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THE NIELSEN RELATION FOR MULTIVALUED MAPS

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The aim of this paper is to generalize the basic notions of the Nielsen fixed point theory onto a class of multivalued maps. In § 1 we define m-maps as upper semi-continuous maps admitting lifts between the universal covering spaces. This property enables us to define the Reidemeister relation induced by the m-maps and then in § 2 the Nielsen relation. In the last paragraph we discuss the case when the considered m-map is from the category admitting the fixed point index. The Nielsen number defined there is the homotopy invariant being the lower bound of fixed points.

1. The Reidemeister Relation. Let X and Y denote topological spaces.

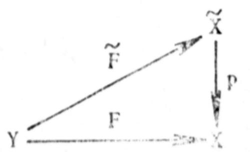
(1.1) Definition. *We will say that a subset $A\subset X$ has $(*)$-property iff it is nonempty, connected and there exists an open neighbourhood U of A such that each loop in U is homotopic (with fixed ends) in X to the constant loop.*

(1.2) Remark. We may demand U to be path-connected provided X is locally path-connected.

(1.3) Definition. *The multivalued map $F\colon Y\to X$ will be called m-map iff it is upper semi-continuous and the image of each point has $(*)$-property in X.*

We will assume that the space X is connected and admits a universal covering (i. e. X is locally path-connected and semi-locally simply connected (see [9]). Let us fix a universal covering $p\colon \tilde{X}\to X$.

(1.4) Definition. *The m-map $\tilde{F}\colon Y\to \tilde{X}$ such that the diagram*



commutes will be called a lift of the m-map $F\colon Y \dashrightarrow X$.

Let us notice that for every $y\in Y$ $p\colon \tilde{F}(y)\to F(y)$ is a homeomorphism.

(1.5) Theorem. *If Y is path-connected and simply connected, then for any m-map $F\colon Y\to X$ and points $y_0\in Y$, $\tilde{x}_0\in \tilde{X}$ such that $p(\tilde{x}_0)\in F(y_0)$ there exists a unique lift $\tilde{F}\colon Y\to\tilde{X}$ satisfying $\tilde{x}_0\in\tilde{F}(y_0)$.*

Proof. (a) $Y=[0, 1]$, $y_0=0$.

Let $t\in[0, 1]$ and let U_t denote the set from (1.1) (for the set $F(t)$) which we assume to be connected. Since F is upper semicontinuous so there exists an open subset $A_t\subset[0, 1]$ containing t for which $F(A_t)\subset U_t$. The family $\{A_t\}_{t\in[0,1]}$ forms an open covering of the interval $[0, 1]$. Let $\lambda>0$ be its Lebesgue number and suppose that $\lambda<1/n$: Serdica Bulgaricae mathematicae publicationes. Vol. 13, 1987, *174—181.* Consider the interval $[0, 1/n]$. There is a set A_{t_1} containing this interval such that $F(A_{t_1})\subset U_{t_1}$. Each loop from U_{t_1} is trivial in X hence $p^{-1}(U_{t_1})$ splits into the sum of disjoint connected components each of them mapped homeomorphically by p onto

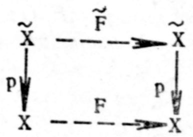
U_{t_1}. Let s_1: $U_{t_1} \to p^{-1}(U_{t_1})$ denote the inverse map onto the component containing $\widetilde{x}_0$. We define for $t \in [0, 1/n]$ $\widetilde{F}(t) = s_1 F(t)$. Then we choose an arbitrary point $\widetilde{x}_1 \in \widetilde{F}(1/n)$ and extend $\widetilde{F}$ the same way onto the interval $[0, 2/n]$. Following this procedure we get the desired lift.

(b) $Y = [0, 1]^2$, $y_0 = (0, 0)$. The proof is similar.

(c) The general case. Choose an arbitrary point $y \in Y$. Let ω be a path in Y joining y_0 with y. We apply (a) to the map $F\omega$: $[0, 1] \to X$ and get a lift $(\widetilde{F\omega})$: $[0, 1] \dashrightarrow \widetilde{X}$ such that $(\widetilde{F\omega})(0) \ni \widetilde{x}_0$. We define $\widetilde{F}(y) = (\widetilde{F\omega})(1)$.

