Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

ON THE $\langle n/\vec{G}/M/1\rangle$ QUEUES AND ERLANG'S LOSS FORMULAS

J. SZTRIK

Abstract. The aim of the present paper is to give the main characteristics of the finite-source $\vec{G}/M/1$ queue in equilibrium under different service disciplines. Here unit i stays in the source for a random time having general distribution function $F_i(x)$. The required service times of all units are assumed to be identically and exponentially distributed random variables with means $1/\mu$, but the service rates depend on the state of the system. It is shown that the solution to this $\vec{G}/M/1$ model is similar in most important respects to that for the $M/M/1$ model.

As an application the Erlang's loss formulas are derived. Finally under preemptive priority service in exponential case we minimize the steady-state probability of refusal.

1. Introduction. The $\langle n/\vec{G}/M/1\rangle$ queue is a closed $\vec{G}/M/1$ queue in which units emanate from a finite source of size n and are served at a service facility according to different service disciplines. Let us suppose if k units are at time t in the source then the customer under service obtains during $(t, t+h)$ an amount of service equal to $r_k h$. Hence the attained service time increases with rate r_k, $0<r_k<\infty$, $k=0, 1, \dots, n-1$. The requred service times of all units are assumed to be identically and exponentially distributed random variables with means $1/\mu$. Therefore the probability that a unit is served in the interval $(t, t+h)$ provided that there are k customers in the source is $\mu_k h+\sigma(h)$, where $\mu_k=\mu r_k$. After completing service, unit i returns to the source and stays there for a random time having general distribution function $F_i(x)$. The service and arrival times are assumed to be mutually independent of each other.

Such a finite-source queueing model is often called the "machine-interference problem." In recent years this model has been effectively used, for example, for the mathematical description of multiprogrammed computer systems, c. f. [1, 9, 11, 13].

Furthermore, the analogy between this problem and the $M/\vec{G}/n/n$ loss system, which is of considerable importance in the field of telephone systems, is well-known, c. f. Takács [15, p. 190]. Since there is a sizable literature on "machine interference" and "Erlang's loss formulas," we refer only to the most related results. Bunday and Scraton [2] have recently proved that the probability distribution of the number of machines running in steady state is the same in the $M/M/r$ and $G/M/r$ cases. Sztrik [14] has generalized this result to the $\vec{G}/M/r$ case. The $G/M/n/n$ Erlang model was treated by Takács [15], while the $M/G/n/n$ for arbitrary service time distribution with finite expectation was discussed by Sevastyanov [12]. These results have been generalized by Fakinos [4] in the case of non-identical servers. For state-dependent arrival rates and homogeneous servers the model can be found in [8]. By allowing different service times and state-dependent arrival rates the validity of Erlang' formulas has been proved by Sahbazov [10]. For time-dependent arrival and service rates the model was treated by Gnedenko [7]. The Erlang loss system $M/G/n/n$ with speeds is discussed in Franken et al. [5, p. 174]. This paper then deals with a generalization of the $G/M/1$ model under first-come, first-served (FCFS), last-come, first-served (LCFS), random selection (RS) service disciplines and gives the main steady-state characteristics of the $\langle n/\vec{G}/M/1\rangle$ queue. As an application the Erlang's loss formulas are derived. Finally, under preemptive priority service in exponential case we minimize the stationary probability of refusal.

2. The mathematical model. 2.1. FCFS service. Let the random variable $\nu(t)$ denote the number of units staying at time t in the source and $(\alpha_1(t), \ldots, \alpha_{\nu(t)}(t))$ indicate their indices ordered lexicographically. Let us denote by $(\beta_1(t), \ldots, \beta_{n-\nu(t)}(t))$ the indices of units waiting or served at the service facility in order of their arrival. Clearly the sets $(\alpha_1(t), \ldots, \alpha_{\nu(t)}(t))$ and $(\beta_1(t), \ldots, \beta_{n-\nu(t)}(t))$ are disjoint. Let us introduce the process

$$\underline{Y}(t) = (\nu(t); \quad \alpha_1(t), \ldots, \alpha_{\nu(t)}(t); \quad \beta_1(t), \ldots, \beta_{n-\nu(t)}(t)).$$

The stochastic process $(\underline{Y}(t),\ t \geqq 0)$ is semi-Markovian.

Let us also introduce the supplementary variables $\xi_{\alpha_l(t)}(t)$ to denote the random time, which unit $\alpha_l(t)$ has been spending in the source till time t, $l = 1, \ldots, \nu(t)$. Define

$$(1) \quad \underline{X}^{(1)}(t) = (\nu(t); \quad \alpha_1(t), \ldots, \alpha_{\nu(t)}(t); \quad \xi_{\alpha_1(t)}(t), \ldots, \xi_{\alpha_{\nu(t)}(t)}(t); \ \beta_1(t), \ldots, \beta_{n-\nu(t)}(t)).$$

Then the process $(\underline{x}^{(1)}(t),\ t \geqq 0)$ satisfies the Markov property.

