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POLYNOMIAL APPROXIMATION OF HYPER-ANALYTIC FUNCTIONS

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A proposition of the type of the classical Runge theorem is proved for hyper-analytic functions. For an arbitrary open subset of the big plane it is shown that every hyper-analytic function is generalized-analytic.

The representation of analytic functions in an open disc as a sum of power series is a fundamental fact in the classical theory. In the case of more general regions we can use the Runge's theorem: if K is a compact polynomially convex set and f is analytic in an open set $U \supset K$, then f is uniformly approximable on K with polynomials. Analogous proposition is true for analytic functions of several complex variables (Oka-Well's theorem [1]), too.

The possibility for a uniformly (polynomial or rational) approximation is connected with the existence of a holomorfic functional calculus for Banach algebras and the introduction of an analytic structure in their spectra [1]. The consideration of these problems and their generalizations can us take out of the class of classical analytic functions. For example, one generalization [2] of Gleason's theorem [3] introduces new structure in the spectrum, namely, a generalized-analytic structure.

Let Γ be a subgroup of the additive group of real numbers R with discrete topology and G be the compact group of all characters on Γ. Each continuous character on G has the form $\chi_p(g)=g(p)$, $g \in G$ for some $p \in \Gamma$. The closure A_G of finite linear combinations of the characters χ_p, $p \geq 0$, with complex coefficients is a uniform algebra on G. The elements A_G are called generalized-analytic functions in R. Arens-A. Singer sense [4]. Each character χ_p, $p \in \Gamma_+ = \Gamma \cap [0, \infty)$ is continuously extendable on the generalized (big) plane $\mathrm{C}_G = [0, \infty) \times G / \{0\} \times G$, equipped with factor-topology, as follows:

$$\tilde{\chi}_p(\lambda, g) = \lambda^p \cdot \chi_p(g) \quad \text{for} \quad \lambda > 0,\ p \neq 0;$$

$$\tilde{\chi}_p(*) = 0 \quad \text{for} \quad p \neq 0, \quad \text{where} \quad * = \{0\} \times G \quad \text{and} \quad \tilde{\chi}_0 = 1.$$

The finite linear combinations with complex coefficients of the functions $\tilde{\chi}_p$, $p \in \Gamma_+$ are called generalized polynomials. Notice that if $\Gamma = \mathrm{Z}$, then $G = S'$, the big plane C_G coincides with the complex plane C, the open generalized disc $\Delta_G = \Delta_G(1) = (0, 1) \times G / \{0\} \times G$ with radius 1 coincides with the open disc $\Delta = \{z \in \mathrm{C} / |z| < 1\}$ and the algebra A_G is the classical algebra $A(S')$.

We consider the case when the group Γ coincides with the additive group of rational numbers Q.

Definition 1. The function f in the open set $U \subset \mathrm{C}_G$ is said to be hyper-analytic in U if it is uniformly approximable in U with functions of the type $h \circ \tilde{\chi}_{1/n}$ where $n \in \mathrm{Z}^+ = \mathrm{Z} \cap (0, \infty)$ and h is analytic in $\tilde{\chi}_{1/n}(U) \subset \mathrm{C}$.

The hyper-analytic functions are defined by T. V. Tonev [5]. It is proved that the algebra of the bounded hyper-analytic functions in Δ_G has no corona [5].

In this paper we consider hyper-analytic functions in an arbitrary open set of the big plane $\mathbf{C}_G$.

The main result is the following

Theorem 1. *Let U be an open set in the generalized plane $\mathbf{C}_G$ and K be a compact polynomially convex subset of U. Then for every hyper-analytic function f in U there exists a sequence of generalized polynomials converging uniformly to f on K.*

In the proof we shall use the classical Runge's theorem. First we need some definitions and lemmas.

Definition 2. *A subset A of $\mathbf{C}_G$ is a polynomial polyhedron if there exist generalized polynomials $P_1, P_2, \dots, P_k$ such that $A=\{(\lambda, g)\in \mathbf{C}_G/\lambda\leq 1$ and $|P_j(\lambda, g)|\leq 1,\ j=1, 2, \dots, k\}$.*

Evidently A is a compact set in $\mathbf{C}_G$.

