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ALMOST CONTACT METRIC MANIFOLDS WITH SOME CONDITIONS CONCERNING THE STRUCTURE TENSOR FIELDS

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The minimal class of the classification scheme of the almost contact metric manifolds, containing the classes of the α-Sasakian and the α-Kenmotsu manifolds is considered. This class is characterized by the Nijenhuis tensor, the differential of the fundamental 2-form. The class under consideration could be characterized precisely by the maximal subgroup of the contact conformal group, preserving it as well.

Introduction. The conditions defining the α-Sasakian and the α-Kenmotsu manifolds appear to be a slight modification of the conditions defining the Sasakian and the Kenmotsu manifolds [2]. These manifolds were characterized by the covariant derivative of the fundamental 2-form in [3, 4].

A classification scheme of the almost contact metric manifolds consisting of twelve basic classes is given in [1]. In that scheme the class W_2 is the class of the α-Sasakian manifolds and the class W_3 is the class of the α-Kenmotsu manifolds. The class $W_2 \oplus W_3$ is the minimal class, containing the classes of the α-Sasakian and the α-Kenmotsu manifolds. Some classes of the scheme in [1] are exactly characterized by the subgroups of the contact conformal group [5].

This paper deals with the class $W_2 \oplus W_3$. Similarly to [3], [4], the necessary and sufficient conditions for an almost contact metric manifold to be in $W_2 \oplus W_3$ by making use of the Nijenhuis tensor, the differential of the structure 1-form and the differential of the fundamental 2-form are given. The same way as in [5] has been applied in order to derive the maximal subgroup C_3 of the contact conformal group, preserving the class $W_2 \oplus W_3$. It has been proved that an almost contact metric manifold will be in $W_2 \oplus W_3$ iff it is contact conformally related by a transformation of C_3 to an α-Sasakian manifold.

1. Preliminaries. Let $M(\varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact metric manifold, i. e. M is a differentiable manifold and (φ, ξ, η, g) is an almost contact metric structure on M, formed by the tensor fields φ, ξ, η of type (1, 1), (1, 0), (0, 1) respectively, and a Riemannian metric g such that

$$(1.1) \qquad \varphi^2 = -id + \eta \otimes \xi, \ \eta(\xi) = 1, \ \varphi(\xi) = 0, \ g(\varphi_x, \varphi_y) = g(x, y) - \eta(x)\eta(y)$$

for all vector fields $x, y \in X(M)$, where by $X(M)$ the Lie algebra of the vector fields on M is denoted. The tangential space to M at a point p is denoted by $T_p M$.

The fundamental 2-form Φ on M is given by

$$\Phi(x, y) = g(x, \varphi_y).$$

M is said to be normal [6], if the almost complex structure I defined on $M \times R$ (R is the real line with a coordinate t) by $I(x, \lambda\, d/dt) = (\varphi_x - \lambda\xi, \eta(x)\, d/dt)$ is integrable. It is well known that M is normal iff the tensor field N of type (1, 2) defined by

$$N = [\varphi, \varphi] + d\,\eta \otimes \xi$$

vanishes identically, where $[\varphi, \varphi]$ is the Nijenhuis tensor field of φ.

Let ∇ be the Riemannian connection on M. The covariant derivative of Φ denoted by F satysfies $F(x, y, z) = -(\nabla_x \Phi)(y, z) = g((\nabla_x \varphi) y, z)$. The tensor F has the following properties

$$F(x, y, z) = -F(x, z, y), \tag{1.2}$$

$$F(x, \varphi y, \varphi z) = -F(x, y, z) + \eta(y) F(x, \xi, z) + \eta(z) F(x, y, \xi).$$

Let $\{l_i\}$, $i = 1, \ldots, 2n+1$ be an orthonormal basis of $T_p M$. The 1-forms f, f^* and ω are defined by

$$f(z) = \sum_{i=1}^{2n+1} F(l_i, l_i, z), \quad f^*(z) = \sum_{i=1}^{2n+1} F(l_i, \varphi l_i, z), \quad \omega(z) = F(\xi, \xi, z). \tag{1.3}$$

The tensors p_1, p_2, p_3 and p_9 of type (0, 3) are defined by

$$p_1(x, y, z) = \eta(x)\eta(y)\omega(z) - \eta(x)\eta(z)\omega(y),$$

$$p_2(x, y, z) = \frac{f(\xi)}{2n}\{\eta(z) g(x, y) - \eta(y) g(x, z)\},$$

$$p_3(x, y, z) = -\frac{f^*(\xi)}{2n}\{\eta(z)\Phi(x, y) - \eta(y)\Phi(x, z)\},$$

$$p_9(x, y, z) = \frac{1}{2(n-1)}\{g(\varphi x, \varphi y) hf(z) - g(\varphi x, \varphi z) hf(y) - \Phi(x, y) hf(\varphi z) + \Phi(x, z) hf(\varphi y)\},$$

where $hf = f - f(\xi)\eta - \omega$. Note that p_1, p_2, p_3 and p_9 satisfy the conditions (1.2).