Let V_k^n and C_k^n denote the sets of all variations and combinations of order k of the integers $1, \ldots, n$ ordered lexicographically, respectively. Then the phase space of the process $\underline{x}^{(1)}(t)$ consists of the sets

$$(i_1, \ldots, i_k; \quad x_1, \ldots, x_k; \ j_1, \ldots, j_{n-k}),$$

$$(i_1, \ldots, i_k) \in C_k^n, \quad (j_1, \ldots, j_{n-k}) \in V_{n-k}^n, \quad x_i \in R_+, \quad i = \overline{0, k}, \quad k = \overline{0, n}.$$

The process is in state $(i_1, \ldots, i_k;\ x_1, \ldots, x_k;\ j_1, \ldots, j_{n-k})$ if k units with indices $(i_1, \ldots, i_k)$ have been staying in the source for times $(x_1, \ldots, x_k)$, respectively, while the rest need service and their indices in order of arrival are $(j_1, \ldots, j_{n-k})$.

For the distribution of $\underline{x}^{(1)}(t)$ consider the functions given below:

$$Q^{(1)}_{0;j_1, \ldots, j_n}(t) = P(\nu(t) = 0; \quad \beta_1(t) = j_1, \ldots, \quad \beta_n(t) = j_n),$$

$$Q^{(1)}_{i_1, \ldots, i_k; j_1, \ldots, j_{n-k}}(x_1, \ldots, x_k;\ t) = P(\nu(t) = k; \quad \alpha_1(t) = i_1, \ldots, \alpha_k(t) = i_k;$$

$$(2) \qquad \xi_{i_1}(t) < x_1, \ldots, \xi_{i_k}(t) < x_k; \quad \beta_1(t) = j_1, \ldots, \beta_{n-k}(t) = j_{n-k}).$$

2.2. LCFS service. In this section we also introduce a stochastic process

$$(3) \quad \underline{x}^{(2)}(t) = (\nu(t); \quad \alpha_1(t), \ldots, \alpha_{\nu(t)}(t); \quad \xi_{\alpha_1(t)}(t), \ldots, \xi_{\alpha_{\nu(t)}(t)}(t); \quad \beta_1(t), \ldots, \beta_{n-\nu(t)}(t)).$$

The only difference between $\underline{x}^{(1)}(t)$ and $\underline{x}^{(2)}(t)$ is the order of service. In $\underline{x}^{(1)}(t)$ unit $\beta_1(t)$ while in $\underline{x}^{(2)}(t)$ unit $\beta_{n-\nu(t)}(t)$ is under service.

For the distribution of $\underline{x}^{(2)}(t)$ let us introduce the functions:

$$Q^{(2)}_{0;j_1, \ldots, j_n}(t) = P(\nu(t) = 0; \quad \beta_1(t) = j_1, \ldots, \beta_n(t) = j_n),$$

$$Q^{(2)}_{i_1, \ldots, i_k; j_1, \ldots, j_{n-k}}(x_1, \ldots, x_k; t) = P(\nu(t) = k; \quad \alpha_1(t) = i_1, \ldots, \alpha_k(t) = i_k;$$

$$(4) \qquad \xi_{i_1}(t) < x_1, \ldots, \xi_{i_k}(t) < x_k; \quad \beta_1(t) = j_1, \ldots, \beta_{n-k}(t) = j_{n-k}).$$

2.3. RS service. Under this discipline if a new unit arrives to the service facility the service is immediately interrupted and the server starts serving a randomly selected unit (of course, the earlier one can be selected, too).

Let us introduce the process

(5) $$\underline{x}^{(3)}(t)=(\nu(t);\quad \alpha_1(t),\ldots,\alpha_{\nu(t)};\quad \xi_{\alpha_1(t)},\ldots,\xi_{\alpha_{\nu(t)}(t)}(t)).$$

Define the distribution of $\underline{x}^{(3)}(t)$ by functions:

(6) $$Q_0^{(3)}(t)=P(\nu(t)=0),$$

$$Q^{(3)}_{i_1,\ldots,i_k}(x_1,\ldots,x_k;\ t)=P(\nu(t)=k;\quad \alpha_1(t)=i_1,\ldots,\alpha_k(t)=i_k;\quad \xi_{i_1}(t)<x_1,\ldots,\xi_{i_k}(t)<x_k)$$

2.4. General treatment. Let λ_i be defined by $1/\lambda_i=\int_0^\infty x dF_i(x)$. Then we have:

Theorem 1. *If $0<1/\lambda_i<\infty$, $i=1,\ldots,n$, then the processes* (1), (3), (5) *possess a unique limiting (stationary) ergodic distribution which is independent of the initial conditions, namely*

(7) $$Q^{(1)}_{0;j_1,\ldots,j_n}=\lim_{t\to\infty}Q^{(1)}_{0;j_1,\ldots,j_n}(t),$$

$$Q^{(1)}_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1,\ldots,x_k)=\lim_{t\to\infty}Q^{(1)}_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1,\ldots,x_k;\ t);$$

(8) $$Q^{(2)}_{0;j_1,\ldots,j_n}=\lim_{t\to\infty}Q^{(2)}_{0;j_1,\ldots,j_n}(t),$$

$$Q^{(2)}_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1,\ldots,x_k)=\lim_{t\to\infty}Q^{(2)}_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1,\ldots,x_k;\ t);$$

(9) $$Q_0^{(3)}=\lim_{t\to\infty}Q_0^{(3)}(t),$$

$$Q^{(3)}_{i_1,\ldots,i_k}(x_1,\ldots,x_k)=\lim_{t\to\infty}Q^{(3)}_{i_1,\ldots,i_k}(x_1,\ldots,x_k;\ t),$$

respectively.

Proof. Notice that the processes belong to the class of piecewise-linear Markov processes subject to discontinuous changes treated in Gnedenko-Kovalenko [6] in details. Our statement follows from a theorem on p. 211 of this monograph.

