Provided for non-commercial research and educational use.
Not for reproduction, distribution or commercial use.

Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

The attached copy is furnished for non-commercial research and education use only.
Authors are permitted to post this version of the article to their personal websites or institutional repositories and to share with other researchers in the form of electronic reprints.
Other uses, including reproduction and distribution, or selling or licensing copies, or posting to third party websites are prohibited.

For further information on
Serdica Bulgaricae Mathematicae Publicationes
and its new series Serdica Mathematical Journal
visit the website of the journal http://www.math.bas.bg/~serdica
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

EXTREMAL PROBLEMS FOR CERTAIN NEVANLINNA ANALYTIC FUNCTIONS

MAXWELL O. READE, PAVEL G. TODOROV

This paper deals in detail with the results announced in our paper [1]. We determine sharp bounds for some basic functionals defined for all analytic functions having the forms $f(z)=\int_{-1}^{1}d\mu(t)/(z-t)$ and $\varphi(z)=\int_{-1}^{1}zd\mu(t)/(1-tz)$, respectively, $\mu(t)$ is a probability measure on $[-1,1]$.

1. Introduction. Let us consider the class N_1 of Nevanlinna analytic functions

$$f(z)=\int_{-1}^{1}\frac{d\mu(t)}{z-t},\ z\notin\{z\,|-1\leqq z\leqq 1\}, \tag{1}$$

where $\mu(t)$ is a probability measure on $[-1,1]$. According to a theorem of Thale (see [2], p. 234-235, Theorem 2.3) the open disc $|z|>1$ is the maximal domain of univalence of the class N_1. If in (1) we replace z by $1/z$, then we obtain the class N_2 of associated functions

$$\varphi(z)=f\left(\frac{1}{z}\right)=\int_{-1}^{1}\frac{zd\mu(t)}{1-tz},\ z\notin\{z\,|\,z\leqq -1,\ z\geqq 1\} \tag{2}$$

for which the open disc $|z|<1$ is the maximal domain of univalence. We have found the radii of starlikeness and convexity of order alpha of the classes N_1 and N_2 in our papers [3] and [4]. We have also proved in [3] and [4] that the functions $\varphi(z)$ of the class N_2 as well as the functions $z\varphi'(z)$ are typically real in the init disc $|z|<1$. G. Goluzin [5], Shu Shao-Peh [6] and M. Remizova [7] have found sharp estimates for the modulus and the arguments of the typically real functions and their derivatives. Note that the extremal functions do not belong to the class N_2. For an arbitrary fixed z, $|z|<1$, sharp bounds on the functionals

$$|\varphi(z)|,\ |\varphi'(z)|,\ \arg\varphi(z),\ \arg\varphi'(z),\ \operatorname{Re}\varphi(z),\ |\operatorname{Im}\varphi(z)|$$

over the class N_2 do exist as well as bounds over the class N_1. With the help of methods due to G. Goluzin [5] we have already found the sharp upper bounds on $|\varphi(z)|$, $|\varphi'(z)|$ and $|\operatorname{Im}\varphi(z)|$ and the sharp lower and upper bounds on $\arg\varphi(z)$ and $\arg\varphi'(z)$ [8]. Now with the help of ideas due to the M. Remizova [7] we find the sharp lower and upper bounds on the functionals

$$|\varphi(z)|,\ \operatorname{Re}\varphi(z),\ |\operatorname{Im}\varphi(z)|,\ |\varphi'(z)|$$

obtaining again some of our earlier results in [8].

2. Estimates for $|\varphi(z)|$, $\arg\varphi(z)$, $|\operatorname{Im}\varphi(z)|$ and $\operatorname{Re}\varphi(z)$. Note that the kernels of the integral representations (1) and (2), i. e. the functions

$$k(z,t)=\frac{1}{z-t},\ l(z,t)=\frac{z}{1-tz}, \tag{3}$$

play a leading role in this paper.

SERDICA Bulgaricae mathematicae publicationes. Vol. 12, 1986, p. 390—398.

Theorem 1. I. *For a fixed* z, $|z|<1$, $\operatorname{Im} z\neq 0$, *the region* Δ *of values of the functional* $\varphi(z)$ *is the convex hull of the curve* $w=l(z,t)$, $-1\leq t\leq 1$, *i. e.* Δ *is the segment of the disc with center at the point* $-i/2\operatorname{Im}(1/z)$ *and radius* $1/2|\operatorname{Im}(1/z)|$ *joining the points of the circle with coordinates*

$$A=\frac{z}{1+z},\ B=\frac{z}{1-z},\ 0\notin \widehat{AB}. \tag{4}$$

II. *For a fixed* z, $|z|<1$, $\operatorname{Im} z=0$, *the region* Δ *of values of the functional* $\varphi(z)$ *is the rectilinear segment* $\overline{AB}$ *of the real axis with endpoints (4).*

Proof. The theorem follows from the now classic Ašnevic-Ulina theorem for the region of values of functionals represented by Stieltjes integrals [9].