The conditions defining the classes W_2, W_3 and $W_2 \oplus W_3$ of almost contact metric manifolds are the following [1].

The class W_2:

$$F = p_2. \tag{1.4}$$

The class W_3:

$$F = p_3. \tag{1.5}$$

The class $W_2 \oplus W_3$:

$$F = p_2 + p_3. \tag{1.6}$$

It is proved in [8] that for an almost contact metric manifold (1.6) will hold iff $M \times R$ with the induced almost Hermitian structure is a conformal Kahler manifold.

2. Classes of the α-Sasakian and the α-Kenmotsu manifolds. Recall that an α-Sasakian manifold M [2] is an almost contact metric manifold such that

$$N = 0, \; d\eta = \alpha\Phi, \; \alpha \in R. \tag{2.1}$$

An almost contact metric manifold M is said to be an α-contact manifold if $d\eta = k\Phi$, where k is a C^∞ differentiable function on M. Taking into account (1.3) we obtain for the function $k = f(\xi)/n$. In other words M is an α-Sasakian manifold if M is a normal, α-contact with k-const manifold. We shall prove that M will be an α-Sasakian manifold iff M is a normal α-contact manifold with closed fundamental 2-form.

Proposition 2.1. *An almost contact metric manifold M will be an α-Sasakian manifold iff*

$$N = 0, \; d\eta = \frac{f(\xi)}{n}\Phi, \; d\Phi = 0. \tag{2.2}$$

Proof. When M is an α-Sasakian manifold, then (2.2) are automatically satisfied. For the inverse taking the exterior differential of $d\eta = \frac{f(\xi)}{n}\Phi$ and applying $d\Phi = 0$ we derive $f(\xi) = \text{const}$.

Same as for the case with the Sasakian manifolds [3] one can prove

Proposition 2.2. *An almost contact metric manifold M is an α-Sasakian manifold iff $M \in W_2$.*

An almost contact metric manifold M is said to be an α-Kenmotsu manifold (cf. [2, 4]) if

$$N=0,\ d\eta=0,\ d\Phi=\frac{f^*(f)}{n}\,\eta\wedge\Phi \tag{2.3}$$

Remark. This definition does not require $f^*(\xi)=\text{const}$.

Proposition 2.3. *Let M be an almost contact metric manifold. M is an α-Kenmotsu manifold iff $M \in W_3$.*

The proof is the same as for the Kenmotsu manifolds.

3. The class $W_2 \oplus W_3$. The propositions 2.2, 2.3 show that the class $W_2 \oplus W_3$ is the minimal class of the classification scheme for the almost contact metric manifolds [1], containing the classes of the α-Sasakian and the α-Kenmotsu manifolds. First we shall characterize the class $W_2 \oplus W_3$ by the Nijenhuis tensor N, the differential $d\eta$ of the structure 1-form η and by the differential $d\Phi$ of the fundamental 2-form Φ.

Theorem 3.1. *Let $M(\varphi, \xi, \eta, g)$ be an almost contact metric manifold. M will be in the class $W_2 \oplus W_3$ iff*

$$N=0, \tag{3.1}$$

$$d\Phi=\frac{f^*(\xi)}{n}\,\eta\wedge\Phi, \tag{3.2}$$

$$d\eta=\frac{f(\xi)}{n}\,\Phi. \tag{3.3}$$

Proof. Let M be a manifold in the class $W_2 \oplus W_3$. Thus using the equalities

$$d\Phi(x,y,z)=F(x,y,z)+F(y,z,x)+F(z,x,y), \tag{3.4}$$

$$d\eta(x,y)=F(x,\varphi_y,\xi)-F(y,\varphi_x,\xi), \tag{3.5}$$

we obtain from (1.6) the equalities (3. 2), (3.3) and

$$F(x,y,z)-F(\varphi_x,\varphi_y,z)-\eta(y)\,F(\varphi_x,\xi,\varphi_z)=0. \tag{3.6}$$

On the other hand, in [7] it is proved that (3.1) is equivalent to (3.6).

