.*, **14**, 1965, 1049-1056.
11. G. N. Watson. A Treatise on the Theory of Bessel Functions. 2nd ed., Cambridge, 1944.

4, Zagoras street
Paradissos, Amaroussion
151 25 Athens, Greece

Received 13. 3. 1985