Theorem 2. *For a given* z *from the disc* $|z|<1$ *and for each function* $\varphi\in N_2$ *we have the following inequalities*

$$\left|\frac{z}{1+z}\right|\leq|\varphi(z)|\leq\left|\frac{z}{1-z}\right|,\ \text{if } \left|z-\frac{1}{2}\right|\leq\frac{1}{2}, \tag{5}$$

$$\frac{|\operatorname{Im} z|}{|1-z^2|}\leq|\varphi(z)|\leq\frac{1}{\left|\operatorname{Im}\frac{1}{z}\right|},\ \text{if } \left|z\pm\frac{1}{2}\right|\geq\frac{1}{2}; \tag{6}$$

$$\left|\frac{z}{1-z}\right|\leq|\varphi(z)|\leq\left|\frac{z}{1+z}\right|,\ \text{if } \left|z+\frac{1}{2}\right|\leq\frac{1}{2}, \tag{7}$$

where for $z\neq 0$ *the equalities hold true only*:

i) *for the functions* $\varphi(z)=l(z,-1)$ *and* $\varphi(z)=l(z,1)$ *on the left-hand side and the right-hand side of (5), respectively;*

ii) *for the functions*

$$\varphi(z)=i\frac{z^2}{1-z^2}\operatorname{Im}\frac{1}{z},\quad \varphi(z)=-\frac{i}{\operatorname{Im}\frac{1}{z}} \tag{8}$$

on the left-hand side and on the right-hand side of (6), respectively;

iii) *for the functions* $\varphi(z)=l(z,1)$ *and* $\varphi(z)=l(z,-1)$ *on the left-hand side and on the right-hand side of (7), respectively.*

Proof. Since from (2) we can obtain

$$\varphi(\bar{z})=\overline{\varphi(z)}, \tag{9}$$

it follows that it is sufficient only to consider the case $\operatorname{Im} z\geq 0$, $|z|<1$. For $\operatorname{Im} z=0$, $|z|<1$, the assertions (5) and (7) follow from Theorem 1 (II) mentioned above. For $\operatorname{Im} z>0$, $|z|<1$, according to the above Theorem 1 (I), the region Δ belongs to the disc

$$\left|w+\frac{i}{2\operatorname{Im}\frac{1}{z}}\right|\leq-\frac{1}{2\operatorname{Im}\frac{1}{z}}. \tag{10}$$

Hence in this case the assertions (5) and (7) follow from the inequalities

$$|A|\leq|\varphi(z)|\leq|B|,\ \text{for } \left[\left|z-\frac{1}{2}\right|\leq\frac{1}{2}\right]\cap[\operatorname{Im} z>0]; \tag{5'}$$

$$|B|\leq|\varphi(z)|\leq|A|,\ \text{for } \left[\left|z+\frac{1}{2}\right|\leq\frac{1}{2}\right]\cap[\operatorname{Im} z>0]; \tag{7'}$$

and (4). The assertion (6) follows from the inequalities

$$|C|\leq|\varphi(z)|\leq|D|,\ \text{for } \left[\left|z+\frac{1}{2}\right|\geq\frac{1}{2}\right]\cap[\operatorname{Im} z>0], \tag{6'}$$

where C is the foot of the perpendicular from the origin to the chord $\overline{AB}$, and D is the point of intersection of the arc $\widehat{AB}$ with the positive imaginary half-axis. In fact, from (10) it follows that the coordinates of D are

$$D = -\frac{i}{\operatorname{Im}\frac{1}{z}}, \tag{11}$$

i. e. we obtain the second extremal function in (8). The modulus of C is

$$|C| = |B-A|\,\left|\operatorname{Im}\frac{A}{B-A}\right| = \frac{\operatorname{Im} z}{|1-z^2|}, \text{ if } \operatorname{Im} z > 0. \tag{12}$$

The coordinates of C are

$$C = \lambda l(z, -1) + (1-\lambda)\, l(z, 1), \tag{13}$$

for some λ, $0 \leq \lambda \leq 1$, which satisfies the condition

$$|\lambda l(z, -1) + (1-\lambda)\, l(z, 1)| = |C|. \tag{14}$$

From (12) and (14) we find that

$$\lambda = \frac{1}{2}\left(1 + \operatorname{Re}\frac{1}{z}\right). \tag{15}$$

Thus from (13) and (15) we obtain

$$C = i\frac{z^2}{1-z^2}\operatorname{Im}\frac{1}{z}, \tag{16}$$

i. e. we obtain the first extremal function in (8).