Now let (3.1), (3.2), (3.3) hold. From (3.2) and (3.4) we have

$$F(x,y,\xi)-F(y,x,\xi)=-\frac{f^*(\xi)}{n}\,\Phi(x,y).$$

(3.5) and (3.6) imply

$$F(\varphi_x,\varphi_y,\xi)+F(y,x,\xi)=\frac{f(\xi)}{n}\,g(\varphi_x,\varphi_y).$$

Further, from (3.1) and (3.6) it follows $F(x,y,\xi)-F(\varphi_x,\varphi_y,\xi)=0$. The last three equalities imply

$$F(x,y,\xi)=\frac{f(\xi)}{2n}\,g(\varphi_x,\varphi_y)-\frac{f^*(\xi)}{2n}\,\Phi(x,y). \tag{3.7}$$

Thus, from (3.2) and (3.4) we get

$$d\Phi(\varphi_x,y,z)+d\Phi(x,\varphi_y,z)=\frac{f^*(\xi)}{n}\{\eta(x)\,g(z,y)-\eta(y)\,g(z,x)\}. \tag{3.8}$$

Then (3.7), (3.8) and the well known relation $g(N(x,y),z)=-d\Phi(\varphi_x,y,z)-d\Phi(x,\varphi_y,z)-\eta(y)\{F(x,\varphi_z,\xi)-F(z,\varphi_x,\xi)\}+\eta(x)\{F(y,\varphi_z,\xi)+F(z,\varphi_y,\xi)\}-2F(z,\varphi_x,y)$ imply (1.6). Theorem 3.1 is completely proved.

Analogously to the α-Sasakian and tke α-Kenmotsu manifolds we characterize the manifolds in the class $W_2 \oplus W_3$ by the following conditions for the structure tensor φ.

Definition ([7]). *The structure tensor* φ *on* $M(\varphi, \xi, \eta, g)$ *is said to be*

a) *of D-antiinvariant type if* $(\nabla_x \varphi) y \parallel \xi$,

b) *of ξ-antiinvariant type if* $(\Delta_\xi \varphi) x \parallel \xi$

for all vectors x and y perpendicular to ξ.

Theorem 3.2. *Let M be an almost contact metric manifold. M will be in the class* $W_2 \oplus W_3$ *iff*

a) φ *is of D-antiinvariant type,*

b) φ *is of ξ-antiinvariant type,*

c) *the integral curves of* ξ *are geodesic,*

d) $(\nabla_x \varphi)\xi$ *is in* span $\{x, \varphi_x\}$, *whenever* $\eta(x)=0$.

Proof. When M is in $M_2 \oplus W_3$ the conditions of the theorem follow by a simple verification. Conversely, let the structure tensor φ have the properties of the theorem 3.2. Note that the structure relations (1.1) imply

$$F(x, y, z)=F(hx, hy, hz)+\eta(x)F(\xi, y, z)+\eta(y)F(x, \xi, z)-\eta(z)F(x, \xi, y)-p_1(x, y, z)$$

for all vectors $x, y, z \in T_pM$, $p \in M$. From the definition above we derive directly

i) φ will be of D-antiinvariant type iff $F(hx, hy, hz)=0$,

ii) φ will be of ξ-antiinvariant type iff $F(\xi, hy, hz)=0$ for all $x, y, z \in T_pM$, where $h=-\varphi^2$. Since the integral curves of ξ are geodesic iff the 1-form ω vanishes, then $p_1=0$. The conditions a), b) and c) imply $F(x, hy, hz)=0$, $p_1=0$ and hence

$$F(x, y, z)=\eta(y)F(x, \xi, z)-\eta(z)F(x, \xi, y). \tag{3.9}$$

The condition d) implies

$$F(x, \xi, z)=\alpha\, g(x, z)+\beta\, g(\varphi x, z) \tag{3.10}$$

for all vectors $x, z \in T_pM$ and $\eta(x)=0$. It is easy to see that $\alpha=-f(\xi)/2n$, $\beta=-f^*(\xi)/2n$. Thus comparing (3.9) and (3.10) we derive (1.6), i. e. $M \in W_2 \oplus W_3$.