Furthermore we have the following result.

Theorem 2. *If $t>\max(x_1,\ldots x_k)$ then the distributions* (2), (4), (6) *possess k-dimensional densities $q^{(1)}_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1,\ldots,x_k;\ t)$, $q^{(2)}_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1,\ldots,x_k;\ t)$, $q^{(3)}_{i_1,\ldots,i_k}(x_1,\ldots,x_k;\ t)$, respectively, independently of the initial conditions.*

Proof. Defining μ by $\mu=\max(\mu_0,\ldots,\mu_{n-1})$, based on the treatment found in Sevastyanov [12] we get

$$P(\nu(t)=k;\quad \alpha_s(t)=i_s;\quad x_s\le\xi_{i_s}(t)<x_s+\Delta_s,\ s=\overline{1,k};\quad \beta_l(t)=j_l,\quad l=\overline{1,n-k})$$

$$\le P(\nu(t)=k;\quad \alpha_s(t)=i_s;\quad x_s\le\xi_{i_s}(t)<x_s+\Delta_s,\ s=\overline{1,k})\le\mu^k\prod_{s=1}^{k}\int_{t-x_s-\Delta_s}^{t-x_s}[1-F_{i_s}(t-u_s)]\,du_s$$

$$=\mu^k\prod_{s=1}^{k}\int_{x_s}^{x_s+\Delta_s}[1-F_{i_s}(y_s)]dy_s\le\mu^k[1-F_{i_1}(x_1)]\ldots[1-F_{i_k}(x_k)].\Delta_1.\Delta_2\ldots\Delta_k,\quad k=1,\ldots,n,$$

from which the statement follows.

Corollary 1. *The steady-state distributions* (7), (8), (9) *possess densities $q^{(1)}_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1,\ldots,x_k)$, $q^{(2)}_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1,\ldots,x_k)$, and $q^{(3)}_{i_1,\ldots,i_k}(x_1,\ldots,x_k)$, respectively.*

P r o o f. If we take the ergodic functions (7), (8), (9) as initial distributions, then these distributions remain unchanged at any time t. By choosing $t>\max(x_1 \ldots, x_k)$, Theorem 2 provides the existence of the corresponding densities, $(x_1, \ldots, x_k)\in R^k_+$, $k=1, \ldots, n$.

In order to formulate the next theorem, introduce some further notation, namely

$$q^{(1)*}_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1,\ldots,x_k)=\frac{q^{(1)}_{i_1,\ldots,i_k;j_1,\ldots,_{n-k}}(x_1,\ldots,x_k)}{[1-F_{i_1}(x)]\ldots[1-F_{i_k}(x_k)]}; \tag{10}$$

$$q^{(2)*}_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1,\ldots,x_k)=\frac{q^{(2)}_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1,\ldots,x_k)}{[1-F_{i1}(x_1)]\ldots[1-F_{i_k}(x_k)]}; \tag{11}$$

$$q^{(3)*}_{i_1,\ldots,i_k}(x_1,\ldots,x_k)=\frac{q^{(3)}_{i_1,\ldots,i_k}(x_1,\ldots,x_k)}{[1-F_{i_1}(x_1)]\ldots[1-F_{i_k}(x_k)]}, \tag{12}$$

which are the so-called normed density functions, $k=1, \ldots, n$. Then we have:

T h e o r e m 3. *The normed density functions* (10), (11), (12) *satisfy the following integro-differential equations* (13), (15), (17) *with boundary conditions* (14), (16) (18), *respectively. For FCFS discipline:*

$$\left[\frac{\partial}{\partial x_1}+\ldots+\frac{\partial}{\partial x_k}\right]^* q^{(1)*}_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1,\ldots,x_k) \tag{13}$$

$$=-\mu_k q^{(1)*}_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1,\ldots,x_k)+\int_0^\infty q^{(1)*}_{i_1,\ldots,j'_{n-k},\ldots,i'_k;j_1,\ldots,j_{n-k-1}}(x'_1,$$

$$\ldots, y', \ldots, x'_k)dF_{j_{n-k}}(y),$$

$$q^{(1)*}_{i1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1,\ldots,x_{l-1}, 0, x_{l+1},\ldots,x_k) \tag{14}$$

$$=\mu_{k-1}q^{(1)*}_{i_1,\ldots,i_{l-1},il+1\ldots,i_k,i_l,j_1,\ldots,j_{n-k}}(x_1,\ldots,x_{l-1}, x_{l+1}\ldots,x_k),$$

$$\text{for } l=1,\ldots,k, \quad k=1,\ldots,n,$$

$$\mu_0 Q^{(1)}_{0;j_1,\ldots,j_n}=\int_0^\infty q^{(1)}_{jn;j_1,\ldots,j_{n-1}}(y)dF_{j_n}(y).$$