This completes the proof ot Theorem 2.

For the functions $\varphi \in N_2$, L. Dundučenko (see [10], p. 38. Theorem 2) obtained the inequalities

$$\frac{|z|}{1+|z|} \leq |\varphi(z)| \leq \frac{|z|}{1-|z|}, \text{ if } |z| < 1, \tag{17}$$

where for $z \neq 0$ the equalities are attained only by the functions $\varphi(z) = z/(1 \pm z)$ at the points $z = \pm r$, $0 < r < 1$, on the left-hand side and at the points $z = \mp r$, $0 < r < 1$, on the right-hand side, respectively. By comparison with Dundučenko's inequalities (17), it is clear that our inequalities (5)—(7) are sharper.

From the geometric considerations used in the proof of Theorem 2 we obtain

Theorem 3. *For a given $z \neq 0$ from the disc $|z| < 1$, and for each function $\varphi \in N_2$, the inequalities*

$$\arg\frac{z}{1+z} \leq \arg\varphi(z) \leq \arg\frac{z}{1-z}, \text{ if } \operatorname{Im} z \geq 0, \tag{18}$$

and

$$\arg\frac{z}{1-z} \leq \arg\varphi(z) \leq \arg\frac{z}{1+z}, \text{ if } \operatorname{Im} z \leq 0, \tag{19}$$

hold true, where for $\operatorname{Im} z \neq 0$ the equalities hold only for the functions $\varphi(z) = l(z, \pm 1)$ respectively.

Theorem 3 was obtained by us in an earlier paper by another method [8].

Theorem 4. *For a given $z \neq 0$ of the disc $|z| < 1$ and for each function $\varphi \in N_2$, the inequalities*

$$\frac{|\operatorname{Im} z|}{|1+z|^2} \leqq |\operatorname{Im} \varphi(z)| \leqq \frac{|\operatorname{Im} z|}{|1-z|^2}, \quad \textit{for } \left|z-\frac{1}{2}\right| \leqq \frac{1}{2}; \tag{20}$$

$$\frac{|\operatorname{Im} z|}{|1+z|^2} \leqq |\operatorname{Im} \varphi(z)| \leqq \frac{1}{\left|\operatorname{Im} \frac{1}{z}\right|}, \quad \textit{for } \left[\left|z-\frac{1}{2}\right| \geqq \frac{1}{2}\right] \cap [\operatorname{Re} z \geqq 0]; \tag{21}$$

$$\frac{|\operatorname{Im} z|}{|1-z|^2} \leqq |\operatorname{Im} \varphi(z)| \leqq \frac{1}{\left|\operatorname{Im} \frac{1}{z}\right|}, \quad \textit{for } \left[\left|z+\frac{1}{2}\right| \geqq \frac{1}{2}\right] \cap [\operatorname{Re} z \leqq 0]; \tag{22}$$

$$\frac{|\operatorname{Im} z|}{|1-z|^2} \leqq |\operatorname{Im} \varphi(z)| \leqq \frac{|\operatorname{Im} z|}{|1+z|^2}, \quad \textit{for } \left|z+\frac{1}{2}\right| \leqq \frac{1}{2}, \tag{23}$$

hold true, where for $\operatorname{Im} z \neq 0$ *the equalities hold true only:*

i) *for the functions* $\varphi(z) \equiv l(z, -1)$ *and* $\varphi(z) \equiv l(z, 1)$ *on the left-hand side and on the right-hand side of (20), respectively;*

ii) *for the function* $\varphi(z) \equiv l(z, -1)$ *and the second function in (8) on the left-hand side and on the right-hand side of (21), respectively; in particular, for* $\operatorname{Re} z = 0$, *again for the functions* $\varphi(z) \equiv l(z, \pm 1)$ *at the points* $z = \pm ir$ *in the left-hand side of (21);*

iii) *for the function* $\varphi(z) \equiv l(z, 1)$ *and the second function in (8) on the left-hand side and on the right-hand side of (22), respectively; in particular, for* $\operatorname{Re} z = 0$, *again for the functions* $\varphi(z) \equiv l(z, \pm 1)$ *at the points* $z = \pm ir$ *on the left-hand side of (22);*

iv) *for the functions* $\varphi(z) \equiv l(z, 1)$ *and* $\varphi(z) \equiv l(z, -1)$ *on the left-hand side and on the right-hand side of (23), respectively.*