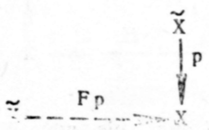
This definition is correct: if ω' is another path joining y_0, with y then they are fixed end homotopic and thanks to (b) $(\widetilde{F\omega})(1) = (\widetilde{F\omega'})(1)$.

1.6 Corollary. *Let F: $X \to X$ be a m-map and let $\widetilde{x}, \widetilde{x}_2 \ldots \in \widetilde{X}$ be such points that $p(\widetilde{x}_2) \in Fp(\widetilde{x}_1)$. Then there exists a unique m-map $\widetilde{F}$: $\widetilde{X} \to \widetilde{X}$ for which the diagram*

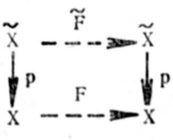


commutes and $\widetilde{x}_2 \in \widetilde{F}(\widetilde{x}_1)$.

Proof. We apply (1.5) to the diagram

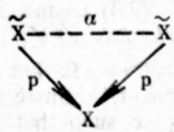


Let us denote by lift F the set of all m-maps F: $X \to X$ for which the diagram



commutes.

We will call the elements of lift F the lifts of m-map F. Let us recall [5] that the set of all (singlevalued) maps α : $\widetilde{X} \to \widetilde{X}$ such that the diagram commutes



and forms a group (isomorphic to the fundamental group of the space X). We will denote this group by ϑ.

(1.7) Corollary. *Let us fix one element* $\widetilde{F} \in \operatorname{lift} F$. *Then each lift of* F *is of the form* $\alpha\widetilde{F}$, *where* $\alpha \in \vartheta$ *and* $\alpha \widetilde{F} = \beta \widetilde{F}$ *iff* $\alpha = \beta$.
Now we define an equivalence relation R on the set lift F

(1.8) $$\widetilde{F} R \widetilde{F}' \quad \text{iff} \quad \widetilde{F}' = \gamma \widetilde{F} \gamma^{-1} \text{ for some } \gamma \in \vartheta.$$

Following the singlevalued case we will call it Reidemeister relation and denote the quotient set

$$\nabla(F) = \operatorname{lift} F/R.$$

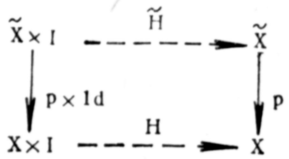
The elements of the set $\nabla(F)$ will be called Reidemeister classes of the m-map F.

(1.9) Remark. The above definition of $\nabla(F)$ depends on the choice of the universal covering. Nevertheless one can prove that the sets of Reidemeister classes got from different universal coverings are in natural one to one correspondence.

The number of elements of the set $\nabla(F)$ will be called the Reidemeister number of the m-map F. Now we are going to check that it is a homotopy invariant.

(1.10) Definition. *Two m-maps* $F, G: Y \dashrightarrow X$ *are called m-homotopic iff there exists m-map* $H: Y \times I \to X$ *such that* $H(y, 0) = F(y)$ *and* $H(y, 1) = G(y)$.

(1.11) Corollary. *Let* $F, G: X \to X$ *be two m-maps and let* $H: X \times I \to X$ *be the m-homotopy joining them. Then for any* $\widetilde{F} \in \operatorname{lift} F$ *there exists a unique m-map* $\widetilde{H}: \widetilde{X} \times I \to \widetilde{X}$ *such that the diagram*



commutes and $\widetilde{H}(\widetilde{x}, 0) = \widetilde{F}(\widetilde{x})$.

Thus $\widetilde{H}(\cdot\,, 1) \in \operatorname{lift} G$ and this way the homotopy H determines a bijection between the sets lift F and lift G. This bijection preserves the Reidemeister relation and induces a one to one correspondence between the sets $\nabla(F)$ and $\nabla(G)$.

