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ON THE RATE OF ESCAPE OF A TRANSIENT RANDOM WALK

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An integral test for the rate of escape of a transient random walk is proved.

1. Introduction. Let $X_1, X_2, \ldots$ be independent, identically distributed symmetric non-lattice random variables with common distribution satisfying

$$P(|X_1|>x)=x^{-1}L(x), \tag{1.1}$$

where $L(x)$ is a slowly varying function for $x\to\infty$. This means that the random variable X_1 is in the domain of (non-normal) attraction of the Cauchy distribution. Then there exists a sequence of real numbers a_n such that $a_n^{-1}S_n$, where $S_n=X_1+\ldots+X_n$, converges in distribution to the Cauchy distribution. The numbers a_n satisfy, for $n\to\infty$,

$$a_n\sim cnL(a_n) \tag{1.2}$$

for some positive constant c. By a local limit theorem of C. Stone [8] we have, for $n\to\infty$,

$$P(|S_n|\leq k_n)\approx a_n^{-1}k_n \quad \text{if } a_n^{-1}k_n\to 0. \tag{1.3}$$

In [3] P. Griffin shows that for each $\delta\in(0, 1)$ there exists a random walk such that

$$\liminf_{n\to\infty} n^{-\alpha}|S_n| = \begin{cases} \infty & \text{a. s. if } \alpha<\delta \\ 0 & \text{a. s. if } \alpha>\delta. \end{cases} \tag{1.4}$$

See example 1 in section 4. In this paper we shall also consider the case $\alpha=\delta$. Griffin makes use of a criterion proved by H. Kesten in [4]. We shall borrow some of the ideas of the proof in [4] and also apply the following extension of the Borel-Cantelli lemma.

Lemma 1.1. *Let* $\{D_n\}$ *be any sequence of events with* $\Sigma P(D_n)=\infty$, *then* $P(\limsup D_n)\geq c^{-1}$ if

$$\liminf\{\sum_{k=1}^{n} P(D_k)\}^{-2}\sum_{k=1}^{n}\sum_{l=1}^{n} P(D_k\wedge D_l)\leq c. \tag{1.5}$$

See for the proof, for example, F. Spitzer [7].

The general form of a slowly varying function is given by

$$L(x)=c(x)\exp\{\int_{x_0}^{x} y^{-1}\varepsilon(y)dy\}, \tag{1.6}$$

where

$$\lim_{x\to\infty} c(x)=c\in(0, \infty) \tag{1.7}$$

and

(1.8) $$\lim_{x\to\infty} \varepsilon(x)=0.$$

See W. Feller [1]. In this paper we shall write

(1.9) $$\varepsilon(x)=\psi(x)\{\log x\}^{-1}.$$

Obviously, (1.8) implies

(1.10) $$\psi(x)=o(\log x) \quad \text{for } x\to\infty.$$

We shall make some technical assumptions on ψ. We want to complete the analysis of the example given by Griffin. Therefore we restrict ourselves to functions ψ which nearly behave like c $\log_3 x$ for $x\to\infty$. First we make the assumption

(1.11) $$\psi^2(x)(\log_2 x)(\log x)^{-1}\to 0 \quad \text{for } x\to\infty.$$

Then we have, for a_n defined by (1.2) and ψ non-decreasing,

$$(L(n))^{-1}L(a_n)=\exp\{\int_n^{a_n} y^{-1}(\log y)^{-1}\psi(y)\,dy\}\leqq \exp\{\psi(a_n)\log(\log a_n/\log n)\}$$
$$\leqq \exp\{\psi(a_n)(\log L(a_n))(\log a_n)^{-1}\}.$$

Obviously we have

(1.12) $$\lim_{n\to\infty}(L(n))^{-1}L(a_n)=1.$$

Let ψ be slowly varying with standard representation. $\psi(x)=d(x)\exp\{\int_{x_0}^{x} y^{-1}\eta(y)dy\}$, where $\lim_{y\to\infty}\eta(y)=0$ and $\lim_{x\to\infty} d(x)=d\in(0,\infty)$.

We also make the assumption

(1.13) $$\psi(x)\{\max_{x^\alpha<y<x}\eta(y)\}\log x \text{ is bounded for large } x.$$

This condition implies that if $\lim_{x\to\infty}\psi(x)=\infty$ we have, for $x\to\infty$, $\{\psi(x)\}^{-1}\{\psi(x^\alpha)\}\to 1$ and

(1.14) $$\psi(x^\alpha)\exp\{-\gamma\psi(x^\alpha)\}\approx\psi(x)\exp\{-\gamma\psi(x)\}.$$

In section 3 we shall prove the following integral test.