Theorem 5. a) *For a given* $z \neq 0$ *from the half-disc* $[z \mid |z| < 1, \operatorname{Im} z \geqq 0]$, *and for each function* $\varphi \in N_2$ *we have the inequalities:*

I. *If* $|z-(1/2)| \leqq 1/2$, *then*

$$\frac{\operatorname{Re} z+|z|^2}{|1+z|^2} \leqq \operatorname{Re} \varphi(z) \leqq \frac{\operatorname{Re} z-|z|^2}{|1-z|^2}, \quad \textit{if } \left|z-\frac{1}{\sqrt{2}} e^{-i\frac{\pi}{4}}\right| \leqq \frac{1}{\sqrt{2}}; \tag{24}$$

$$\frac{\operatorname{Re} z+|z|^2}{|1+z|^2} \leqq \operatorname{Re} \varphi(z) \leqq \frac{1}{2\left|\operatorname{Im} \frac{1}{z}\right|}, \quad \textit{for} \tag{25}$$

$$\left[\left|z+\frac{1}{\sqrt{2}} e^{-i\frac{\pi}{4}}\right| \geqq \frac{1}{\sqrt{2}}\right] \cap \left[\left|z-\frac{1}{\sqrt{2}} e^{-i\frac{\pi}{4}}\right| \geqq \frac{1}{\sqrt{2}}\right] \cap \left[\left|z^2-\frac{1}{2}\right| \leqq \frac{1}{2}\right];$$

$$\frac{\operatorname{Re} z-|z|^2}{|1-z|^2} \leqq \operatorname{Re} \varphi(z) \leqq \frac{1}{2\left|\operatorname{Im} \frac{1}{z}\right|}, \quad \textit{for} \tag{26}$$

$$\left[\left|z+\frac{1}{\sqrt{2}} e^{-i\frac{\pi}{4}}\right| \geqq \frac{1}{\sqrt{2}}\right] \cap \left[\left|z-\frac{1}{\sqrt{2}} e^{-i\frac{\pi}{4}}\right| \geqq \frac{1}{\sqrt{2}}\right] \cap \left[\left|z^2-\frac{1}{2}\right| \geqq \frac{1}{2}\right];$$

$$\frac{\operatorname{Re} z-|z|^2}{|1-z|^2} \leqq \operatorname{Re} \varphi(z) \leqq \frac{\operatorname{Re} z+|z|^2}{|1+z|^2}, \quad \text{if } \left|z+\frac{1}{\sqrt{2}} e^{-i\frac{\pi}{4}}\right| \leqq \frac{1}{\sqrt{2}}, \tag{27}$$

where the equalities hold true only:

i) *for the functions* $\varphi(z) \equiv l(z, -1)$ *and* $\varphi(z) \equiv l(z, 1)$ *on the left-hand side and on the right-hand side of (24), respectively;*

ii) *for the function* $\varphi(z) \equiv l(z, -1)$ *on the left-hand side of (25) and for the function*

$$\varphi(z) \equiv -\frac{1}{\sqrt{2} \operatorname{Im} \frac{1}{z}} e^{i\frac{\pi}{4}} \tag{28}$$

on the right-hand side of (25);

iii) *for the function* $\varphi(z)\equiv l(z,1)$ *on the left-hand side of* (26) *and for the function (28) on the right-side of (26);*

iv) *for the functions* $\varphi(z)\equiv l(z,1)$ and $\varphi(z)\equiv l(z,-1)$ *on the left-hand side and on the right-hand side of* (27), *respectively.*