(1.12) Theorem. *The Reidemeister numbers of the m-homotopic m-maps are equal.*

2. The Nielsen Relation. In this paragraph we will study the fixed point set of the m-map $F: X \to X$. We will denote it by

$$\operatorname{Fix} F = \{x \in X: \ x \in F(x)\}.$$

(2.1) Definition. *Let* $x, x' \in \operatorname{Fix} F$. *We will say that* x *and* x' *are Nielsen equivalent iff there exists a lift* $\widetilde{F} \in \operatorname{lift} F$ *such that* $x, x' \in p(\operatorname{Fix} \widetilde{F})$. *We will write then* $x \underset{N}{\sim} x'$ *and denote the quotient set by* $\Phi'(F) = \operatorname{Fix} F/N$.

(2.2) Remark. When F is singlevalued then the above definition coincides with the classical Nielsen relation [5].

(2.3) Lemma: *If* $x \in \operatorname{Fix} F$, *then there exists an open subsets* V_x *containing* x *such that* $y \in V_x \cap \operatorname{Fix} F$ *implies* $x \underset{N}{\sim} y$.

Proof: Let $x \in \operatorname{Fix} F$ and let U_x be the corresponding neighbourhood of $F(x) \subset X$ from (1.1). Since F is upper semi-continuous there exists an open subset V_x containing x such that $F(V_x) \subset U_x$. We may assume that V_x is path-connected and that $V_x \subset U_x$. Let $y \in V_x \cap \operatorname{Fix} F$. We will show that $x \underset{N}{\sim} y$. Let $p: \widetilde{X} \to X$ denote again a universal covering and let us fix a point $\tilde{x} \in p^{-1}(x)$. From (1.6) we get a lift $\tilde{F}$: $\tilde{X} \to \tilde{X}$ such that $\tilde{x} \in \tilde{F}(x)$. Consider the restriction of $\tilde{F}$

$$\tilde{F}\colon p^{-1}(V_x) \to p^{-1}(U_x).$$

Two above sets are disjoint sums of connected components, each of them mapped homeomorphically by p onto V_x and U_x, respectively. Denote by $\tilde{V}_{\tilde{x}}$, $\tilde{U}_{\tilde{x}}$ the components containing $\tilde{x}$. We get a commutative diagram

$$\begin{array}{ccc} \tilde{V}_{\tilde{x}} & \xrightarrow{\tilde{F}} & \tilde{U}_{\tilde{x}} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ V_x & \xrightarrow{F} & U_x \end{array}$$

where the vertical lines are homeomorphisms and $\tilde{V}_{\tilde{x}} \subset \tilde{U}_{\tilde{x}}$. Now it is obvious that if $y \in \operatorname{Fix} F \cap V_x$ and $\tilde{y} \in \tilde{V}_x$ satisfy $p(\tilde{y}) = y$ then $\tilde{y} \in \tilde{F}(y)$. Thus $x, y \in p(\operatorname{Fix} \tilde{F})$ so $x \underset{N}{\sim} y$.

(2.4) Lemma: *Let F: $X \to X$ be m-map. Then*

(a)
$$\operatorname{Fix} F = \bigcup_{\tilde{F} \in \text{lift } F} p(\operatorname{Fix} \tilde{F});$$

(b) *for any two lifts $\tilde{F}$, $\tilde{F}'$ of F the sets $p(\operatorname{Fix} \tilde{F})$, $p(\operatorname{Fix} \tilde{F}')$ are either equal or disjoint;*

(c) $p(\operatorname{Fix} \tilde{F}) = p(\operatorname{Fix} \tilde{F}') \neq \emptyset$ *implies* $\tilde{F} \underset{R}{\sim} \tilde{F}'$.

Proof: Similar to the singlevalued case [5].

Let $x \in \operatorname{Fix} F$. Let us consider $L_x = \{\tilde{F} \in \text{lift } F : \ x \in p(\operatorname{Fix} \tilde{F})\}$. Then (2.4a) implies that L_x is nonempty and it follows from (2.4 b, c) that L_x is exactly one Reidemeister class. On the other hand (2.4 b, c) implies that $L_x = L_y$ iff $x \underset{N}{\sim} y$. Thus we get injective map

$$\nu\colon \Phi'(F) \to \nabla(F)$$

given by the formula $\nu[x] = L_x$.