Theorem. *Take* $\alpha\in(0,1)$ *and* $\gamma=\log(1/\alpha)$. *Let* $X_1, X_2, \ldots$ *be i. i. d. symmetric non-lattice random variables with common distribution satisfying* (1.1), *where* L *(and* ψ*) are given as above. Assume that* ψ *is non-decreasing satisfying* (1.11) *and* (1.13) *and* $\exp\{-\gamma\psi(x)\}$ *is slowly varying for* $x\to\infty$. *Let*

(1.15) $$I(\psi)=\int^{\infty} x^{-1}(\log x)^{-1}\psi(x)\exp\{-\gamma\psi(x)\}\,dx.$$

Then we have

$$\liminf n^{-\alpha}|S_n|=\begin{cases} 0 \text{ a. s.} & \textit{according as } I(\psi)=\infty \\ \infty \text{ a. s.} & <\infty. \end{cases}$$

In section 4 we consider two examples and in section 5 we make some remarks.

2. The proof of the theorem. In the proof we consider estimates in which $L(x)$ only appears for large values of x. Therefore it is no restriction to assume that in (1.7) we have $c(x)\equiv c=1$.

Define the subsequence n_k by

$$n_k=k^{\frac{1}{1-\alpha}}\varphi(k), \tag{2.1}$$

where φ is a slowly varying function at infinity with

$$\lim_{x\to\infty}\varphi(x)=\infty. \tag{2.2}$$

One easily checks

$$n_k^\alpha\{(n_{k+1}-n_k)L(n_{k+1}-n_k)\}^{-1}\to 0 \quad \text{for } k\to\infty. \tag{2.3}$$

Define the stopping times T_k by

$$T_k=\inf\{l:\ l>n_k \text{ and } |S_l|\leqq l^\alpha\}. \tag{2.4}$$

Take $c>1$. For $n>n_k$ we have

$$P(|S_n|<cn^\alpha)\geqq\sum_{l=n_k+1}^{n}P(T_k=l\wedge|S_n|<cn^\alpha)=\sum_{l=n_k+1}^{n}P(T_k=l)P(|S_n|<cn^\alpha\,|\,|S_l|<l^\alpha).$$

Let F_n be the distribution function of S_n. Take $S_0=0$ a. s. Consider, for $l<n$,

$$P(|S_l|<l^\alpha\wedge|S_n|<cn^\alpha)=\int_{-l^\alpha}^{l^\alpha}\int_{-cn^\alpha-x}^{cn^\alpha-x}dF_{n-l}(y)dF_l(x)$$

$$\geqq\int_{-l^\alpha}^{l^\alpha}\int_{-(c-1)n^\alpha}^{(c-1)n^\alpha}dF_{n-l}(y)dF_l(x)=P(|S_l|<l^\alpha)P(|S_{n-l}|<(c-1)n^\alpha).$$

Thus we obtain

$$P(|S_n|<cn^\alpha)\geqq\sum_{l=n_k+1}^{n}P(T_k=l)P(|S_{n-l}|<(c-1)n^\alpha).$$

We want to estimate $\sum_{l=n_k+1}^{n_{k+1}}P(T_k=l)$. We have

$$\sum_{n=n_k+1}^{n_{k+2}}P(|S_n|<cn^\alpha)\geqq\sum_{n=n_k+1}^{n_{k+2}}\sum_{l=n_k+1}^{n}P(T_k=l)P(|S_{n-l}|<(c-1)n^\alpha)$$

$$\geqq\sum_{l=n_k+1}^{n_{k+1}}P(T_k=l)\sum_{n=l}^{n_{k+2}}P(|S_{n-l}|<(c-1)n^\alpha)$$

$$=\sum_{l=n_k+1}^{n_{k+2}}P(T_k=l)\sum_{m=0}^{n_{k+2}-l}P(|S_m|<(c-1)n^\alpha)\geqq\sum_{l=n_k+1}^{n_{k+1}}P(T_k=l)\sum_{m=0}^{n_{k+2}-n_{k+1}}P(|S_m|<(c-1)n_k^\alpha).$$