4. The subgroup C_3 of the contact conformal group. Let M be an almost contact metric manifold and let (φ, ξ, η, g) be an almost contact metric structure on M. The transformation

$$\tilde{\varphi}=\varphi, \quad \tilde{\xi}=\frac{1}{\sqrt{\lambda+\mu}}\xi, \quad \tilde{\eta}=\sqrt{\lambda+\mu}\cdot\eta, \quad \tilde{g}=\lambda\cdot g+\mu\cdot\eta\oplus\eta \tag{4.1}$$

where λ, μ are C^∞ differentiable functions on M such that $\lambda>0$, $\lambda+\mu>0$, is said to be a contact conformal transformation of the structure (φ, ξ, η, g) [5]. The structures (φ, ξ, η, g) and $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ are said to be contact conformally related. It follows from (1.1) and (4.1) that $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is also an almost contact metric structure on M. One can verify that the set C of all contact conformal transformations on M forms a group which we call a contact conformal group.

We consider the transformations of C, defined by the following additional conditions for the functions λ and μ

$$\begin{aligned} C_2 &: d\lambda=0,\ d\mu=0, \\ C_3 &: d\lambda=\xi(\lambda)\eta,\ d\mu=\xi(\mu)\eta. \end{aligned} \tag{4.2}$$

The immediate verification shows that C_2 and C_3 are subgroups of the contact conformal group C. We note that the transformations of C_2 are the usual D-homothetic transformations.

In [5] it is proved that the classes W_2, W_3 and $W_2 \oplus W_3$ are not contact conformally invariant. It is proved also that C_2, respectively C_3, are the maximal subgroups of C, preserving the class W_2 of the α-Sasakian manifolds, respectively the class W_3 of the α-Kenmotsu manifolds.

To obtain the maximal subgroup of the contact conformal group, preserving the class $W_2 \oplus W_3$ is of great importance. For this reason we need the following

Lemma. ([5]). *Let M be an almost contact metric manifold and let* (φ, ξ, η, g) *and* $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ *be contact conformally related by* (4.1). *Let* $\{F, f, f^*, \omega, p_1, p_2, p_3, p_9\}$, *respectively* $\{\tilde{F}, \tilde{f}, \tilde{f}^*, \tilde{\omega}, \tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_9\}$ *be the tensors corresponding to the structure* (φ, ξ, η, g), *respectively to* $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$. *Then*

$$2\tilde{F}(x, y, z) = 2\lambda F(x, y, z) - 2\{(\lambda+\mu) p_1(x, y, z) + \lambda p_3(x, y, z) + \lambda p_9(x, y, z)\} + 2\{\tilde{p}_1(x, y, z) + \tilde{p}_3(x, y, z) + \tilde{p}_9(x, y, z)\} + \mu\{\eta(y) F(x, \xi, z) - \eta(z) F(x, \xi, y) + \eta(y) F(\varphi_z, \xi, \varphi_x)$$

$$- \eta(z) F(\varphi_y, \xi, \varphi_x) + \eta(x)[F(y, \xi, z) - F(z, \xi, y) - F(\varphi_y, \xi, \varphi_z) + F[\varphi_z, \xi, \varphi_y)]\}, \tag{4.3}$$

$$\tilde{\omega}(z) = \omega(z) - (2(\lambda+\mu))^{-1}\{d\lambda(\varphi z) + d\mu(\varphi z)\}, \tag{4.4}$$

$$\tilde{f}(z) = f(z) - (2\lambda(\lambda+\mu))^{-1}\{[(2n-1)\lambda + 2(n-1)\mu] d\lambda(\varphi z) + \lambda d\mu(\varphi_z)\} + \lambda^{-1}\mu f(\xi).\eta(z) \tag{4.5}$$

$$d\lambda(\varphi_z) = \lambda(n-1)^{-1}\{hf(z) - h\tilde{f}(z)\}, \tag{4.6}$$

$$d\lambda(\xi) = n^{-1}\lambda\{\tilde{f}^*(\xi) - f^*(\xi)\}, \tag{4.7}$$

$$\tilde{p}_2 = (\lambda+\mu) p_2, \tag{4.8}$$

for all $x, y, z \in X(M)$.