Lemma 1. *If B is a polynomial polyhedron in the complex plane $\mathbf{C}$, then $\tilde{\chi}_p^{-1}(B)$ is a polynomial polyhedron in the big plane $\mathbf{C}_G$ for every $p\in Q^+$.*

Proof. Let $B=\{z\in \mathbf{C}/\ |z|\leq 1$ and $|P_j(z)|\leq 1,\ j=1, 2, \dots, k\}$ where P_j is a polynomial of single complex variable.

We shall see that $\tilde{\chi}_p^{-1}(B)=\{(\lambda, g)/\lambda\leq 1$ and $|(P_j\circ\tilde{\chi}_p)(\lambda, g)|\leq 1,\ j=1, 2, \dots, k\}$.

If $(\lambda, g)=\tilde{\chi}_p^{-1}(z)$; $z\in B$ we have:

$$|\tilde{\chi}_p(\lambda, g)|=|z|,\ \text{i. e. } \lambda^p=|z|\leq 1 \text{ or } \lambda\leq 1$$

$$|(P_j\circ\tilde{\chi}_p)(\lambda, g)|=|P_j(z)|\leq 1,\ j=1, 2, \dots, k.$$

On the other side, let $(\lambda, g)\in\{(\lambda, g)/\lambda\leq 1$ and $|(P_j\circ\tilde{\chi}_p)(\lambda, g)|\leq 1,\ j=1, 2, \dots, k\}$ and $z=\tilde{\chi}_p(\lambda, g)$. Then

$$|z|=\lambda^p\leq 1 \text{ and } |P_j(z)|=|(P_j\circ\tilde{\chi}_p)(\lambda, g)|\leq 1,\quad j=1, 2, \dots, k.$$

Hence $z\in B$ and $(\lambda, g)\in\tilde{\chi}_p^{-1}(B)$.

As we shall see later, it is not always true that the image of a polynomial polyhedron in $\mathbf{C}_G$ by $\tilde{\chi}_p$ is a polynomial polyhedron in $\mathbf{C}$. But for a suitable $p\in Q^+$ it holds.

Lemma 2. *If A is a polynomial polyhedron in $\mathbf{C}_G$, then there exists a $m\in \mathbf{Z}^+$ such that $\tilde{\chi}_{1/m}(A)$ is a polynomial polyhedron in $\mathbf{C}$.*

Proof. Let $A=\{(\lambda, g)\in\mathbf{C}_G/\lambda\leq 1$ and $|P_j(\lambda, g)|\leq 1,\ j=1, 2, \dots, k\}$ be a polynomial polyhedron in $\mathbf{C}_G$. If $P_j=\sum_{s=0}^{n_j}\alpha_s^{(j)}\cdot\tilde{\chi}_{p(j,s)}$ and $p(j,s)=\gamma(j,s)/\beta(j,s)$, $j=1,2,\dots,k$; $s=0, 1, \dots, n_j$, we denote $m=\prod_{j=1}^{k}\prod_{s=0}^{n_j}\beta(j,s)\in\mathbf{Z}^+$. Every generalized polynomial can be represented in the form:

$$P_j=\sum_{s=0}^{n_j}\alpha_s^{(j)}\cdot\tilde{\chi}_{p(j,s)}=\sum_{s=0}^{n_j}\alpha_s^{(j)}\cdot(\tilde{\chi}_{1/m})^{k(j,s)}=P'_j\circ\tilde{\chi}_{1/m},$$

where $k(j,s)=m\cdot p(j,s)$ and $P'_j(z)=\sum_{s=0}^{n_j}\alpha_s^{(j)}\cdot z^{k(j,s)}$ is a polynomial of a single complex variable.

We shall prove that $\tilde{\chi}_{1/m}(A)=\{z\in\mathbf{C}/\ |z|\leq 1$ and $|P'_j(z)|\leq 1,\ j=1, 2, \dots, k\}$.

If $z \in \tilde{\chi}_{1/m}(A)$, then $z=\tilde{\chi}_{1/m}(\lambda, g)$ where $\lambda \leq 1$, $|P_j(\lambda, g)| \leq 1$ for $j=1, 2, \ldots, k$ and:

$$|z| = |\tilde{\chi}_{1/m}(\lambda, g)| = \lambda^{1/m} \leq 1 \quad |P'_j(z)| = (P'_j \circ \tilde{\chi}_{1/m})(\lambda, g)| = |P_j(\lambda, g)| \leq 1.$$

On the contrary, let z belongs to the polynomial polyhedron $\{z \in \mathbf{C} / |z| \leq 1$ and $|P'_j(z) \leq 1$, $j=1, 2, \ldots, k\}$ and (λ, g) is an arbitrary point of $\mathbf{C}_G$ for which $\tilde{\chi}_{1/m}(\lambda, g) = z$. Then $\lambda^{1/m} = |z| \leq 1$ i. e. $\lambda \leq 1$

$$|P_j(\lambda, g)| = |(P'_j \circ \tilde{\chi}_{1/m})(\lambda, g)| = |P'_j(z)| \leq 1.$$

We obtain that $(\lambda, g) \in A$.