This implies

$$(2.5)\qquad \sum_{l=n_k+1}^{n_{k+1}} P(T_k=l) \leqq \{\sum_{n=n_k+1}^{n_{k+2}} P(|S_n|<cn^\alpha\}\{\sum_{m=0}^{n_{k+2}-n_{k+1}} P(|S_m|<(c-1)n_k^\alpha\}^{-1}.$$

Applying (1.2), (1.3) and (1.12), we obtain

$$(2.6)\qquad P(|S_n|<n^\alpha \quad \text{for some } n_k<n\leqq n_{k+1})$$

$$\leqq (n_{k+2}-n_{k+1})cn_{k+1}^{\alpha-1}\{L(n_{k+1})\}^{-1}\{\sum_{m=0}^{n_{k+2}-n_{k+1}} P(|S_m|\leqq(c-1)n_k^\alpha\}^{-1}.$$

Now we estimate

$$(2.7)\qquad \sum_{m=0}^{n_{k+2}-n_{k+1}} P(|S_m|<(c-1)n_k^\alpha).$$

We choose the sequence of positive real numbers b_k such that

$$(2.8)\qquad b_k=n_k^\alpha\,\varphi_2(k)/L(n_k^\alpha),$$

where $\varphi_2(k)\to\infty$ for $k\to\infty$ and

$$(2.9)\qquad \varphi^{1-\alpha}(k)\,L\,(n_k^\alpha)\{\varphi_2(k)\}^{-1}\sim \exp\{c\log n_k^\alpha/\psi(n_k^\alpha)\}.$$

This sequence has the following properties for $k\to\infty$

$$(2.10.\text{i})\qquad b_k=o(n_k^\alpha \sum_{[b_k]}^{n_{k+2}-n_{k+1}} m^{-1}(L(m))^{-1},$$

$$(2.10.\text{ii})\qquad n_k^\alpha=o(b_kL(b_k)),$$

$$(2.10.\text{iii})\qquad L(n_{k+2}-n_{k+1})/L(b_k)\to\text{constant},$$

$$(2.10.\text{iv})\qquad P(|S_n|<n^\alpha \quad \text{for some } n_k<n\leqq n_{k+1})$$

$$\leqq c\int_{n_{k+1}}^{n_{k+2}} x^{-1}(\log x)^{-1}\,\psi(x)\exp(-\gamma\psi(x))\,dx.$$

We shall prove these assertions in the Appendix.

The first part of the theorem follows from the convergence part of the Borel-Cantelli lemma.

If the integral diverges we define the subsequence n_k by

$$(2.11)\qquad n_k=k^{\frac{1}{1-\alpha}}\varphi(k),$$

where φ satisfies

$$(2.12)\qquad \varphi^{1-\alpha}(k)=(1-\alpha)\log n_k\exp(\gamma\psi(n_k))(L(n_k)\,\psi\,(n_k))^{-1}.$$

It follows that $n_{k+1}-n_k\sim k^{\alpha/(1-\alpha)}\,\varphi(k)$ for $k\to\infty$. Using that $\exp(\gamma\psi)$ is a slowly varying function, we obtain

$$\int_{n_k}^{n_{k+1}} (x\log x)^{-1}\,\psi(x)\exp(-\gamma\psi(x))dx\approx\psi(n_k)\exp(-\gamma\psi(n_k))\,\log(\log n_{k+1}/\log n_k)$$

$$\approx \psi\,(n_k)\,\exp\,(-\gamma\psi\,(n_k))\,(n_{k+1}-n_k)\,(n_k\log n_k)^{-1}$$

$$\approx \psi(n_k)\exp(-\gamma\psi(n_k))k^{\frac{\alpha}{1-\alpha}}\varphi(k)(1-\alpha)^{-1}(n_k\log n_k)^{-1}$$

$$\approx \psi(n_k)\exp(-\gamma\psi(n_k))\,n_k^\alpha(1-\alpha)^{-1}\varphi^{1-\alpha}(k)(n_k\log n_k)^{-1}\approx n_k^\alpha(n_k L(n_k))^{-1}.$$

Define the events D_k, $k=1, 2, \ldots$, by

$$D_k : |S_{n_k}| \leq n_k^\alpha. \tag{2.13}$$

Then we have $\Sigma P(D_k)=\infty$. The Hewitt—Savage zero-one law (see [5] p. 80) implies that $P(D_k$ i. o.) is either zero or one. Then it suffices to show that the liminf in (1.5) is finite.