Theorem 4.1. *The subgroup C_3 is the maximal subgroup of the contact conformal group, preserving the class $W_2 \oplus W_3$.*

Proof. Let $M(\varphi, \xi, \eta, g)$ be a manifold in $W_2 \oplus W_3$ and let the structures (φ, ξ, η, g), $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ be contact conformally related to the functions λ, μ, satisfying (4.2). From (1.6) it follows that

$$\eta(y) F(x, \xi, z) - \eta(z) F(x, \xi, y) + \eta(y) F(\varphi_z, \xi, \varphi_x) - \eta(z) F(\varphi_y, \xi, \varphi_x)$$
$$+ \eta(x)\{F(y, \xi, z) - F(z, \xi, y) - F(\varphi_y, \xi, \varphi_z) + F(\varphi_z, \xi, \varphi_y)\} = 2p_2(x, y, z).$$

Moreover, using (4.4) and (4.6) we have $\tilde{\omega} = 0$, $h\tilde{f} = 0$ and hence $\tilde{p}_1 = 0$, $\tilde{p}_9 = 0$. Thus (4.3) and (4.8) imply $\tilde{F} = \tilde{p}_2 + \tilde{p}_3$ that is $M(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g}) \in W_2 \oplus W_3$. So we have proved that the subgroup C_3 of the contact conformal group preserves the class $W_2 \oplus W_3$.

Now, let $M(\varphi, \xi, \eta, g)$ and $M(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ be two manifolds in $W_2 \oplus W_3$ which are contact conformally related by (4.1). Since $F = p_2 + p_3$ and $\tilde{F} = \tilde{p}_2 + \tilde{p}_3$ we have $\omega = \tilde{\omega} = 0$, $hf = h\tilde{f} = 0$. Further, (4.4) and (4.6) imply (4.2). So it is proved that C_3, is the maximal subgroup of the contact conformal group, preserving the class $W_2 \oplus W_3$.

In [5] we have proved that an almost contact metric manifold M will be in W_2 iff M is contact conformally related to a Sasakian manifold by a transformation of the subgroup C_2, and $M \in W_3$ iff M is contact conformally related to a cosymplectic manifold by a transformation of the subgroup C_3. The class $W_2 \oplus W_3$ has also an exact characteristic by the subgroup C_3 and the class of the α-Sasakian manifolds. The main result of the present paper is

Theorem 4.2. *An almost contact metric manifold M will be in the class $W_2 \oplus W_3$ iff M is contact conformally related to a manifold in W_2 with a transformation of C_3.*

Proof. If $M(\varphi, \xi, \eta, g)$ is in the class W_2, then by an arbitrary transformation of C_3 we will obtain $M(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g}) \in W_2 \oplus W_3$. The proof is the same as for theorem 4.1.

For the inverse, let $M(\varphi, \xi, \eta, g)$ be in the class $W_2 \oplus W_3$. We denote

$$\lambda = |f(\xi)|. \tag{4.9}$$

After the exterior differentiation (3.3), (3.2) implies $(d \ln \lambda + n^{-1} f^*(\xi)\, \eta) \wedge \Phi = 0$. Then it follows $d \ln \lambda + n^{-1} f^*(\xi)\, \eta = 0$. Therefore

$$\lambda^{-1}\xi(\lambda) + n^{-1} f^*(\xi) = 0, \quad d\lambda \circ \varphi = 0. \tag{4.10}$$

Setting

$$\mu = k - \lambda, \quad k = \text{const} > 0, \tag{4.11}$$

we consider the contact conformal transformation of (φ, ξ, η, g) with the functions λ and μ, determined by (4.9) and (4.11). It is clear that λ, μ satisfy (4.2) and the contact conformal transformation considered is in C_3. Making use of theorem 4.1 we get $\widetilde{F} = \widetilde{p}_2 + \widetilde{p}_3$.

Moreover, (4.7) and (4.10) imply $\widetilde{f}^*(\xi) = 0$. Thus $\widetilde{p}_3 = 0$ and M having the structure $(\widetilde{\varphi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g})$ is in W_2.

REFERENCES

1. V. Alexiev, G. Ganchev. On the classification of the almost contact metric manifolds. Mathematics and Education in Mathematics'86. Sofia, 1986, 155-161.
2. D. Jansses, L. Vanhecke. Almost contact structure and curvatuve tensors. *Kodai Math. J.*, **4**, 1981, 1-27.
3. D. Blair. Contact manifolds in Riemannian geometry. Berlin, 1976.
4. K. Kenmotsu. A class of almost contact Riemannian manifolds. *Tohoku Math. J.*, **24**, 1972, 93-103.
5. V. Alexiev, G. Ganchev. Constant conformal transformations of the almost contact metric structures. (in press)
6. S. Sasaki, Y. Hatakeyma On the differentiable manifolds with certain structures which are closely related to almost contact structure. *Tohoku Math. J.*, **13**, 1961, 281-294.
7. G. Ganchev, V. Alexiev. Some classes of almost contact metric manifolds. Mathematics and Education in Mathematics'86. Sofia, 1986, 186-191.
8. J. Oubina. A classification for almost contact structures. (in press)

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