It is easy to see that Lemma 2 holds true if instead of the chosen m we take some multiple of m. The number m can be also the least common multiple of the numbers $\beta(j, s)$, $j=1, 2, \ldots, k$; $s=0, \ldots, n_j$ or some its multiple.

Definition 3. *Let K be a bounded subset in $\mathbf{C}_G$. We define the polynomially convex hull $\hat{K}$ or K by:*
$\hat{K} = \{(\lambda, g) \in \mathbf{C}_G \mid |P(\lambda, g)| \leq \max_K |P|$ *for every generalized polynomial* $P\}$. *K is said to be polynomially convex if $K = \hat{K}$.*

Lemma 3. *Every polynomial polyhedron in $\mathbf{C}_G$ is a compact polynomially convex set.*

Proof. Let $A = \{(\lambda, g) \in \mathbf{C}_G \mid \lambda \leq 1$ and $|P_j(\lambda, g)| \leq 1$, $j=1, 2, \ldots, k\}$ be a polynomial polyhedron in $\mathbf{C}_G$. For $(\lambda_0, g_0) \in \hat{A}$ it holds $|P(\lambda_0, g_0)| \leq \sup_A |P(\lambda, g)|$ for every generalized polynomial. Then $|P_j(\lambda_0, g_0)| \leq \sup_A |P_j| \leq 1$ for $j=1, 2, \ldots, k$ and $\lambda_0 = |\tilde{\chi}_1(\lambda_0, g_0)| \leq \sup_A |\tilde{\chi}_1| \leq \sup_{\lambda \leq 1} \lambda = 1$ i. e. $(\lambda_0, g_0) \in A$.

Notice that Lemma 1 is true for polynomially convex sets in $\mathbf{C}$. Let B be a polynomially convex set in $\mathbf{C}$ and $p \in Q^+$. We denote $A = \tilde{\chi}_p^{-1}(B)$. If $(\lambda_0, g_0) \in \hat{A} \setminus A$, then $z_0 = \tilde{\chi}_p(\lambda_0, g_0) \notin B$ and for every polynomial P_1 of single complex variable we obtain: $|P_1(z_0)| = |(P_1 \circ \tilde{\chi}_p)(\lambda_0, g_0)| \leq \sup_A |P_1 \circ \tilde{\chi}_p| \leq \sup_B |P_1|$, i. e. $z_0 \in \hat{B} \setminus B$.

Lemma 4. *Let K be a compact polynomially convex subset of Δ_G and U be an open set containing K. Then there exists a polynomial polyhedron A such that $K \subset \operatorname{int} A \subset A \subset U$.*

Proof. Since K is polynomially convex and $K \subset U$ then for every point $z = (\lambda, g) \in \bar{\Delta}_G \setminus U$ we can find a generalized polynomial Q_z such that $|Q_z(z)| > \max_K |Q_z|$.

1) if $\max_K |Q_z| = 0$ we denote $P_z = \varepsilon_1 . Q_z$, where $\varepsilon_1 > 0$ and $\varepsilon_1 . |Q_z(z)| > 1$. For the generalized polynomial P_z it holds $|P_z(z)| > 1$ and $\max_K |P_z| = \varepsilon_1 . \max_K |Q_z| = 0 < 1$.

2) if $\max_K |Q_z| > 0$ we denote $P_z = (\varepsilon_2 . Q_z)/\max_K |Q_z|$, where $0 < \varepsilon_2 < 1$ and $\varepsilon_2 . |Q_z(z)| > \max_K |Q_z|$. For the generalized polynomial P_z we have

$$|P_z(z)| = \varepsilon_2 . |Q_z(z)| / \max_K |Q_z| > 1 \text{ and}$$

$$|P_z| = \varepsilon_2 . |Q_z| / \max_K |Q_z| < |Q_z| / \max_K |Q_z| \leq 1 \text{ on } K.$$

Since $\bar{\Delta}_G \setminus U$ is a compact set and the generalized polynomials are continuous, there exist points $z_1, z_2, \ldots, z_r$, their neighbourhoods $V_{z_1}, V_{z_2}, \ldots, V_{z_r}$ and generalized polynomials $P_{z_1}, P_{z_2}, \ldots, P_{z_r}$ such that $\bigcup_{j=1}^{r} V_{z_j} \supset \bar{\Delta}_G \setminus U$, $|P_{z_j}|>1$ on V_{z_i} and $\max_K |P_{z_i}|<1$ for $j=1, 2, \ldots, k$.