For LCFS discipline:

$$\left[\frac{\partial}{\partial x_1}+\ldots+\frac{\partial}{\partial x_k}\right]^* q^{(2)}_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1,\ldots,x_k) \tag{15}$$

$$=-\mu_k q^{(2)*}_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1,\ldots,x_k)$$

$$+\int_0^\infty q^{(2)*}_{i'_1,\ldots,j'_{n-k},\ldots,i'_k;j_1,\ldots,j_{n-k-1}}(x'_1,\ldots,y',\ldots,x'_k)dF_{j_{n-k}}(y),$$

$$q^{(2)*}_{i_1,\ldots,i_k;j_1,\ldots,j_{n-k}}(x_1,\ldots,x_{l-1}, 0, x_{l+1},\ldots,x_k) \tag{16}$$

$$=\mu_{k-1}q^{(2)*}_{i_1,\ldots,i_{l-1},i_{l+1}\ldots,i_k;j_1,\ldots,j_{n-k},i_l}(x_1,\ldots,x_{l-1}, x_{l+1},\ldots,x_k),$$

$$\text{for } l=1,\ldots,k, \quad k=1,\ldots,n,$$

$$\mu_0 Q^{(2)}_{0;\,j_1,\ldots,j_n}=\int\limits_0^\infty q^{(2)*}_{j_n;\,j_1,\ldots,j_{n-1}}(y)dF_{j_n}(y).$$

For RS discipline:

(17) $$\Big[\frac{\partial}{\partial x_1}+\ldots+\frac{\partial}{\partial x_k}\Big]^* q^{(3)*}_{i_1,\ldots,i_k}(x_1,\ldots,x_k)=-\mu_k q^{(3)*}_{i_1,\ldots,i_k}(x_1,\ldots,x_k)$$

$$\times\sum_{j\neq i_1,\ldots,i_k}\int\limits_0^\infty q^{(3)*}_{i'_1,\ldots,j',\ldots,i'_k}(x'_1,\ldots,y',\ldots,x'_k)dF_j(y),$$

$$q^{(3)*}_{i_1,\ldots,i_k}(x_1,\ldots,x_{l-1},0,x_{l+1},\ldots,x_k)=\mu_{k-1}/n-k+1$$

$$\times q^{(3)*}_{i_1,\ldots,i_{l-1},i_{l+1},\ldots,i_k}(x_1,\ldots,x_{l-1},x_{l+1},\ldots,x_k);$$

$$for\ l=1,\ldots,k,\quad k=1,\ldots,n,$$

(18) $$\mu_0 Q^{(3)}_0=\sum_{j=1}^n\int\limits_0^\infty q^{(3)*}_j(y)dF_j(y).$$

The symbol [ ] will be explained later while $(i'_1,\ldots,j'_{n-k},\ldots,i'_k)$ denotes the lexicographical order of indices $(i_1,\ldots,i_k,j_{n-k})$ and $(x'_1,\ldots,y',\ldots x'_k)$ indicates the corresponding times.*

Proof. Since the processes (1), (3), (5) are Markovian, their densities must satisfy the Chapman-Kolmogorov equations. A derivation is based on the examination of the sample paths of the processes during an infinitesimal interval of width h. The following relations hold:

(19) $$q^{(1)}_{i_1,\ldots,i_k;\,j_1,\ldots,j_{n-k}}(x_1+h,\ldots,x_k+h)=q^{(1)}_{i_1,\ldots,i_k;\,j_1,\ldots,j_{n-k}}(x_1,\ldots,x_k)$$

$$\times(1-\mu_k h)\prod_{l=1}^k\frac{1-F_{i_l}(x_l+h)}{1-F_{i_l}(x_l)}+\prod_{l=1}^k\frac{1-F_{i_l}(x_l+h)}{1-F_{i_l}(x_l)}$$

$$\times\int\limits_0^\infty q^{(1)}_{i'_1,\ldots,j'_{n-k},\ldots,i'_k;\,j_1,\ldots,j_{n-k-1}}(x'_1,\ldots,y',\ldots,x_k)\frac{F_{j_{n-k}}(y+h)-F_{j_{n-k}}(y)}{1-F_{j_{n-k}}(y)}dy+o(h),$$

$$q^{(1)}_{i_1,\ldots,i_k;\,j_1,\ldots,j_{n-k}}(x_1+h,\ldots,x_{l-1}+h,0,x_{l+1}+h,\ldots,x_k+h)h$$

$$=\mu_{k-1}h\,q^{(1)}_{i_1,\ldots,i_{l-1},i_{l+1},\ldots,i_k;\,i_l,j_1,\ldots,j_{n-k}}(x_1,\ldots,x_{l-1},x_{l+1},\ldots,x_k)$$

(20) $$\times\prod_{\substack{s=1\\s\neq l}}^k\frac{1-F_{i_s}(x_s+h)}{1-F_{i_s}(x_s)}+o(h),$$

$$Q^{(1)}_{0;\,j_1,\ldots,j_n}=Q^{(1)}_{0;\,j_1,\ldots,j_n}(1-\mu_0 h)$$

$$+\int\limits_0^\infty q^{(1)}_{j_n;\,j_1,\ldots,j_{n-1}}(y)\frac{F_{j_n}(y+h)-F_{j_n}(y)}{1-F_{j_n}(y)}dy+o(h).$$