3. Generalized Fundamental Groupoid. There are two equivalent definitions of the Nielsen relation for singlevalued maps [5]. The first uses the universal covering; we have generalized it here as definition (2.1). Let us recall now the more popular one:

(3.1) Definition: *Let f: $X \to X$ denote a singlevalued self map of a topological space X. Then two points x, $x' \in \operatorname{Fix} f$ are called equivalent iff there is a path ω: $I \to X$ joining them such that ω and $f\omega$ are fixed end homotopic.*

The last approach can not be simply applied to multivalued case since the composition $F\omega$ is generally no longer a path. Nevertheless we will show how to generalize this definition onto the case of m-maps. This approach seems to be more convenient in calculations.

Let us recall

(3.2) Definition. *Let X be a topological space. The category which objects are points of X and morphisms from x to x' are the fixed end homotopy classes of paths joining these points is called the fundamental groupoid of the space X* [9].

We denote the set of morphisms between the points x and x' by $\Pi\ (X\colon x, x')$ and the whole fundamental groupoid by $\Pi(X)$.

(3.3) Remark. Any continuous singlevalued map $f\colon X \to Y$ induces a functor $\Pi(f)\colon \Pi(X) \to \Pi(Y)$ by formulae $\Pi(f)(x) = f(x)$, $\Pi(f)[\omega] = [f\omega]$.

Using these notation we may formulate an obvious

(3.4) Lemma. *Let $f\colon X \to X$ be a continuous singlevalued self-map. Two points $x, x' \in \operatorname{Fix} f$ are equivalent in sense of* (3.1) *iff the map*

$$\Pi(f)\colon \Pi(X;\ x,\ x') \to \Pi(X;\ x,\ x')$$

has a fixed point.

The aim of this paragraph is to generalize the notion of fundamental groupoid to extend the definition (3.1) onto the case of m-maps. Then we will check that this extension coincides with (2.1).

Let X denote again a connected, locally path-connected, semi-locally simply-connected topological space and let A_0, A_1 be two subsets of X satisfying property $(*)$ of (1.1). Then the sets $\Pi(X\colon a_0,\ a_1)$ for $(a_0, a_1) \in A_0 \times A_1$ may be identified as follows: let U_i be a path-connected neighbourhood of A_i from (1.1), let $a_i, a'_i \in A_i$ and let ω_i be a path in U_i joining the points a_i and a'_i $(i = 0, 1)$. Then we identify $\Pi(X;\ a_0,\ a_1) \ni [\alpha]$ with $[\omega_0^{-1} * \alpha * \omega_1] \in \Pi(X;\ a'_0,\ a'_1)$ and define the quotient set

$$\widehat{\Pi}(X;\ A_0,\ A_1) = \bigcup_{\substack{a_0 \in A_0 \\ a_1 \in A_1}} \Pi(X;\ a_0,\ a_1)/\sim. \tag{3.5}$$

Let $(a_0, a_1) \in A_0 \times A_1$ and denote by

$$i_{a_0,a_1}\colon \Pi(X;\ a_0,\ a_1) \to \Pi(X;\ A_0,\ A_1)$$

the natural bijection.

(3.6) Remark. When A_0 and A_1 are single points then (3.5) agrees with (3.2).

(3.7) Definition. *The generalized fundamental groupoid of the space X is the category which objects are subsets of X satisfying $(*)$-property and $\widehat{\Pi}(X;\ A_0, A_1)$ is the set of morphisms between the objects A_0 and A_1. We will denote this category by $\widehat{\Pi}(X)$.*

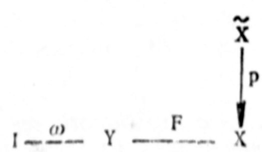
(3.8) Remark. $\Pi(X)$ may be regarded as a subcategory of $\widehat{\Pi}(X)$.

(3.9) Lemma. *Let X be a connected space admitting a universal covering and let Y be an arbitrary topological space. Then each m-map $F\colon Y \to X$ induces a functor*

$$\widehat{\Pi}(F)\colon \Pi(Y) \to \widehat{\Pi}(X)$$

which coincides with $\Pi(F)\colon \Pi(Y) \to \Pi(X)$ when F is singlevalued map.