II. *If* $|z\pm(1/2)|\geq 1/2$, *then*

$$\frac{\operatorname{Re} z-|z|^2}{|1-z|^2}\leq \operatorname{Re}\varphi(z)\leq \frac{1}{2\left|\operatorname{Im}\frac{1}{z}\right|},\ for \tag{29}$$

$$\left[\left|z-\frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}\right|\leq\frac{1}{\sqrt{2}}\right]\cap\left[\left|z+\frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}}\right|\geq\frac{1}{\sqrt{2}}\right];$$

$$\frac{\operatorname{Re} z-|z|^2}{|1-z|^2}\leq \operatorname{Re}\varphi(z)\leq \frac{\operatorname{Re} z+|z|^2}{|1+z|^2},\ for \tag{30}$$

$$\left[\left|z-\frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}\right|\leq\frac{1}{\sqrt{2}}\right]\cap\left[\left|z+\frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}}\right|\leq\frac{1}{\sqrt{2}}\right];$$

$$\frac{1}{2\operatorname{Im}\frac{1}{z}}\leq \operatorname{Re}\varphi(z)\leq \frac{\operatorname{Re} z+|z|^2}{|1+z|^2},\ for \tag{31}$$

$$\left[\left|z-\frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}\right|\geq\frac{1}{\sqrt{2}}\right]\cap\left[\left|z+\frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}}\right|\leq\frac{1}{\sqrt{2}}\right],$$

where the equalities hold true only:

i) *for the function* $\varphi(z)\equiv l(z,1)$ *on the left-hand side of (29) and for the function (28) on the right-hand side of (29);*

ii) *for the functions* $\varphi(z)\equiv l(z,1)$ *and* $\varphi(z)\equiv l(z,-1)$ *on the left-hand side and on the right-hand side of (30), respectively;*

iii) *for the function*

$$\varphi(z)\equiv\frac{1}{\sqrt{2}\operatorname{Im}\frac{1}{z}}e^{-i\frac{\pi}{4}} \tag{32}$$

on the left-hand side of (31) and for the function $\varphi(z)\equiv l(z,-1)$ *on the right-hand side of (31).*

III. *If* $|z+(1/2)|\leq 1/2$, *then*

$$\frac{\operatorname{Re} z+|z|^2}{|1+z|^2}\leq \operatorname{Re}\varphi(z)\leq \frac{\operatorname{Re} z-|z|^2}{|1-z|^2},\ for\ \left|z+\frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}\right|\leq\frac{1}{\sqrt{2}}; \tag{33}$$

$$\frac{1}{2\operatorname{Im}\frac{1}{z}}\leq \operatorname{Re}\varphi(z)\leq \frac{\operatorname{Re} z-|z|^2}{|1-z|^2},\ for \tag{34}$$

$$\left[\left|z-\frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}\right|\geq\frac{1}{\sqrt{2}}\right]\cap\left[\left|z+\frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}\right|\geq\frac{1}{\sqrt{2}}\right]\cap\left[z^2-\frac{1}{2}\right|\leq\frac{1}{2}];$$

$$\frac{1}{2\operatorname{Im}\frac{1}{z}}\leq \operatorname{Re}\varphi(z)\leq \frac{\operatorname{Re} z+|z|^2}{|1+z|^2},\ for \tag{35}$$

$$\left[\left|z-\frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}\right|\geq\frac{1}{\sqrt{2}}\right]\cap\left[\left|z+\frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}\right|\geq\frac{1}{\sqrt{2}}\right]\cap\left[\left|z^2-\frac{1}{2}\right|\geq\frac{1}{2}\right];$$

(36) $$\frac{\operatorname{Re} z-|z|^2}{|1-z|^2}\leq \operatorname{Re}\varphi(z)\leq \frac{\operatorname{Re} z+|z|^2}{|1+z|^2},\ \textit{if}\ \left|z-\frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}\right|\leq\frac{1}{\sqrt{2}},$$

where the equalities hold true only:

i) *for the functions* $\varphi(z)\equiv l(z,-1)$ *and* $\varphi(z)\equiv l(z,1)$ *on the left-hand side and on the right-hand side of (33), respectively;*

ii) *for the function (32) on the left-hand side of (34) and for the function* $\varphi(z)\equiv l(z,1)$ *on the right-hand side of (34);*

iii) *for the function (32) on the left-hand side of (35) and for the function* $\varphi(z)\equiv l(z,-1)$ *on the right-hand side of (35);*

iv) *for the functions* $\varphi(z)\equiv l(z,1)$ *and* $\varphi(z)\equiv l(z,-1)$ *on the left-hand side and on the right-hand side of (36), respectively.*

b). *For a given* $\bar z\neq 0$ *from the half-disc* $[|\bar z|<1]\cap[\operatorname{Im}\bar z\leq 0]$ *and for each function* $\varphi(\bar z)\in N_2$ *we obtain some relations which are obtained from those in section a) by replacing z by* $\bar z$ *and taking into account that* $\operatorname{Re}\varphi(\bar z)=\operatorname{Re}\varphi(z)$.