Similarly,

$$q^{(2)}_{i_1,\ldots,i_k;\,j_1,\ldots,j_{n-k}}(x_1+h,\ldots,x_k+h)=q^{(2)}_{i_1,\ldots,i_k;\,j_1,\ldots,j_{n-k}}(x_1,\ldots,x_k)$$

$$(21)\qquad \times(1-\mu_k h)\prod_{l=1}^{k}\frac{1-F_{i_l}(x_l+h)}{1-F_{i_l}(x_l)}+\prod_{l=1}^{k}\frac{1-F_{i_l}(x_l+h)}{1-F_{i_l}(x_l)}$$

$$\times\int_0^\infty q^{(2)}_{i'_1,\dots,j'_{n-k},\dots,i'_k;j_1,\dots,j_{n-k-1}}(x'_1,\dots,y',\dots,x'_k)\frac{F_{j_{n-k}}(y+h)-F_{j_{n-k}}(y)}{1-F_{j_{n-k}}(y)}\,dy+o(h),$$

$$q^{(2)}_{i_1,\dots,i_k;j_1,\dots,j_{n-k}}(x_1+h,\dots,x_{l-l}+h,\ 0,\ x_{l+1}+h,\dots,x_k+h)h$$

$$=\mu_{k-1}hq^{(2)}_{i_1,\dots,i_{l-1},i_{l+1},\dots,i_k;j_1,\dots,j_{n-k},i_l}(x_1,\dots,x_{l-1},x_{l+1},\dots,x_k)$$

$$(22)\qquad \times\prod_{\substack{s=1\\ s\neq l}}^{k}\frac{1-F_{is}(x_s+h)}{1-F_{i_s}(x_s)}+o(h),$$

$$Q^{(2)}_{0;j_1,\dots,j_n}=Q^{(2)}_{0;j_1,\dots,j_n}(1-\mu_0 h)+\int_0^\infty q^{(2)}_{j_n;j_1,\dots;j_{n-1}}(y)\frac{F_{j_n}(y+h)-F_{j_n}(y)}{1-F_{j_n}(y)}dy+o(h).$$

Finally,

$$(23)\qquad q^{(3)}_{i_1,\dots,i_k}(x_1+h,\dots,x_k+h)=q^{(3)}_{i_1,\dots,i_k}(x_1,\dots,x_k)(1-\mu_k h)\prod_{l=1}^{k}\frac{1-F_{i_l}(x_l+h)}{1-F_{i_l}(x_l)}$$

$$+\prod_{l=1}^{k}\frac{1-F_{il}(x_l+h)}{1-F_{il}(x_l)}\cdot\sum_{j\neq i_1,\dots,i_k}\int_0^\infty q^{(3)}_{i'_1,\dots,j',\dots,i'_k}(x'_1,\dots,y',$$

$$\dots,x'_k)\frac{F_j(y+h)-F_j(y)}{1-F_j(y)}dy+o(h),$$

$$q^{(3)}_{i_1,\dots,i_k}(x_1+h,\dots,x_{l-1}+h,\ 0,\ x_{l+1}+h,\dots,x_k+h)h,$$

$$=\frac{\mu_{k-1}h}{n-k+1}q^{(3)}_{i_1,\dots,i_{l-1},i_{l+1},\dots,i_k}(x_1,\dots,x_{l-1},x_{l+1},\dots,x_k)$$

$$(24)\qquad \times\prod_{\substack{s=1\\ s\neq l}}^{k}\frac{1-F^{*}_{i_s}(x_s+h)}{1-F_{i_s}(x_s)}+o(h),$$

$$Q^{(3)}_0=Q^{(3)}_0(1-\mu_0 h)+\sum_{j=1}^{n}\int_0^\infty q^{(3)}_j(y)\frac{F_j(y+h)-F_j(y)}{1-F_j(y)}dy+o(h).$$

Hence the derivation of the equations (13), (15), (17) and of the boundary conditions (14), (16), (18) is quite simple. Dividing the left-hand side of eq. (19)—(24) by factor $\Pi[1-F_l(x_l+h)]$, taking the limits as $h\to 0$ and taking into account the definition of normed density functions, we get the desired result.

In the left-hand side of (13), (15), (17), used for the notation of the limit in the right-hand side, the usual notation for partial differential quotients has been applied. Strictly considering, this is not allowed since the existence of the individual partial differential quotient is not assured. This is why the operator is noted by [ ]*. Actually this is a $(1, 1, \dots, 1)\in R^k_+$ directional derivative. (See Cohen [3, p. 252)].

Let $Q^{(1)}_{i_1,\dots,i_k;j_1,\dots,j_{n-k}}$, $Q^{(2)}_{i_1,\dots,i_k;j_1\dots,j_{n-k}}$ denote the steady-state probabilities that units with indices $(i_1,\dots,i_k)$ are in the source and order of arrival of the rest to the service facility is $(j_1, \ldots, j_{n-k})$, respectively. Furthermore, denote by $Q^{(s)}_{i_1, \ldots, i_k}$, $\widehat{Q}^{(s)}_k$ the steady-state probabilities that units with indices $(i_1, \ldots, i_k)$ are staying in the source, k units are in the source, $(i_1, \ldots, i_k) \in C^n_k$ $k=0, 1, \ldots, n$, $s=1, 2, 3$, respectively. Finally, let $Q_{i_1}, \ldots, i_k$, $\widehat{Q}_k$ denote the steady-state probabilities that units with indices $(i_1, \ldots, i_k)$ are staying in the source, k units are in the source irrespective of the service discipline, respectively.

In the sequel we solve eqs. (13), (15), (17) subject to boundary conditions (14), (16), (18) to determine the steady-state distribution

$$(\widehat{Q}^{(s)}_k),\ k=0, 1, \ldots, n,\ s=1, 2, 3.$$

Theorem 4. *If* $0<1/\lambda_i<\infty$, $i=1, \ldots, n$, *then*

$$\widehat{Q}^{(s)}_k = \widehat{Q}_k = (n-k)!\ S_k / \sum_{j=0}^{n} (n-j)!\ S_j, \tag{25}$$

where $s_0=1$, $s_j = \mu_0 \ldots \mu_{j-1} \sum_{(i_1, \ldots, i_j) \in C^n_j} \frac{1}{\lambda_{i_1} \ldots \lambda_{i_j}}$, $j=1, \ldots, n$, $k=0, 1, \ldots, n$, $s=1, 2, 3$.