Proof. We define $\widehat{\Pi}(F)(y) = F(y)$ for each $y \in Y$. Let $[\omega] \in \Pi(Y;\ y_0,\ y_1)$. Let us fix a universal covering $p\colon \widetilde{X} \to X$ and points $x_0 \in F(y_0)$, $x_1 \in F(y_1)$, $\widetilde{x}_0 \in p^{-1}(x_0)$. Then thank to (1.5) the diagram



admits a unique lift $(\widetilde{F\omega})$ such that $\widetilde{x}_0 \in (\widetilde{F\omega})(0)$. Let $\{\widetilde{x}_1\} = (\widetilde{F\omega})(1) \cap p^{-1}(x_1)$ and let ϵ be a path in $\widetilde{X}$ joining $\widetilde{x}_0$ with $\widetilde{x}_1$. We define

(3.10) $$\widehat{\Pi}(F)[\omega] = i_{(x_0, x_1)}[p\tau] \in \widehat{\Pi}(X;\ F(y_0),\ F(y_1)).$$

One can check that the above definition does not depend on the choice of the covering $\widetilde{X}$, the points x_0, x_1, $\widetilde{x}_0$ and the path τ. Thus we get the desired functor $\widehat{\Pi}(F)$. Now we are able to modify (3.1) (compare (3.4)).

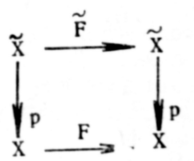
(3.11) Definition. *Two fixed points x, x' of the m-map $F: X \to X$ are in $\widetilde{N'}$ relation iff the maps*

$$\widehat{\Pi}(F),\ i_{(x,x')}:\ \Pi(X;\ x,\ x') \to \widehat{\Pi}(X;\ F(x),\ F(x'))$$

have a coincidence point.

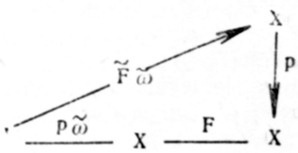
(3.12) Theorem. *The relations $\widetilde{N}$, $\widetilde{N'}$ are equal.*

Proof. Let $x \underset{\widetilde{N}}{} x'$. Then there exists a lift $\widetilde{F}$



such that $x,\ x' \in p(\operatorname{Fix} \widetilde{F})$.

Let us choose two points $\widetilde{x}$, $\widetilde{x}' \in X$ such that $p(\widetilde{x}) = x$, $p(\widetilde{x}') = x'$, and $\widetilde{x}$, $\widetilde{x}' \in \operatorname{Fix} \widetilde{F}$. Let $\widetilde{\omega}$ denote the path in $\widetilde{X}$ joining the points $\widetilde{x}$ and $\widetilde{x}'$. Then the commutative diagram

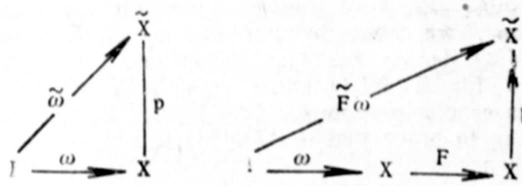


gives us $\widehat{\Pi}(F)(p\widetilde{\omega}) = i_{(x,x')}[p\widetilde{\omega}]$ hence $x \underset{\widetilde{N'}}{} x'$.

Assume now that $x \underset{\widetilde{N'}}{} x'$. Then there exists a path ω joining x with x' in X such that

(3.13) $$\widehat{\Pi}(F)[\omega] = i_{(x,x')}[\omega] \in \widehat{\Pi}(X,\ F(x),\ F(x')).$$