Remark. *The curve* $|z^2-(1/2)|=1/2$ *is a Bernoulli lemniscate.*

Proof. a). Let $z\neq 0$ be a fixed point of the half-disc $[|z|<1]\cap[\operatorname{Im} z\geq 0]$. According to Theorem 1 we have

I. if $|z-(1/2)|\leq 1/2$, then

(24′) $$\operatorname{Re} A\leq \operatorname{Re}\varphi(z)\leq \operatorname{Re} B,\ \text{where}\ \arg B\leq\frac{\pi}{4},$$

(25′) $$\operatorname{Re} A\leq \operatorname{Re}\varphi(z)\leq \operatorname{Re} E,\ \text{where}\ \arg A\leq\frac{\pi}{4},\ \arg B\geq\frac{\pi}{4},\ \operatorname{Re} A\leq \operatorname{Re} B,$$

where E is the point;

(28′) $$E=\frac{1}{2\operatorname{Im}\frac{1}{z}}-\frac{i}{2\operatorname{Im}\frac{1}{z}},$$

which is realized by the extremal function (28);

(26′) $$\operatorname{Re} B\leq \operatorname{Re}\varphi(z)\leq \operatorname{Re} E,\ \text{where}\ \arg A\leq\frac{\pi}{4},\ \arg B\geq\frac{\pi}{4},\ \operatorname{Re} B\leq \operatorname{Re} A;$$

(27′) $$\operatorname{Re} B\leq \operatorname{Re}\varphi(z)\leq \operatorname{Re} A,\ \text{where}\ \arg A\geq\frac{\pi}{4},$$

where the equalities in (24′ — 27′) hold true only for those functions indicated in i) — iv) of Theorem 5 (a, I);

II. if $|z+(1/2)|\geq 1/2$, then

(29′) $$\operatorname{Re} B\leq \operatorname{Re}\varphi(z)\leq \operatorname{Re} E,\ \text{where}\ \arg A\leq\frac{\pi}{4},\ \arg B\leq\frac{3\pi}{4};$$

(30′) $$\operatorname{Re} B\leq \operatorname{Re}\varphi(z)\leq \operatorname{Re} A,\ \text{where}\ \arg A\geq\frac{\pi}{4},\ \arg B\leq\frac{3\pi}{4};$$

(31′) $$\operatorname{Re} F\leq \operatorname{Re}\varphi(z)\leq \operatorname{Re} A,\ \text{where}\ \arg A\geq\frac{\pi}{4},\ \arg B\geq\frac{3\pi}{4};$$

where F is the point

(32′) $$F=\frac{1}{2\operatorname{Im}\frac{1}{z}}-\frac{i}{2\operatorname{Im}\frac{1}{z}},$$

which is realized by the extremal function (32), where the equalities in (29′ — 31′) hold true only for those functions indicated in i) — iii) of Theorem 5 (a, II);

III. if $|z+(1/2)|\leq 1/2$, then

$$\operatorname{Re} A\leq \operatorname{Re}\varphi(z)\leq \operatorname{Re} B, \text{ where } \arg A\geq \frac{3\pi}{4}; \tag{33′}$$

$$\operatorname{Re} F\leq \operatorname{Re}\varphi(z)\leq \operatorname{Re} B, \text{ where} \tag{34′}$$

$$\arg A\leq \frac{3\pi}{4}, \ \arg B\geq \frac{3\pi}{4}, \ \operatorname{Re} A\leq \operatorname{Re} B;$$

$$\operatorname{Re} F\leq \operatorname{Re}\varphi(z)\leq \operatorname{Re} A, \text{ where} \tag{35′}$$

$$\arg A\leq \frac{3\pi}{4}, \ \arg B\geq \frac{3\pi}{4}, \ \operatorname{Re} B\leq \operatorname{Re} A;$$

$$\operatorname{Re} B\leq \operatorname{Re}\varphi(z)\leq \operatorname{Re} A, \text{ where } \arg B\leq \frac{3\pi}{4}, \tag{36′}$$

where the equalities in (33′ — 36′) hold true only for those function indicated cases in i) — iv) of Theorem 5 (a. III).

This completes the proof of Theorem 5.