Proof. If we set $Q^{(s)}_{0; j_1, \ldots, j_n} = C_0$, $q^{(s)*}_{i_1, \ldots, i_k; j_1, \ldots, j_{n-k}}(x_1, \ldots, x_k) = c_k$, $s=1, 2$

$$Q^{(3)}_0 = n!\ C_0,\ q^{(3)*}_{i_1, \ldots, i_k}(x_1, \ldots, x_k) = (n-k)!\ c_k,$$

then they satisfy the equations (13), (15), (17) with boundary conditions (14), (16), (18). respectively. Moreover, it is easy to see that $c_k = \mu_0 \ldots \mu_{k-1} C_0$, $k=1, \ldots, n$. By the help of this relation it can be readily verified that

$$Q^{(s)}_{i_1, \ldots, i_k; j_1, \ldots, j_{n-k}} \frac{\mu_0 \ldots \mu_{k-1}}{\lambda_{i_1} \ldots \lambda_{i_k}} \cdot C_0,\ s=1, 2,$$

and

$$Q^{(s)}_{i_1, \ldots, i_k} = Q_{i_1, \ldots, i_k} = \frac{\mu_0 \ldots \mu_{k-1}}{\lambda_{i_1} \ldots \lambda_{i_k}} (n-k)!\ C_0,\ s=1, 2, 3.$$

So we obtain that

$$\widehat{Q}^{(s)}_k = \widehat{Q}_k = \sum_{(i_1, \ldots, i_k) \in C^n_k} Q_{i_1, \ldots, i_k} = (n-k)!\ S_k \cdot C_0,$$

where C_0 can be determined by the norming condition $\Sigma\, \widehat{Q}_k = 1$, $s=\overline{1,3}$.

It is worth noting that this distribution is insensitive to the form of the distribution functions governing the system, depending only on their means.

In particular, if

$$r_k = \begin{cases} r, & 0 \le k < n-r, \\ n-k, & n-r \le k \le n-1,\ r \in \mathrm{N},\ r \le n, \end{cases}$$

then

$$\mu_0 \ldots \mu_{k-1} = \begin{cases} (r\mu)^k, & 0 \le k < n-r, \\ \frac{r!}{(n-k)!} r^{n-k} \cdot \mu^k, & n-r \le k \le n. \end{cases}$$

Defining Λ_k by $\Lambda_k = \sum_{c^n_k} \frac{1}{\lambda_{i_1} \ldots \lambda_{i_k}}$ we obtain

$$Q_{i_1,\ldots,i_k}=\frac{(n-k)!\,r^k\mu^k}{r!\,r^{n-r}\mu^n}\cdot\frac{1}{\lambda_{i_1}\ldots\lambda_{i_k}}\Big/\Big[\sum_{j=0}^{n-r-1}\frac{(n-j)!\,r^j\mu^j}{r!\,r^{n-r}\mu^n}\Lambda_j+\sum_{j=n-r}^{n}\frac{(n-j)!\,r!\,r^{n-r}\mu^k}{r!\,r^{n-r}\mu^n}\cdot\frac{r!}{(n-j)}\Lambda_j\Big],$$

$$\text{for } 0\leq k<n-r,$$

$$Q_{i_1,\ldots,i_k}=\frac{(n-k)!\,r!\,r^{n-r}\mu^k}{r!\,r^{n-r}\cdot\mu^n}\,\frac{r!}{(n-k)!}\,\frac{1}{\lambda_{i_1},\ldots\lambda_{i_k}}$$

$$\times\Big[\sum_{j=0}^{n-r-1}\frac{(n-j)!\,r^j\mu^j}{r!\,r^{n-r}\mu^n}\Lambda_j+\sum_{j=n-r}^{n}\frac{(n-j)!\,r!\,r^{n-r}\mu^k}{r!\,r^{n-r}\mu^n}\,\frac{r!}{(n-j)!}\Lambda_j\Big]^{-1}$$

$$\text{for } n-r\leq k\leq n.$$

After elementary calculations it can be seen that the processes (1), (3), (5) are stochastically equivalent to the process treated by Sztrik [14].

Before determining the steady-state characteristics of the system we need one additional theorem. In order to formulate it we introduce some further notation. Let $Q^{(i)}$ denote the stationary probability that unit i is in the source, $i=1,\ldots,n$. It is clear that the process

$$\underline{Y}^*(t)=(\nu(t);\quad \alpha_1(t),\ldots,\ \alpha_{\nu(t)}(t))$$

is semi-Markovian with state space

$$\bigcup_{\substack{(i_1,\ldots,i_k)\in C_k^n\\ k=1,\ldots,n}}(i_1,\ldots,\ i_k)+\{0\}.$$

Let H_i be the event that unit i is in the source and $Z_{H_i}(t)$ its characteristic function, i. e.

$$Z_{H_i}(t)=\begin{cases}1 & \text{if } \underline{Y}^*(t)\in H_i\\ 0 & \text{otherwise.}\end{cases}$$

Then we have:

Theorem 5.