Let us fix a point $\widetilde{x} \in p^{-1}(x)$ and suppose that the lifts



satisfy $\omega(0)=\tilde{x}\in(\widetilde{F\omega})(0)$. Then by (3.13) and (3.10) $\tilde{\omega}(1)\in\widetilde{F\omega}(1)$. Take the lift $\tilde{F}$ such that $\tilde{x}\in \operatorname{Fix}\tilde{F}$. Then $\tilde{x}\in\tilde{F}(\tilde{x})=\tilde{F}\tilde{\omega}(0)$ and we get two lifts $\widetilde{F\omega}$, $\tilde{F}\tilde{\omega}$ of the map $F\omega$ such that $\widetilde{F\omega}(0)\cap\tilde{F}\tilde{\omega}(0)\neq\emptyset$ hence $\widetilde{F\omega}=\tilde{F}\tilde{\omega}$. In particular $\tilde{\omega}(1)\in\widetilde{F\omega}(1)=\tilde{F}\tilde{\omega}(1)$ so $\tilde{\omega}(1)\in\operatorname{Fix}\tilde{F}$. The equality $p\tilde{\omega}(1)=\omega(1)=x'$ implies $x, x'\in p(\operatorname{Fix}\tilde{F})$ so $x \underset{N}{\sim} x'$.

(3.14) Corollary. *Let $F\colon X\to X$ denote a m-map and let $x, x'\in\operatorname{Fix}F$. Suppose that there exists a path joining them such that the composition $F\omega\colon I\to X$ admits a continuous singlevalued selector τ satisfying: $\tau(0)=x$, $\tau(1)=x'$ and the paths ω, τ are fixed end homotopic. Then $x \underset{N}{\sim} x'$.*

Proof. It follows from the definition of the induced map (3.10) that $\hat{\Pi}(F)[\omega]=i_{(x,x')}[\tau]\in\hat{\Pi}(X, F(x), F(x'))$ hence $x \underset{N'}{\sim} x'$ and the theorem (3.12) gives us $x \underset{N}{\sim} x'$.

4. The Nielsen Number. Let X denote a metric ANR. In this paragraph we will consider the m-map $F\colon X\to X$ satisfying

(4.1) F is upper semi-continuous,

(4.2) the image of each point is Q-acyclic continuum (see [4]),

(4.3) the image of F is relatively compact in X.

We are going to define the Nielsen number of such map.

Let U be such open subset of X that $U\cap\operatorname{Fix}F$ is compact. We will denote by $i(X, F, U)\in Z$ the fixed point index introduced in [3] (F denotes here the decomposition consisting of the unique acyclic function F: see also [7]).

Let us recall its necessary properties:

(4.4) if U and V are such open subsets that $U\cap\operatorname{Fix}F=V\cap\operatorname{Fix}F$ is compact, then $i(X, F, U)=i(X, F, V)$;

(4.5) if $H\colon U\times I\to X$ is a multivalued homotopy satisfying (4.1), (4.2), (4.3), then $i(X, H_0, U)=i(X, H_1, U)$ (where $H_t\colon U\to X$ is given by the formula $H_t(x)=H(x, t)$ $x\in U$, $t\in I$;

(4.6) if $i(X, F, U)\neq 0$, then $\operatorname{Fix}F\cap U\neq\emptyset$;

(4.7) if $f\colon X\to X$ is a continuous singlevalued selector of F (i. e. $f(x)\in F(x)$) and $\operatorname{Fix}F\cap U$ is compact then so is $\operatorname{Fix}f\cap U$ and $i(X, F, U)=i(X, f, U)$ (see [7] Proposition (4.1)).

Let us consider a m-map $F\colon X\to X$ satisfying (4.1), (4.2) and (4.3). Let $A\subset\operatorname{Fix}F$ be one of its Nielsen classes. Let us choose an open subset $U\subset X$ for which $U\cap\operatorname{Fix}F=A$.

(4.8) Definition. *The class A will be called essential iff $i(X, F, U)\neq 0$.*

The above definition does not depend of the choice of the neighbourhood U (property (4.4)).

The compactness of the set $\operatorname{Fix}F$ and lemma (2.3) imply that the number of all Nielsen classes is finite.

(4.9). Definition. *The number of essential classes of the m-map F is called Nielsen number and is denoted by $N(F)$.*

(4.10) Theorem. *Let $H\colon X\times I\to X$ be a homotopy satisfying the conditions* (4.1), (4.2), (4.3). Then $N(H_0)=N(H_1)$.

