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Българско математическо списание

ON SYSTEMS OF RANDOM ORDINARY DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

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In this paper we obtain theorems on the existence of solutions of the Cauchy problem for systems (linear and semilinear) of random ordinary differential equations of the first order.

1. Introduction. The literature concerning the existence theory of random differentiai equations is quite extensive (see e. g. the monographs [1, 5, 4] and the references therein). In this paper, using the existence theory for evolution equations in Banach spaces (see [2, 8, 9]), we obtain some existence theorems for the above mentioned Cauchy problem.

In order to formulate the problem in question we need some notation. Let $(\Omega, \Gamma, \boldsymbol{P})$ be a complete probability space. By $L^p(\Omega, R)$ ($p \in [1, \infty]$ being a constant, $R=$ the set of all real numbers) we denote the Banach space consisting of all random variables $x:\Omega \to R$ with finite norm

$$\| x \| = [\int_\Omega | x(\omega)|^p \boldsymbol{P}(d\omega)]^{1/p} \text{ if } p \in [1, \infty)$$

and

$$\| x \| = \operatorname*{ess\,sup}_{\omega \in \Omega} | x(\omega) | \text{ if } p = \infty.$$

Denote by Z the set of all complex numbers. For any $\lambda \in Z$ the real (imaginary) part of λ is denoted by $\operatorname{Re} \lambda$ ($\operatorname{Im} \lambda$). For a random variable $y:\Omega \to Z$ the functions $|y|$ $\operatorname{Re} y$, $\operatorname{Im} y$ are defined as usual. We shall use the complex Banach space

$$Y = X + jX = \{y = x_1 + jx_2 : x_1, x_2 \in X\} = L^p(\Omega, Z)$$

with norm $|\!|\!| y |\!|\!| = \| \, |y| \, \|$, $y \in Y$. Let n be a fixed positive integer. The product spaces X^n, Y^n provided with norms

$$\| x \|_n = \| x_1 \| + \ldots + \| x_n \|, \quad x = (x_1, \ldots x_n), \quad x_i \in X,$$

$$|\!|\!| y_n |\!|\!| = |\!|\!| y_1 |\!|\!| + \ldots + |\!|\!| y_n |\!|\!|, \quad y = (y_1, \ldots, y_n), \quad y_i \in Y,$$

respectively, are Banach spaces too. The limit, continuity and derivatives of random functions with values in the Banach spaces introduced above are always understood in the strong sense, and integrals of these functions are taken in the Bochner sense.

We shall consider the Cauchy problem

$$u'(t) + A(t)u(t) = f(t), \quad 0 < t \le t_0, \tag{1.1}$$

$$u(0) = u_0, \tag{1.2}$$

SERDICA Bulgaricae mathematicae publicationes. Vol. 13, 1987, p. 366—374.

where $t_0>0$ is a constant, $u=(u_1, \ldots, u_n), u'=(u'_1, \ldots, u'_n)$ and $f=(f_1, \ldots, f_n)$ are vector functions with values in X^n (or Y^n) and $u_0=(u_{01}, \ldots u_{0n})$ is a given element of X^n (or Y^n). Finally, $A(t)$ is a linear operator (in general unbounded) determined by a matrix $[a_{ik}(t)]_{n\times n}$, where a_{ik} are functions with values in X (or Y). So we have

$$A(t)u(t)=(\sum_{k=1}^{n} a_{1k}(t)u_k(t), \ldots, \sum_{k=1}^{n} a_{nk}(t)u_k(t)).$$

As a corollary of appropriate theorems of [2] and [9] we obtain a theorem on the existence of a unique solution of the problem (1.1), (1.2).

Next we consider the semilinear system

$$u'(t)+A(t)u(t)=(Bu)(t), \quad 0<t\leqq t_0, \tag{1.3}$$

where $Bu=(B_1u, \ldots, B_nu)$ and B_i are some nonlinear operators. Using the appropriate results of papers [8] and [9] we obtain two existence theorems for the problem (1.3), (1.2).