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T Z_{H_i}(t)dt=\frac{1/\lambda_i}{1/\lambda_i+T_i}=Q^{(i)},$$

where T_i denotes the mean sojourn time of unit i at the service facility.

Proof. The statement is a special case of a theorem concerning the expected sojourn time for semi-Markov processes, see Tomkó [16, p. 297].

Sometimes we need the long-run fraction of time the unit i spends in the source. This happens e. g. in the "machine interference model." In that case for the utilization of machine i we have

$$u_i=Q^{(i)}=\sum_{k=1}^{n}\ \sum_{i\in(i_1,\ldots,i_k)\in C_k^n}Q_{i_1,\ldots,i_k}.$$

3. Performance measures. (i) Utilization. As usual, using renewal-theoretic arguments for the server utilization, we have $u_s=1-\widehat{Q}_n$. In particular, if $F_i(x)=1-\exp(-\lambda_i x)$, $i=1,\ldots,n$, then the following relation is also valid

$$1-\widehat{Q}_n=M\partial\,.\,[M\eta^*+M\partial]^{-1},$$

where $\eta^*=\min(\eta_1,\ldots,\eta_n)$, random variable η_i denotes the source-time of unit i, $i=1,\ldots,n$, and $M\partial$ the average busy period of the server, respectively. Thus the expected length of the busy period is given by

$$M\partial=\frac{1-\widehat{Q}_n}{\widehat{Q}_n}\ \frac{1}{\Sigma\ \lambda_i}\,.$$

(ii) Mean sojourn times. By the virtue of Theorem 5 we obtain $Q^{(i)}=(1+\lambda_i T_i)^{-1}$. Consequently, the average sojourn time of unit i at the service facility is

$$T_i=(1-Q^{(i)})(\lambda_i Q^{(i)})^{-1},\quad \text{for } i=1,\ldots,n.$$

Since $\Sigma_{i=1}^{n}(1-Q^{(i)})=\overline{n}$, where $\overline{n}$ denotes the mean number of units staying at the service facility, by reordering and adding we have

$$\sum_{i=1}^{n}\lambda_i T_i Q^{(i)}=\overline{n}.$$

This is Little's formula for the finite-source $\vec{G}/M/1$ queue. In particular, if $r_k=1$, viz. $\mu_k=\mu$, $k=0, 1,\ldots, n-1$ for the mean waiting time of unit i we obtain $W_i=T_i-1/\mu$.

4. Erlang's loss formulas. Let us consider an $M/\vec{G}/n/n$ loss system in which the probability that a customer calls for service in the time interval $(t, t+h)$, provided that at time t k requests are under service, is $\mu_k h+o(h)$. An arriving unit randomly selects among the idle servers and it is lost if all servers are busy. The servers are independent and stochastically different; the i-th server is characterized by $1/\lambda_i$ mean service time; $i=1,\ldots, n$. Notice that there is an analogy between the 2.3 case and this problem. Therefore for the steady-state probability that k servers are busy we obtain (25), which can also be written as

$$\widehat{Q}_k=\frac{(n-k)!}{n!}S_k\widehat{Q}_0.$$

This coincides with the result derived by Sahbazov [10].
Consequently, the probability of refusal can be given by

$$\widehat{Q}_n=\frac{\mu_0\ldots\mu_{n-1}}{\lambda_1\ldots\lambda_n}\ \frac{\widehat{Q}_0}{n!}\,.$$

The busy periods can now be considered for individual servers or for the system as a whole. Considering the system as a whole first, it will be empty only when there are no customers in the system, and will be busy otherwise. The average length of busy period for the system can be derived by $M\partial=(1-\widehat{Q}_0)(\mu_0\widehat{Q}_0)^{-1}$, since the system utilization is

$$U=1-\widehat{Q}_0=M\partial/[1/\mu_0+M\partial].$$

When individual servers are considered, they will alternate between busy and idle periods. Applying Theorem 5 for the utilization of server i, which is the long-run fraction of time the i-th server is busy, we have

$$U_i=Q^{(i)}=\sum_{k=1}^{n}\ \sum_{i\in(i_1,\ldots,i_k)\in C_k^n}Q_{i_1,\ldots,i_k}=\frac{1/\lambda_i}{1/\lambda_i+I_i},$$

where I_i denotes the expected idle period length of server i. Hence

$$I_i=(1-Q^{(i)})(\lambda_i Q^{(i)})^{-1},\quad \text{for } i=1,\ldots,n.$$

Clearly the total utilization is $U_t=\Sigma\,U_i$, which is the mean number of busy servers.

Corollary 2. *If* $\lambda_1=\ldots=\lambda_n=\lambda$ *then* (**25**) *implies that*

$$\widehat{Q}_k=\frac{\mu_0\ldots\mu_{k-1}}{k!\,\lambda^k}\widehat{Q}_0,$$

which can be found in [8], p. 152.

Corollary 3. *If* $\mu_0 = \ldots = \mu_{n-1} = \mu$ *then* (25) *can be written as*

$$\widehat{Q}_k = \frac{(n-k)!}{n!} \mu^k \Lambda_k \widehat{Q}_0,$$

which is due to Fakinos [4].

