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ON SOME NEW PROPERTIES OF HARMONIC MAPPINGS

THEMISTOCLES M. RASSIAS

This paper is devoted to some properties of harmonic mappings which can be found useful in the theory of minimal surfaces. The final part of the paper deals with a number of research problems which are of current interest in function theory and related subjects.

1. On a generalization of the Riemann mapping theorem. Riemann's problem of mapping a simply connected plane region whose boundary consists of more than a single point conformally on a circle as normal region can be reduced to the study of two problems: (1) the interior problem that concerns the map of the interior points and (2) the boundary problem that concerns the behaviour of the map on the boundary. It was Riemann who studied the first problem by using techniques of the Dirichlet principle and Schwarz and Neumann who gave proofs for the case of regions with restricted boundaries. Later, Osgood gave a satisfactory answer to the general case using methods due to Poincare. The second problem was solved for analytic boundaries by Schwarz and in other special cases by Picard. The general case was treated by W. Osgood [5] and by C. Carathéodory [1]. It is my purpose now to indicate how I can apply the theory of minimal surfaces (see for example [1, 2, 8, 9, 11, 13 14]) for a generalization of the Riemann mapping theorem.

Consider the minimal surface equation (Lagrange [4]) given by

$$(1)\qquad (1+\varphi_y^2)\varphi_{xx}-2\varphi_x\varphi_y\varphi_{xy}+(1+\varphi_x^2)\varphi_{yy}=0.$$

The surface is assumed in the non-parametric form and Plateau's problem is regarded as a generalized Dirichlet problem, with (1) replacing Laplace's equation. According to K. Weierstrass [14] a parametric form of the solution for the minimal surface equation (1) is given by

$$x=\operatorname{Re} F_1(w),\ y=\operatorname{Re} F_2(w),\ z=\operatorname{Re} F_3(w)$$

where $F_1(w)$, $F_2(w)$, $F_3(w)$ are any analytic functions satisfying

$$F_1'^2(w)+F_2'^2(w)+F_3'^2(w)=0.$$

Set $\psi_1(w)=\frac{1}{2}(F_1'(w)+iF_2'(w))$ and $\psi_2(w)=\frac{1}{2}F_3'(w)$.

It follows that

$$x+iy=\int\psi_1(w)dw-\int\overline{\frac{\psi_2^2(w)}{\psi_1(w)}}\,dw.$$

Denote $U(\psi_1)=\int\psi_1(w)dw-\int\overline{(\psi_2^2(w)/\psi_1(w))}\,dw$ then we can state the following theorem.

Theorem 1.1. *Let Γ be a simple closed analytic curve in the z-plane. Then there exists a regular function $\psi_1(w)$ defined in $\Omega=\{w: |w|\geq 1\}$, such that $U(\psi_1)$ maps Ω simply onto the closed domain exterior to Γ and such that infinity is mapped into infinity, for a fixed regular function $\psi_2(w)$ defined in Ω.*

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Remark: If $\psi_2(w)=0$, for any w, then the above theorem implies the Riemann mapping theorem, as a special case.

2. Following techniques from Morse theory and complex analysis (see for example [13]) one can give global analytic proofs of the following fundamental theorems:

Theorem 2.1. *A smooth Jordan curve* Γ *of total curvature at most* 6π *bounds only a finite number of minimal surfaces of the type of the disk.*

Theorem 2.2. *Let* Γ *be an arbitrary smooth simple closed curve lying in the smooth boundary of a uniformly convex subset of* R^3. *Then* Γ *bounds a smoothly embedded minimal disk of least area among all embedded disks having* Γ *as boundary.*

Theorem 2.3. *In Euclidean space of three dimensions, let* Γ_1, Γ_2 *be any two Jordan curves not intersecting one another. If the minimal surfaces* M_1 *and* M_2 *determined by* Γ_1 *and* Γ_2 *taken separately have in common a point Q that is regular for both of them, then there exists a doubly-connected minimal surface M bounding by* Γ_1, Γ_2.

Remark: The previous theorem solves Plateau's problem for two contours in R^3 and thus derives the corresponding result of J. Douglass [2].

3. On one-to-one harmonic mappings. In the following we state some properties of one-to-one harmonic mappings which have been proved to be very useful for a further development of the theory of minimal surfaces in Euclidean space of three dimensions. Let $D_z=\{z:z=x+iy$ and $|z|<1\}$ and $D_w=\{w:w=u+iv$ and $|w|<1\}$.

Proposition 3.1. *Let* $z:D_w\to C$ *be a complex-valued harmonic function in* D_w *such that* $z(w)=x(w)+iy(w)$. *Suppose* $z(0)=0$ *and* $|z(w)|<1$ *for* $|w|<1$. *Then*

$$|z(w)|\leq\frac{4}{\pi}\tan^{-1}|w| \text{ in } D_w.$$

Theorem 3.2. Let $z:D_w\to D_z$ *be a one-to-one harmonic mapping of* D_w *onto D such that* $z(0)=0$. *Then*

$$\left|\frac{\partial z}{\partial u}\right|^2+\left|\frac{\partial z}{\partial v}\right|^2\geq\frac{2}{\pi^2} \quad in \quad D_w,$$

where $z=x+iy$ *and* $w=u+iv$.