Solutions of the above problems are taken in the classical sense, i. e. u is a solu tion of the problem (1.1), (1.2) (or (1.3), (1.2)) if u is continuous in $[0, t_0]$, u' is con tinuous in $(0, t_0]$, and u satisfies pointwise the problem in question.

Note that all the results of the present paper remain true if we replace the interval $[0, t_0]$ by $[t_1, t_2]$. Then, of course, the initial conditions are taken at t_1.

2. Assumptions concerning operators $A(t)$ and some lemmas. First we consider the operators $A(t)$ appearing in (1.1) in the real case, i. e. in the space X^n. We introduce the following assumption:

(2.I) $a_{ik}: [0, t_0]\times\Omega\to R$, $i, k=1, \ldots, n$ are such random functions that $a_{ik}(t)=a_{ik}(t, \cdot)\in X$ $t\in[0, t_0]$. Moreover, the set

$$D=\{x=(x_1, \ldots, x_n)\in X^n: \sum_{k=1}^{n} a_{ik}(t)x_k\in X, \ i=1, \ldots, n\}$$

is independent of $t\in[0, t_0]$.

Under the above assumption the matrix $[a_{ik}(t)]_{n\times n}$, $t\in[0, t_0]$ determines operators

$$A(t): D\to X^n, \quad A_Y(t): D+jD\to Y^n$$

by the formulas

$$A(t)x=(\sum_{k=1}^{n} a_{1k}(t)x_k, \ldots, \sum_{k=1}^{n} a_{nk}(t)x_k), \quad x=(x_1, \ldots, x_n)\in D,$$

$$A_Y(t)y=A(t)(\mathrm{Re}\, y)+jA(t)(\mathrm{Im}\, y), \quad y=(y_1, \ldots, y_n)\in D+jD,$$

where $\mathrm{Re}\, y=(\mathrm{Re}\, y_1, \ldots, \mathrm{Re}\, y_n)$, $\mathrm{Im}\, y=(\mathrm{Im}\, y_1, \ldots, \mathrm{Im}\, y_n)$. It is clear that

$$A_Y(t)y=(\sum_{k=1}^{n} a_{1k}(t)y_k, \ldots, \sum_{k=1}^{n} a_{nk}(t)y_k), \quad y\in D+jD.$$

Lemma 2.1. *If assumption* (2.I) *is satisfied, then D and $D+jD$ are linear dense sets of the spaces X^n and Y^n, respectively. Moreover, $A(t)$ and $A_Y(t)$ are linear closed operators.*

Proof. The linearity of D, $D+jD$, $A(t)$ and $A_Y(t)$ is obvious. It follows from [6] (Section 3) that X_0^n is a dense set of X^n, where X_0 denotes the set of all simple real valued random functions. Hence, in view of $X_0^n\subset D$, D is a dense set of X^n. Consequently, $D+jD$ is a dense set of Y^n. Arguing like in the proof of Lemma 5.3 of [8] one can show that $A(t)$ are closed operators. Hence, it easily follows the closedness of $A_Y(t)$.

Now we introduce the following assumption.

(2.II) For any $t \in [0, t_0]$ the resolvent $R(\lambda, A_Y(t))$ exists for all $\lambda \in Z$ with $\operatorname{Re} \lambda \leq 0$ and there holds the inequality

$$(2.1) \qquad \| R(\lambda, A_Y(t)) \|_n \leq K_1(1+|\lambda|)^{-1}$$

$K_1>0$ being a constant.