3. Estimates for $|\varphi'(z)|$ and $\arg\varphi'(z)$. From (2) we obtain

$$\varphi'(z)=\int_{-1}^{1}\frac{d\mu(t)}{(1-tz)^2}, \quad \varphi\in N_2, \tag{37}$$

Theorem 6. I. *For a fixed z, $|z|<1$, $\operatorname{Im} z\neq 0$, the region Δ' of values of the functional $\varphi'(z)$ is bounded by the curve*

$$w=\frac{1}{(1-tz)^2}, \quad -1\leq t\leq 1, \tag{38}$$

and the rectilinear segment

$$\overline{A'B'}: w=\frac{\lambda}{(1+z)^2}+\frac{1-\lambda}{(1-z)^2}, \quad 0\leq\lambda\leq 1, \tag{39}$$

where the points A' and B' are the points

$$A'=\frac{1}{(1+z)^2} \text{ and } B'=\frac{1}{(1-z)^2}. \tag{40}$$

The curve (38) is the arc $\widehat{A'B'}$ of the cardioid $w=\xi^2$ when ξ describes the arc $\widehat{A_1B_1}$, $0\notin\widehat{A_1B_1}$, of the circle

$$\left|\xi-\frac{i\bar z}{2\operatorname{Im} z}\right|=\frac{|z|}{2|\operatorname{Im} z|}, \tag{41}$$

where the points A_1 and B_1 are the points

$$A_1=\frac{1}{1+z} \text{ and } B_1=\frac{1}{1-z}. \tag{42}$$

The arc $\widehat{A'B'}$ of the cardioid $w=\xi^2$ does not pass through the origin.

II. *For a fixed z, $|z|<1$, $\operatorname{Im} z=0$, the region Δ' of values of the functional $\varphi'(z)$ is the rectilinear segment $\overline{A'B'}$ of the real axis whose ends are the points (40).*

Proof. The theorem follows from the classic Ašnevic-Ulina theorem on the region of values of functionals represented by a Stieltjes integral [9].

Theorem 7. *For a given $z \neq 0$ from the disc $|z|<1$ and for each function $\varphi \in N_2$, the inequalities*

$$\frac{1}{|1+z|^2} \leq |\varphi'(z)| \leq \frac{1}{|1-z|^2}, \text{ if } \left|z-\frac{1}{2}\right| \leq \frac{1}{2}, \tag{43}$$

$$\frac{|\operatorname{Im} z|(1-|z|^2)}{|z||1-z^2|^2} \leq |\varphi'(z)| \leq \frac{1}{\left|z \operatorname{Im}\frac{1}{z}\right|}, \text{ if } \left|z+\frac{1}{2}\right| \geq \frac{1}{2}; \tag{44}$$

$$\frac{1}{|1-z|^2} \leq |\varphi'(z)| \leq \frac{1}{|1+z|^2}, \text{ if } \left|z+\frac{1}{2}\right| \leq \frac{1}{2}, \tag{45}$$

hold true, where the equalities hold true only:

i) *for the functinos $\varphi(z) \equiv l(z, -1)$ and $\varphi(z) \equiv l(z, 1)$ on the left-hand side and on the right-hand side of (43), respectively;*

ii) *for the function*

$$\varphi(z) \equiv \lambda\frac{z}{1+z} + (1-\lambda)\frac{z}{1-z}, \text{ where } \lambda = \frac{1}{2} + \frac{(1+|z|^2)}{4|z|^2}\operatorname{Re} z, \tag{46}$$

on the left-hand side of (44) and for the function $\varphi(z) \equiv l(z, t)$, $t = \operatorname{Re}(1/z)$ on the right-hand side of (44);

iii) *for the functions $\varphi(z) \equiv l(z, 1)$ and $\varphi(z) \equiv l(z, -1)$ on the left-hand side and on the right-hand side of (45), respectively.*

Proof. From (37) we obtain the relation

$$\varphi'(\bar{z}) = \overline{\varphi'(z)} \tag{47}$$

so that it is sufficient to consider only the case $\operatorname{Im} z \geq 0$, $|z|<1$. For $\operatorname{Im} z = 0$, $|z|<1$, the inequalities (43) and (45) follow from Theorem 6 (II). For $\operatorname{Im} z > 0$, $|z|<1$, according to Theorem 6 (I), we obtain

$$|A'| \leq |\varphi'(z)| \leq |B'|, \text{ where } \arg B_1 \leq \arg D_1; \tag{43'}$$

$$|B'| \leq |\varphi'(z)| \leq |A'|, \text{ where } \arg A_1 \geq \arg D_1, \tag{45'}$$

where D_1 is the point

$$D_1 = 1 + i\frac{\operatorname{Re} z}{\operatorname{Im} z}. \tag{48}$$

Thus we obtain the assertions (43) and (45). The assertion (44) follows from the inequalities

$$|C'| \leq |\varphi'(z)| \leq |D'|, \tag{44'}$$

$$\arg B_1 \geq \arg D_1, \text{ and } \arg A_1 \leq \arg D_1,$$

where C' is the foot of the perpendicular from the origin to the chord $\overline{A'B'}$, and D' is the point $D' = D_1^2$, i. e.