We consider the polynomial polyhedron:

$$A=\{(\lambda, g)\in C_G \mid \lambda\leq 1 \text{ and } |P_{z_j}(\lambda, g)|\leq 1, \quad j=1, 2, \ldots, k\}.$$

If $z=(\lambda, g)\in K$, then $\lambda<1$ and $|P_j(z)|\leq \max_K |P_{z_j}|<1$. Hence $z\in \operatorname{int} A$ and we obtain $K\subset \operatorname{int} A$. If $z=(\lambda, g)\in \bar{\Delta}_G\setminus U$, then this point belongs to V_{z_i} for some j and $|P_{z_j}(\lambda, g)|>1$, i. e. $z\notin A$. If $z\notin U$ and $z\notin \bar{\Delta}_G$ it is clear that $z\notin A$. Hence $A\subset U$.

Example. In [6] it is proved that the compact set $K=\{(1, e_s)/s\in[0, 2\pi]\}$ is polynomially convex in C_G. By Lemma 4 there exists a polynomial polyhedron A in C_G which contains K and $*\notin A$. Then $\tilde{\chi}_1(A)$ is a compact in C containing the unit circle $\tilde{\chi}_1(K)$. Since $*\notin A$, then $0\notin \tilde{\chi}_1(A)$. Hence $C\setminus\tilde{\chi}_1(A)$ is not connected and therefore $\tilde{\chi}_1(A)$ is not polynomially convex in C. Then $\tilde{\chi}_1(A)$ is not a polynomial polyhedron in C.

Notice that in the same way as in [6] it can be proved the polynomial convexity in C_G of every compact of the type $\{(1, e_s)/s\in[\alpha, \beta]\}$.

Let $[\alpha, \beta]$ be a finite closed interval in R, $g_0\in G$ and $\lambda_0>0$ be arbitrary. The set $D=\{(\lambda_0, g_0 . e_s)/s\in[\alpha, \beta]\}$ is said to be an arc in C_G. Every arc is a polynomially convex set in G_G. In fact, the map $\tau: C_G\to C_G$ defined as follows: $\tau(\lambda, g)=(\lambda/\lambda_0, g . g_0^{-1})$ is a homeomorphism and $\tau(D)$ is the compact set $K=\{(1, e_s)/s\in[\alpha, \beta]\}$ which is polynomially convex. If $(\lambda_1, g_1)\notin D$, then $\tau(\lambda_1, g_1)=(\lambda_1/\lambda_0, g_1 . g_0^{-1})\notin K$ and there exists a generalized polynomial $P'=\Sigma\alpha_j\tilde{\chi}_{p(j)}$, for which $|P'(\lambda_1/\lambda_0, g_1 . g_0^{-1})|>\max_K|P'|$. For the generalized polynomial $P(\lambda, g)=(P'\circ\tau)(\lambda, g)=P'(\lambda/\lambda_0, g . g_0^{-1})=\Sigma\,\alpha_j\lambda_0^{-p(j)} . g_0^{-1}(p(j))$ $\tilde{\chi}_{p(j)}(\lambda, g)=\Sigma\beta_j\tilde{\chi}_{p(j)}(\lambda, g)$, where $\beta_j=\alpha_j . \lambda_0^{-p(j)} . g_0^{-1}(p(j))$ we obtain:

$$|P(\lambda_1, g_1)|=|P'(\lambda_1/\lambda_0, g_1\, g_0^{-1})|>\max_K|P'|=\max_D|P|.$$

Hence $(\lambda_1, g_1)\notin\hat{D}$ and D is polynomially convex in C_G.

Lemma 5. *If f is a hyper-analytic function in the open set $U\subset C_G$, then there exists a sequence $\{h_{n_k}\circ\tilde{\chi}_{1/n_k}\}_{k=1}^{\infty}$ which uniformly approximates f on U such that:*

1) h_{n_k} *is analytic in* $\tilde{\chi}_{1/n_k}(U)$, $n_k\in Z^+$ *for every* k.