Consider, for $k<l$,

$$P(D_k\wedge D_l)=P(|S_{n_k}|\leq n_k^\alpha\wedge|S_{n_l}|\leq n_l^\alpha) = \int_{-n_k^\alpha}^{n_k^\alpha}\int_{-n_l^\alpha-x}^{n_l^\alpha-x} dF_{n_l-n_k}(y)dF_{n_k}(x)$$

$$\leq \int_{-n_\mu^\alpha}^{n_k^\alpha}\int_{-2n_l^\alpha}^{2n_l^\alpha} dF_{n_l-n_k}(y)\,dF_{n_k}(x)=P(|S_{n_k}|\leq n_k^\alpha)\,P(|S_{n_l-n_k}|\leq 2n_l^\alpha).$$

One easily checks that $\{(n_{k+1}-n_k)L(n_{k+1}-n_k)\}^{-1}n_{k+1}^\alpha=o(1)$ for $k\to\infty$. Therefore we have

$$P(D_k\wedge D_l)\leq c\, n_l^\alpha\{(n_l-n_k)L(n_l-n_k)\}^{-1}P(D_k). \tag{2.14}$$

For $l\geq 2k$ we have $\{(n_l-n_k)L(n_l-n_k)\}^{-1}n_l^\alpha\leq c_1 n_l^\alpha\{n_l L(n_l)\}^{-1}$ for some constant c_1. This implies that there exists some constant c_2 (independent of k and l) such that, for $l\geq 2k$,

$$P(D_k\wedge D_l)\leq c_2 P(D_k)P(D_l). \tag{2.15}$$

Next we shall show that there exists a constant c_3 (independent of k) such that

$$\sum_{l=1}^{k-1} P(D_k\wedge D_{k+l})\leq c_3 P(D_k). \tag{2.16}$$

By (2.14) and (2.12) we have

$$\sum_{l=1}^{k-1}P(D_k\wedge D_{k+l})\leq c_4 P(D_k)\sum_{l=1}^{k-1}n_{k+l}^\alpha\{(n_{k+l}-n_k)L(n_{k+l}-n_k)\}^{-1}$$

$$\leq c_4 P(D_k)\sum_{l=1}^{k-1}(n_{k+l}-n_k)^{-1}L(n_{k+l})\{L(n_{k+l}-n_k)\}^{-1}\psi(n_{k+l})\{\log n_{k+l}\}^{-1}$$

$$\cdot\exp\{-\gamma\psi(n_{k+l})\}\cdot(n_{k+l+1}-n_{k+l}).$$

From the definition of n_k and using the fact that φ is slowly varying, we obtain

$$(n_{k+l}-n_k)^{-1}(n_{k+l+1}-n_{k+l})<c_5 l^{-1} \tag{2.17}$$

for some c_5 (independent of k and l) and all k and l.
Using the representation of L, we obtain

$$\{L(n_{k+l}-n_k)\}^{-1}L(n_{k+l})=\exp\{\int_{n_{k+l}-n_k}^{n_{k+l}} y^{-1}(\log y)^{-1}\psi(y)\,dy\}$$

$$\leqq \exp\{\psi(n_{k+l})\log(\log n_{k+l}/\log(n_{k+l}-n_k))\}$$
$$\leqq c_6 \exp\{-\psi(n_{k+l})\log\{1-(1-\alpha)(1-(\log l)/(\log k)\}\}.$$

Then we have

$$\sum_{l=1}^{k-1} P(D_k \wedge D_{k+l}) \leqq c_7 P(D_k) \sum_{l=1}^{k-1} l^{-1}\psi(n_{k+l})\{\log n_{k+l}\}^{-1}\exp\{-\psi(n_{k+l})$$
$$\cdot \log\{1+(\alpha^{-1}-1)(\log l)/(\log k)\}\}$$
$$\leqq c_7 P(D_k) \sum_{l=1}^{k-1} l^{-1}\psi(n_{k+l})\{\log n_{k+l}\}^{-1}\exp\{-c_8\psi(n_{k+l})(\log l)(\log k)^{-1}\}$$
$$\leqq c_9 P(D_k) \sum_{l=1}^{k-1} l^{-1}\psi(n_k)\{\log n_k\}^{-1}\exp\{-c_{10}\psi(n_k)(\log l)(\log n_k)^{-1}\}.$$

The last series converges as can be seen if we compare the series with the integral $\int_0^{c\psi(n_k)} \exp\{-c_{10}z\}dz$.
Now it easily follows that the liminf in (1.5) is finite.