Corollary 4. *If* $\lambda_1 = \ldots = \lambda_n = \lambda$ *and* $\mu_0 = \ldots = \mu_{n-1} = \mu$ *then* (25) *coincides with the result of* Sevastyanov [12], *e. g.*

$$\widehat{Q}_k = \frac{(\mu/\lambda)^k/k!}{\sum_{j=0}^{n} (\mu/\lambda)^j/j!}.$$

5. Optimization problem. In this section the service times in an $M/\vec{G}/n/n$ system are supposed to be exponentially distributed random variables. We assume that each customer when he arrives goes to the available server with lowest index and remains there until his service is completed. If $n=2$ the stationary state equations are as follows:

$$Q_0\mu_0 = Q_1\lambda_1 + Q_2\lambda_2, \qquad Q_2(\lambda_2+\mu_1) = Q_{1.2}\lambda_1,$$
$$Q_1(\lambda_1+\mu_1) = \mu_0 Q_0 + Q_{1.2}\lambda_2, \qquad Q_{1.2}(\lambda_1+\lambda_2) = Q_1\mu_1 + Q_2\mu_1.$$

It can be readily verified that the solution to this system is

$$Q_1 = \frac{\mu_0}{\lambda_1} \cdot \frac{\lambda_1+\lambda_2+\mu_1}{2\mu_1+\lambda_1+\lambda_2} Q_0, \qquad Q_2 = \frac{\mu_1\mu_0}{\lambda_2} \frac{1}{2\mu_1+\lambda_1+\lambda_2} Q_0,$$
$$Q_{1.2} = \frac{\lambda_2+\mu_1}{\lambda_1} \cdot \frac{\mu_1\mu_0}{\lambda_2} \cdot \frac{1}{2\mu_1+\lambda_1+\lambda_2} Q_0,$$

where Q_0 can be obtained with the aid of the norming condition

$$Q_0 + Q_1 + Q_2 + Q_{1.2} = 1.$$

So

$$\widehat{Q}_1 = \frac{\mu_0}{2\mu_1+\lambda_1+\lambda_2} \frac{(\mu_1+\lambda_2)(\lambda_1+\lambda_2)}{\lambda_1\lambda_2} \widehat{Q}_0,$$
$$\widehat{Q}_2 = \frac{(\mu_1+\lambda_2)\mu_0\mu_1}{\lambda_1\lambda_2(2\mu_1+\lambda_1+\lambda_2)} \widehat{Q}_0.$$

Clearly, this distribution differs from (25).

In practical applications the probability of loss is of great importance, therefore we have:

Theorem 6. *If* $n \geq 2$ *and* $\mu_0 = \ldots = \mu_{n-1} = \mu$ *then* $\widehat{Q}_n$ *is minimal when*

$$\lambda_1 > \lambda_2 > \ldots > \lambda_n.$$

Proof. Notice that this model corresponds to an exponential finite-source preemptive priority queueing system with homogeneous service, in which the i-th customer has priority over customers of index higher than i. Since

$$U_s = 1 - \widehat{Q}_n = M\partial[M\partial + 1/\sum_{i=1}^{n} \lambda_i]^{-1},$$

our statement follows from Theorem 3 in Asztalos [1].

REFERENCES

1. D. Asztalos. Optimal control of finite source priority queues with Computer system applications. *Comp. Maths. Appl.*, **6**, 1980, 425-431.
2. B. D. Bunday, R. E. Scraton. The $G/M/r$ machine interference model. *Eur. J. Operations Res.*, **4**, 1980, 399-402.
3. J. W. Cohen. The multiple phase service network with generalized processor sharing. *Acta Informatica*, **12**, 1979, 245-284.
4. D. Fakinos. The $M/G/k$ blocking system with heterogeneous servers. *J. Operat. Res. Soc.*, **31**, 1980, 919-927.
5. P. Franken, D. König, U. Arndt, V. Schmidt. Queues and Point Processes. Berlin, 1981.
6. B. V. Gnedenko, I. N. Kovalenko. Lectures on Queuing Systems. Kiev, 1963 (in Russian).
7. B. V. Gnedenko. On a generalization of Erlang's formula. *Zast. Mat.*, **12**, 1971, 239-242 (in Russian).
8. G. I. Ivcsenko, V. A. Kastanov, I. N. Kovalenko. Theory of Queueing Systems. M. 1982 (in Russian).
9. H. Kameda. A Finite-Source Queue with Different Customers. *J. ACM*, **29**, 1982, 478-491.
10. A. A. Sahbazov. Loss systems with different service rates. *Zast. Mat.*, **18**, 1984, 177-186 (in Russian).
11. P. Schatte. On the finite population $GI/M/1$ queue and its application to multiprogrammed computers. *Elektr. Inform. Kybern.*, **16**, 1980, 433-441.
12. B. A. Sevastyanov. An ergodic theorem for Markov processes and its application to telephone system with refusals. *Theory Prob. Appl.*, **2**, 1957, 104-112 (in Russian).
13. J. Sztrik. Probability model for non-homogeneous multiprogramming computer system. *Acta Cybernetica*, **6**, 1983, 93-101.
14. J. Sztrik. On the finite-source $\vec{G}/M/r$ queue. *Eur. J. Operations Res.*, **20**, 1985, 261-268.
15. L. Takács. Introduction to the Theory of Queues. New York, 1962.
16. J. Tomkó. On sojourn times for semi-Markov processes, 14-th European Meeting of Statisticians. Wroclaw, Poland, 1981.

Department of Mathematics
University of Debrecen
Pf. 12., Debrecen, Hungary, H-4010

Received 13. 5. 1985