Remark: It can be proved that *there exists no harmonic homeomorphism of the open unit disk in* R^2 *onto* R^2.

Conjecture 3.3. *There exists no harmonic homeomorphism of the open unit ball B in* R^3 *onto* R^3, *i. e. there are no harmonic functions* f_1, f_2, f_3 *defined in* $B=\{z=(z_1, z_2, z_3):|z|<1\}$, *such that the mapping* $z\to(f_1, f_2, f_3)$ *is a homeomorphism of B onto all of* R^3.

4. Let S_H be the class of all complex-valued, harmonic, orientation-preserving, univalent mappings f defined on the open unit disk D_z, such that $f(0)=0$ and $f_z(0)=1$. It follows that $f=h+\bar{g}$, where $h(z)=z+\Sigma_{k=2}^{\infty}a_kz^k$ and $g(z)=\Sigma_{k=1}^{\infty}b_kz^k$ are analytic functions in D_z.

Theorem 4.1. *A function f in* S_H *maps* D_z *onto a convex domain if and only if the analytic function* $h-e^{2i\theta}g$ *is univalent and maps* D_z *onto a domain convex in the direction* θ *for all* θ, $0<\theta<\pi$.

Remark. Harmonic mappings cannot be determined up to normalization by their image domains.

5. Some research problems. A function $u(z)$ defined in a domain D of the plane is said to be *subharmonic* in D if

(a) $u(z)$ is upper semi-continuous in D,
(b) $-\infty \leq u(z) < +\infty$, and $u(z) \not\equiv -\infty$ in D,
(c) For every z_0 in D and all sufficiently small r (depending on z_0) we have

$$u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta.$$

In a space of higher dimension subharmonic functions are defined in a similar way. If $u(z)$ and $-u(z)$ are subharmonic functions then $u(z)$ is a harmonic function. If $f(z)$ is regular in a domain D, and $f(z) \neq 0$, then $u(z) = \log|f(z)|$ is a subharmonic function in D.

Problem 1: (W. K. Hayman [3]). Suppose that $u(z)$ is a subharmonic function and $u(z) < 0$ in the half plane $|\theta| < \frac{\pi}{2}$, where $z = re^{i\theta}$.

Suppose also that $A(r) = \inf\{u(re^{i\theta}) : |\theta| < \frac{\pi}{2}\} \leq -k$, $0 < r < \infty$. Is it true that then $u(r) \leq -\frac{1}{2} k$, $0 < r < \infty$?

Remark: I have proved that this is not true. In fact I have constructed examples of functions satisfying the given conditions and such that $u(r) > -\frac{1}{2} k$ for $0 < r < \infty$.

Problem 2: (L. Zalcman). Let $u(z)$ be a real bounded continuous function on $D = \{|z| < 1\}$, and suppose that to each $z \in D$ corresponds a real number $r(z)$ with $0 < r(z) < 1 - |z|$ such that $(2\pi)^{-1} \int_0^{2\pi} u(z + r(z)e^{i\theta}) d\theta = u(z)$. Must $u(z)$ be a harmonic function on D?

Problem 3: (T. Ganelius). Let K_1, K_2, K_3 be disjoint closed sets in the extended complex plane, and c_1, c_2, c_3 be constants. Let $\rho_n(f)$ be the best rational approximation to the function f which equals c_1 on K_1, c_2 on K_2 and c_3 on K_3; i. e.

$$\rho_n(f) = \inf_{g \in R_n} \max_{z \in \bigcup_i K_i} |f(z) - g(z)|,$$

where R_n is the class of rational functions f of order at most n. Find a geometric characterization of $\lim_{n \to \infty} (\rho_n)^{1/n}$.

Problem 4: (L. A. Rubel). Let $u: R^n \to R$ be a continuous real-valued function. If we want to know whether a homeomorphism $\varphi: R^n \to R^n$ and a harmonic function $v: R^n \to R$ exist such that $v(x) = u(\varphi(x))$ is it necessary and sufficient that there should exist mappings $\mu_2, \mu_3, \ldots, \mu_n$ such that $F = (u, \mu_2, \mu_3, \ldots, \mu_n)$ is a light open mapping. of R^n into R^n?

Remark 1. The problem has been solved for $n = 2$ by S. Stoilow and the solution can be found in Whyburn's "Topological analysis".

Remark 2. More research problems in the theory of harmonic mappings can be found in Th. M. Rassias [7, 10, 11, 12, 13].

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Department of Mathematics
University of La Verne

P. O. Box 51105, Kifissia
Athens, Greece

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