3. Examples.

Example 1. (See P. Griffin [3]). Take $\psi(x)=\begin{cases}(1/\gamma)\log_3 x & x>x_0\\ 0 & \text{otherwise.}\end{cases}$

Then we have $L(x)=c(\log x)^{-1/\gamma}\exp\{(1/\gamma)\log_2 x\log_3 x\}$.
Define the slowly varying function H by $H(x)=L(x)\log x/\log_3 x$. Griffin noticed that

$$\frac{d}{dx}\left(-\frac{1}{L(x)}\right)=\frac{1}{xH(x)}. \tag{3.1}$$

He considers a random walk with $P(|X_1|>x)=x^{-1}H(x)$. Using (3.1) it is relatively easy to estimate the series in (2.7). Then we have

$$\liminf n^{-\alpha}|S_n|=\begin{cases}\infty & \text{a. s. if } \alpha<e^{-1/\gamma}\\ 0 & \text{a. s. } \alpha>e^{-1/\gamma}.\end{cases}$$

Example 2. Take
$\psi(y)=(1/\gamma)\{\log_3 y+2\log_4 y+\log_5 y+\cdots+(1+\delta)\log_k y\}$, $k\geqq 5$. We have

$$I(\psi)\begin{cases}<\infty & \text{for } \delta>0\\ =\infty & \text{for } \delta\leqq 0.\end{cases}$$

Conseguently we have

$$\liminf n^{-\alpha}|S_n|=\begin{cases}\infty & \text{a. s. according as } \delta>0\\ 0 & \text{a. s. } \delta\leqq 0.\end{cases}$$

Remark 1. Take $\varepsilon>0$. Define ψ_i, $i=1, 2$, by $\psi_1(t)=\gamma^{-1}(1-\varepsilon)\log_3 t$ and $\psi_2(t)=\gamma^{-1}(1+\varepsilon)\log_3 t$.

Define also $\widehat{\psi}(t)=\min(\max(\psi(t), \psi_1(t)), \psi_2(t))$. The following assertions hold true

i $I(\psi)<\infty \Rightarrow I(\widehat{\psi})<\infty$
ii $I(\psi)=\infty \Rightarrow I(\widehat{\psi})=\infty$
iii if $I(\psi)<\infty \Rightarrow \psi(t)>\psi_1(t)$ for t sufficiently large.

The proofs of these assertions are similar to the proofs of corresponding assertions in the generalized law of the iterated logarithm. See, for example, the proof of lemma 3.3.1 in [6]. The lower index of a transient random walk is defined to be the unique $\delta \geqq 0$ such that

$$\liminf n^{-\alpha} |S_n| = \begin{cases} 0 & \text{a. s. if } \alpha > \delta \\ \infty & \text{a. s. if } \alpha < \delta. \end{cases}$$

In example 6.4 of [2] we have

$$L(x) = \exp\{(\log_2 x)(\log_3 x)(\eta + \sin(\log_3 x))\}$$

with $\eta \in (\sqrt{2}, \infty)$. By differentation we find

$$\psi(x) = (\log_3 x)\{\eta + \sin(\log_3 x) + \cos(\log_3 x)\} + \sin(\log_3 x) + \eta.$$

Griffin shows that the lower index of the corresponding random walk is equal to $\exp\{-(\eta - \sqrt{2})^{-1}$. In view of the foregoing assertion it is clear that one should compare the example above with

$$L^*(x) = \exp\{(\log_2 x)(\log_3 x)(\eta - \sqrt{2})\}$$

instead to compare with

$$L^{**}(x) = \exp\{\eta(\log_2 x)(\log_3 x)\}.$$

See also the remark that the lower index is not monotone in the tails on p. 80 of [2].

Remark 2. One can extend example 1 in another way. We can ask for an integral test for

$$\liminf \{\varphi(n)\}^{-1} |S_n| = \begin{cases} 0 & \text{a. s.} \\ \infty & \text{a. s.,} \end{cases}$$

where φ is some regularly varying function. This result is proved, under stronger conditions, by Griffin (personal communication).