$$D' = \left(1 + i\frac{\operatorname{Re} z}{\operatorname{Im} z}\right)^2. \tag{49}$$

In fact the point (49) is realized by the derivative of the function $\varphi(z) \equiv l(z, t)$ for $t = \operatorname{Re}(1/z)$. The modulus of C' is

$$|C'| = |B'-A'|\left|\operatorname{Im}\frac{A'}{B'-A'}\right| = \frac{\operatorname{Im} z\,(1-|z|^2)}{|z||1-z^2|^2}, \text{ where } \operatorname{Im} > 0. \tag{50}$$

The coordinates of C' are

(51) $$C' = \frac{\lambda}{(1+z)^2} + \frac{1-\lambda}{(1-z)^2},$$

for some λ, $0 \leqq \lambda \leqq 1$, satisfying the condition

(52) $$|C'| = \left| \frac{\lambda}{(1+z)^2} + \frac{1-\lambda}{(1-z)^2} \right|.$$

From (50) and (52) we find the number λ in (46) for which (51) becomes

(51′) $$C' = -\frac{iz \operatorname{Im} z\,(1-|z|^2)}{|z|^2(1-z^2)^2}.$$

The derivative of the function (46) yields the point (51′).

This completes the proof of Theorem 7.

From the geometric argument used in the proof of Theorem 7 the following result follows immediately.

Theorem 8. *For each given z from the disc $|z|<1$ and for each function $\varphi \in N_2$, the inequalities*

(53) $$\arg \frac{1}{(1+z)^2} \leqq \arg \varphi'(z) \leqq \arg \frac{1}{(1-z)^2}, \text{ for } \operatorname{Im} z \geqq 0,$$

and

(54) $$\arg \frac{1}{(1-z)^2} \leqq \arg \varphi'(z) \leqq \arg \frac{1}{(1+z)^2}, \text{ for } \operatorname{Im} z \leqq 0,$$

hold true, where for $\operatorname{Im} z \neq 0$ the equalities hold true only for the functions $\varphi(z) \equiv l(z, \pm 1)$, respectively.

Theorem 8 was obtained as in our paper [8] by means of another method.

4. Results for the class N_1. These results can be obtained from the preceding theorems for the class N_2 by replacing z with $1/z$.

REFERENCES

1. M. O. Reade, P. G. Todorov. Extremal problems for certain analytic (Nevanlinna) functions *Rend. Circ. Mat., Palermo, Ser. II*, **34**, 1985, 275-282.
2. J. S. Thale. Univalence of Continued Fractions and Stieltjes Transforms. *Proc. Amer. Math. Soc.*, 7, 1956, 232-244.
3. M. O. Reade, P. G. Todorov. The radii of starlikeness and convexity of certain Nevanlinna analytic functions. *Proc. Amer. Math. Soc.*, **83**, 1981, 289-295.
4. P. G. Todorov. The radii of starlikenss and convexity of order alpha of certain Nevanlinna analytic functions (submitted).
5. Г. М. Голузин. О типично вещественных функциях. *Мат. Сб.*, 27 (69), 1950, № 2, 201 — 218.
6. Shu Shao-Peh. The minimum modulus and minimum distortion for the class T_r of typically real, functions. *Acta Math. Sinica*, **6**, 1956, 313-319.
7. М. П. Ремизова. Экстремальные задачи в классе типично-вещественных функций. *Изв. ВУЗ, Математика*, 1 (32), 1985, 59-64.
8. M. O. Reade, P. G. Todorov. On certain analytic (Nevanlinna) functions. *Mich. Math. J.*, **32** 1985, 59-64.
9. И. Я. Ашневиц, Г. В. Улина. Об областях значений аналитических функций, представимых интегралом Стильтеса. *Вестник Ленингр. унив.*, **11**, 1955, 31 — 42.
10. Л. Е. Дундученко. Об одном обобщении классов аналитических функций, рассмотренных Л. Чакаловым. *Изв. Мат. инст., БАН*, **5**, 1961, 35 — 41.

Department of Mathematics
University of Michigan, Ann Arbor,
Michigan 48109, U.S.A.

Department of Mathematics,
Paissii Hilendarski University,
4000 Plovdiv, Bulgaria

Received 22. 11. 1985