2) $1/n_m=1/(n_s . \beta_{m,s})$; $\beta_{m,s}\in Z^+$ *for every* $m>s$.

Proof. Since f is a hyper-analytic function in U then there exists a sequence $\{h_{n'_k}\circ\tilde{\chi}_{1/n'_k}\}_{k=1}^{\infty}$ which uniformly approximates f in U and $h_{n'_k}$ is analytic in $\tilde{\chi}_{1/n'_k}(U)$, $n'_k\in Z^+$ for every k. We denote $n_1=n'_1$ and $n_2=n_1 . n'_2>n_1$. Then the function $h_{n_2}=h_{n'_2}\circ\varphi_2$, where $\varphi_2(z)=z^{n_1}$ is analytic in $\tilde{\chi}_{1/n_2}(U)$ and in U we have: $h_{n'_2}\circ\tilde{\chi}_{1/n'_2}=h_{n'_2}\circ(\tilde{\chi}_{1/n_2})^{n_1}$ $=h_{n'_2}\circ\varphi\circ\tilde{\chi}_{1/n_2}=h_{n_2}\circ\tilde{\chi}_{1/n_2}$. If already we have a $n_k\in Z^+$, h_{n_k} — analytic in $\tilde{\chi}_{1/n_k}(U)$ such that n_k is obtained from n_{k-1} by multiplication with a positive integer and $h_{n'_k}\circ\tilde{\chi}_{1/n'_k}=h_{n_k}\circ\tilde{\chi}_{1/n_k}$, then we do the following step as above. Denote $n_{k+1}=n_k . n'_{k+1}$ $>n_k$ the function $h_{n_{k+1}}=h_{n'_{k+1}}\circ\varphi_{k+1}$ is analytic in $\tilde{\chi}_{1/n_{k+1}}(U)$, where $\varphi_{k+1}(z)=z^{n_k}$ and we have: $h_{n'_{k+1}}\circ\tilde{\chi}_{1/n'_{k+1}}=h_{n'_{k+1}}\circ(\tilde{\chi}_{1/n_{k+1}})^{n_k}=h_{n'_{k+1}}\circ\varphi_{k+1}\circ\tilde{\chi}_{1/n_{k+1}}=h_{n_{k+1}}\circ\tilde{\chi}_{1/n_{k+1}}$.

In this way we obtain the sequence $\{h_{n_k} \circ \widetilde{\chi}_{1/n_k}\}_{k=1}^{\infty}$ which satisfies the conditions 1) and 2). Since $h_{n_k} \circ \widetilde{\chi}_{1/n_k} = h_{n'_k} \circ \widetilde{\chi}_{1/n'_k}$ in U for every k, then the sequence is uniformly convergent to f in U.

Let t be an arbitrary positive integer. It is easy to see from the proof of Lemma 5 that we can choose the indices $n_1 < n_2 < \cdots$ to be multiple of t.

Proof of Theorem 1. Let K be compact polynomially convex subset of Δ_G. By Lemma 4 there exists a polynomial polyhedron A in C_G such $K \subset A \subset U$. Applying now Lemma 2 we can find a positive integer t such that $\widetilde{\chi}_{1/t}(A)$ is a polynomial polyhedron in C.

Let f be a hyper-analytic function in U. Then we choose the sequence $\{h_{n_k} \circ \widetilde{\chi}_{1/n_k}\}_1^{\infty}$ as in Lemma 5. We may assume that n_k is multiple to t for every k. For an arbitrary positive ε there exists a n_{k_0} such that $\max_U |f(\lambda, g) - (h_{n_{k_0}} \circ \widetilde{\chi}_{1/n_{k_0}})(\lambda, g)| < \varepsilon/2$.