Proof. For a subset $Z\subset X\times I$ and a number $t\in I$ we denote $Z_t=\{x\in X\colon (x, t)\in Z\}$. For a m-self map F we denote by $\Phi'(F)$ the set of its Nielsen classes.

Let $H\colon X\times I\to X\times I$ denote the "fat" homotopy, i. e. $\mathsf{H}(x, t)=(H(x, t), t)$. It is also the m-map and for $A\in\Phi'(\mathsf{H})$ either $A_t\in\Phi'(H_t)$ or $A_t=\emptyset$ and moreover for $B\in\Phi'(H_t)$ there exists exactly one class $A\in\Phi'(\mathsf{H})$ such that $A_t=B$.

Now we are going to prove that if $A\in\Phi'(\mathsf{H})$ and U is such open subset of $X\times I$ that $U\cap\operatorname{Fix}\mathsf{H}=A$, then

$$i(X, H_0, U_0)=i(X, H_1, U_1).$$

It is enough to show that the number $i(X, H_t, U_t)$ is locally constant function of $t\in[0, 1]$.

Let us fix $t_0 \in [0, 1]$. The compactness of A gives us neighbourhoods V and W such that: $t_0 \in V \subset [0, 1]$, $A_{t_0} \subset W \subset U_{t_0}$ and $A \cap (X \times V) \subset W \times V \subset U$. We may assume V to be connected. Then for arbitrary $t \in V$ we get from (4.4) and (4.5)

$$i(X, H_t, U_t) = i(X, H_t, W) = i(X, H_{t_0}, W) = i(X, H_{t_0}, U_{t_0}).$$

Let $B \in \Phi'(H_0)$ be an essential class and let $A \in \Phi'(\mathrm{H})$ be the only class satisfying $B = A_0$. Then $A_1 \in \Phi(H_1)$ is also essential and it proves that

$$N(H_0) \leq N(H_1).$$

The same way we prove the apposite inequality.

(4.11) Theorem. *The m-map F satisfying* (4.1), (4.2), (4.3) *has at least $N(F)$ fixed points.*

Let $f: X \to X$ be a singlevalued continuous selector of F. Then $\mathrm{Fix} f \subset \mathrm{Fix} F$ and for fixed points $x, y \in \mathrm{Fix} f$ $x \underset{N}{\sim} y$ with respect to f iff $x \underset{N}{\sim} y$ with respect to F. Thus if $A \in \Phi'(F)$, then $A \cap \mathrm{Fix} f \in \Phi'(f)$. Moreover, for any open subset $U \subset X$ such that $U \cap \mathrm{Fix} = A$ we get from (4.7)

$$i(X, F, U) = i(X, f, U).$$

The last equality implies

(4.12) Corollary. *If F is a m-map satisfying the conditions* (4.1), (4.2), (4.3) *and if f is its continuous selector, then $N(F) = N(f)$.* If f, f' *are selectors of F, then $N(f) = N(f')$.*

(4.13) Example. Let X be a metric ANR and let $F: X \to X$ be a compact upper semi-continuous map such that $F(x)$ is nonempty compact AR $(x \in X)$. The definition (4.9) may be applied to this map.

To prove this let us notice that the conditions (4.1), (4.2), (4.3) are evident so the rest follows from:

(4.14) Lemma. *Let X be a metric ANR and let A be a nonempty, compact AR contained in X. Then A has $(*)$-property in X.*

Proof. Since X is an ANR so it is a retract of an open subset U of a normed space E. We may assume that $X \subset U \subset E$. Let $r: U \to X$ denote this retraction. On the other hand since A is AR there exists a retraction $r_1: X \to A$. Let us denote $U_1 = \{x \in U : [x, r_1 r(x)] \subset U\}$ ($[a, b]$ denotes the interval joining the points $a, b \in E$). U_1 is open subset of U containing A. Let us put $V_1 = X \cap U_1$. Then the formula

$$H(x, t) = r((1-t)x + t r_1(x))$$

gives the homotopy between the inclusion $i: V_1 \to X$ and the retraction $r_1: V_1 \to A \subset X$. This way each loop from V_1 may be deformed in X into A and there it is contractible.

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