Appendix. We shall prove (2.10.ii).

$$n_k^\alpha\{b_k L(b_k)\}^{-1} = L(n_k^\alpha)\{\varphi_2(k) L(b_k)\}^{-1} = \{\varphi_2(k)\}^{-1} \exp\{\int_{b_k}^{n_k^\alpha} y^{-1}(\log y)^{-1}\psi(y)dy\}$$

$$\leqq \{\varphi_2(k)\}^{-1} \exp\{\psi(n_k^\alpha)\log(\log n_k^\alpha/\log b_k)\} \approx (\varphi_2(k))^{-1}\exp\{\psi(n_k^\alpha)(\log L(n_k^\alpha) - \log\varphi_2(k))(\log n_k^\alpha)^{-1}\}$$

From (2.9) we obtain $\log L(n_k^\alpha) - \log\varphi_2(k) \approx c(\log n_k^\alpha)(\psi(n_k^\alpha))^{-1} - (1-\alpha)\log\varphi(k)$.

Then we have $n_k^\alpha\{b_k L(b_k)\}^{-1} \approx c\{\varphi_2(k)\}^{-1}$ because we can always choose φ such that $\log\varphi(k) = o(\log n_k^\alpha)$ for $k \to \infty$. This proves (2.10.ii). Next we shall prove (2.10.iii).

$$L(n_{k+2} - n_{k+1})/L(b_k) = \exp\{\int_{b_k}^{n_{k+2}-n_{k+1}} y^{-1}(\log y)^{-1}\psi(y)dy\}$$

$$\leq \exp\{\psi(n_{k+2} - n_{k+1})\log(\log(n_{k+2} - n_{k+1})/\log b_k)\}$$

$$\approx \exp\{\psi(n_{k+2} - n_{k+1})\log(\log((1-\alpha)^{-1} n_k^\alpha \varphi^{1-\alpha}(k))/\log(n_h^\alpha \varphi_2(k)/L(n_k^\alpha))\}$$

$$\approx \exp\{\varphi(n_{k+2} - n_{k+1})\log((1-\alpha)^{-1}\varphi^{1-\alpha}(k) L(n_k^\alpha)\varphi_2(k))/\log n_k^\alpha\} \approx \exp\{c\}.$$

Now we consider

$$n_k^\alpha \sum_{[b_k]}^{n_{k+2}-n_{k+1}} m^{-1}\{L(m)\}^{-1} \geqq n_k^\alpha \{L(n_{k+2}-n_{k+1})\}^{-1} \log\{(n_{k+2}-n_{k+1})/[b_k]\}$$

$$\geqq n_k^\alpha \{L(b_k)\}^{-1} \log\{(1-\alpha)^{-1} \varphi^{1-\alpha}(k) L(n_k^\alpha)/\varphi_2(k)\} \approx c\, n_k^\alpha \{L(b_k)\}^{-1} \log n_k^\alpha/\psi(n_k^\alpha).$$

Therefore we have

$$b_k\{n_k^\alpha \sum_{m=[b_k]}^{n_{k+2}-n_{k+1}} (m\,L(m)^{-1}\}^{-1} \leqq \varphi_2(k)\,\{L(n_k^\alpha)\}^{-1}\, L(b_k)\,\psi\,(n_k^\alpha)/\log n_k^\alpha \approx \varphi_2(k)\,\psi\,(n_k^\alpha)/\log n_k^\alpha = o\,(1)$$

for $k\to\infty$, since we can choose φ_2 such that $\varphi_2(k)\psi(n_k^\alpha)=o(\log n_k^\alpha)$.
Thus we proved (2.10.i.). Now we can estimate

$$\sum_{m=0}^{n_{k+2}-n_{k+1}} P(\,|\,S_m\,|<(c-1)n_k^\alpha) = \sum_{m=0}^{[b_k]} + \sum_{[b_k]+1}^{n_{k+2}-n_{k+1}}$$

$$= O([b_k] + cn_k^\alpha \sum_{[b_k]+1}^{n_{k+2}-n_{k+1}} \{mL(m)\}^{-1} \geqq cn_k^\alpha \log n_k^\alpha \{\psi(n_k^\alpha) L(b_k)\}^{-1}.$$

Then, using (2.6) and (1.14),

$$P(\,|\,S_n\,|<n^\alpha \text{ for some } n_k<n\leqq n_{k+1})$$

$$\leqq (n_{k+2}-n_{k+1})n_{k+1}^{-1} L(b_k)\{L(n_{k+1})\}^{-1}\,\psi(n_k^\alpha)\{\log n_k^\alpha\}^{-1}$$

$$\leqq c \int_{n_{k+1}}^{n_{k+2}} (x\log x)^{-1}\,\psi(x)\,e^{-\gamma\psi(x)}dx.$$

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