In accordance with the notice to Lemma 2 the set $\widetilde{\chi}_{1/n_{k_0}}(A)$ is a polynomial polyhedron in C. For the function $h_{n_{k_0}}$, which is analytic in the neighbourhood $\widetilde{\chi}_{1/n_{k_0}}(U)$ of the compact polynomially convex set $\widetilde{\chi}_{1/n_{k_0}}(A)$, we may apply the classicall Runge's theorem. Hence there exists a polynomial P such that:

$$\max_{\widetilde{\chi}_{1/n_{k_0}}(A)} |h_{n_{k_0}}(z) - P(z)| < \varepsilon/2.$$

For the function f and the generalized polynomial $P_G = P \circ \widetilde{\chi}_{1/n_{k_0}}$ we obtain that $|f(\lambda, g) - P_G(\lambda, g)| \le |f(\lambda, g) - (h_{n_{k_0}} \circ \widetilde{\chi}_{1/n_{k_0}})(\lambda, g)| + |(h_{n_{k_0}} \circ \widetilde{\chi}_{1/n_{k_0}})(\lambda, g) - (P \circ \widetilde{\chi}_{1/n_{k_0}})(\lambda, g)| < \varepsilon$ for every $(\lambda, g) \in A$. Hence f is uniformly approximable with generalized polynomials on A. Then this is true on $K \subset A$, too.

Let K be an arbitrary compact polynomially convex set in C_G. We consider a homeomorphism $\tau: \mathsf{C}_G \to \mathsf{C}_G$ defined as: $\tau(\lambda, g) = (\lambda/\lambda_0, g)$, where $\lambda_0 > \max\{\lambda/(\lambda, g) \in K\}$. If $U_1 = \tau(U)$, $K_1 = \tau(K) \subset \Delta_G$, $F = f \circ \tau^{-1}$ and $\{H_{n_k} = h_{n_k} \circ \varphi_k\}_{k=1}^{\infty}$, where $\varphi_k(z) = \lambda_0^{1/n_k} . z$ it is easy to see that H_{n_k} is analytic in $\widetilde{\chi}_{1/n_k}(U_1)$ for every k and F is uniformly approximable in U_1 by the sequence $\{H_{n_k} \circ \widetilde{\chi}_{1/n_k}\}_{k=1}^{\infty}$. The compact set K_1 is polynomially convex in the big plane C_G (see the example). For the hyper-analytic function F in U_1 we apply the proved above. In the opposite direction we obtain that f is uniformly approximable with generalized polynomials on K.

Definition 4. *The function f in the open set $U \subset \mathsf{C}_G$ is said to be generalized-analytic in U if it is locally a uniform limit of generalized polynomials.*

Evidently every hyper-analytic function in the big disc Δ_G is generalized-analytic in Δ_G. For an arbitrary open set this cannot be seen directly. In fact, let f be a hyper-analytic function in U, (λ_0, g_0) is an arbitrary point in U and the sequence $\{h_{n_k} \circ \widetilde{\chi}_{1/n_k}\}_{k=1}^{\infty}$ is uniformly convergent to f in U. Let k be fixed. In the open set $V_k = \widetilde{\chi}_{1/n_k}(U)$ we can find a neighbourhood of the point $z_0^k = \widetilde{\chi}_{1/n_k}(\lambda_0, g_0)$ where h_{n_k} is uniformly approximable with polynomials. Then the function $h_{n_k} \circ \widetilde{\chi}_{1/n_k}$ is uniformly approximable with generalized polynomials in the open set $U_k = \widetilde{\chi}_{1/n_k}^{-1}(V_k) \cap U$. The point (λ_0, g_0) belongs to U_k for every k but it is not clear is the set $\bigcap_k U_k$ a neighbourhood of (λ_0, g_0) or not.

Theorem 2. *If U is an open set in C_G, then every hyper-analytic function in U is generalized-analytic in U.*

Proof. Since $U=\bigcup_{j=1}^{\infty}(\Delta_G(j)\cap U)$ it is sufficient to prove the theorem for an open bounded set. On the other side, every open bounded set can be homeomorphically reflected onto an open subset of Δ_G. By means of this homeomorphism to a generalized polynomial corresponds a generalized polynomial and to a hyper-analytic or generalized-analytic function — the same type of function. Hence it is sufficient to consider the case when U is an open subset, which is containing in Δ_G.

Let f be a hyper-analytic function in U and (λ_0, g_0) is an arbitrary point in U. Then there exists an arc $D\subset U$ and $(\lambda_0, g_0)\in D$. Since D is a polynomially convex set in C_G, then by Lemma 4 we can find a polynomial polyhedron A such that $D\subset \mathrm{int}\,A\subset A\subset U$. By Lemma 3 A is a compact polynomially convex set and applying Theorem 1 we obtain that f is uniformly approximable on A with generalized polynomials. Hence f is uniformly approximable with generalized polynomials in the open set $V=\mathrm{int}\,A$, which contains the point (λ_0, g_0). It means that f is a generalized-analytic function in U. The theorem is proved.

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