Assumptions (2.I), (2.II), Lemma 2.1, and Theorem 2 of [9] imply that for any $s \in [0, t_0]$ there exists in X^n a strongly continuous semigroup of operators $e^{-tA(s)}$, $t \geq 0$. In view of assumption (2.II) there exists a bounded inverse operator

$$A_Y^{-1}(t): Y^n \to D+jD, \; t \in [0, t_0]$$

and there holds the equality

$$A_Y^{-1}(t)y = A^{-1}(t)(\operatorname{Re} y)+jA^{-1}(t)(\operatorname{Im} y), \; t \in [0, t], \; y \in Y^n.$$

The closedness of the operators $A(s)$, $s \in [0, t_0]$ and the boundedness of the operators $A^{-1}(t)$, $t \in [0, t_0]$ yield the boundedness of $A(s)A^{-1}(t)$. This enables us to make the following assumption.

(2.III) There are constants $K_2>0$ and $\alpha \in (0.1)$ such that

$$\| [A(s)-A(t)]A^{-1}(\tau) \|_n \leq K_2 |s-t|^{\alpha}, \; s, t, \tau \in [0, t_0].$$

Now we state some particular case of the matrix $[a_{ik}(t)]_{n\times n}$ involving assumption 2.I)-(2.III). Put

$$(2.2) \qquad a_{ik}(t, \omega) = a(\omega)c(t)b_{ik}(\omega), \; t \in [0, t_0], \; \omega \in \Omega, \; i, k=1, \ldots, n$$

and introduce the following assumptions.

(2.IV) $c:[0, t_0] \to (0, \infty)$ is a uniformly Hölder continuous function of exponent $\alpha \in (0, 1)$.

(2.V) $a \in X$ is such a random variable that

$$\operatorname*{ess\,inf}_{\omega \in \Omega} a(\omega) \geq K_3,$$

$K_3>0$ being a constant.

(2.VI) $b_{ik} \in L^{\infty}(\Omega, R)$, $i, k=1, \ldots, n$ and eigenvalues $\mu_k(\omega)$, $k=1, \ldots, n$ of the matrix $B(\omega)=[b_{ik}(\omega)]_{n\times n}$ satisfy the inequalities

$$(2.3) \qquad \operatorname*{ess\,inf}_{\omega \in \Omega} (\operatorname{Re} \mu_k(\omega)) \geq K_4, \; k=1, \ldots, n,$$

where K_4 is a positive constant.

According to Gerszgorin theorem [3] (p. 415), the inequalities (2.3) are satisfied if

$$(2.4) \qquad \operatorname*{ess\,inf}_{\omega \in \Omega} [b_{ii}(\omega) - \sum_{\substack{k=1 \\ k\neq i}}^{n} |b_{ik}(\omega)|] \geq K_4, \; i=1, \ldots, n.$$

Lemma 2.2. *Assumptions (2.IV)-(2.VI) imply assumptions (2.I), (2. III) in the case (2.2)*

Proof. The implication (2.IV)-(2.VI)$\Rightarrow$(2.I) is obvious. To prove (2.II) note that there exists a set $\Omega_0 \in \Gamma$ with $P(\Omega_0)=1$ and positive constants K_5, K_6, K_7 such that

$$(2.5) \qquad |b_{ik}(\omega)| \leq K_5, \; \operatorname{Re} \mu_k(\omega) \geq K_4, \; |\mu_k(\omega)| \leq K_6$$

for any $\omega\in\Omega_0$, $i, k=1,\ldots,n$ and

(2.6) $$c(t)a(\omega)\geqq K_7,\quad t\in[0, t_0],\quad \omega\in\Omega_0.$$

For any $t\in[0,t_0]$, $\omega\in\Omega_0$ we have

(2.7) $\det(\lambda I-A(t,\omega))=[a(\omega)c(t)]^n\det(\mu I-B(\omega))=[a(\omega)c(t)]^n(\mu-\mu_1(\omega))\ldots(\mu-\mu_n(\omega)),$

where $\mu=\lambda[c(t)a(\omega)]^{-1}$, I denotes the unit matrix (of the n-th order), and det is the abbreviation of the word determinant. It follows from (2.5) that

$$|\mu'-\mu_k(\omega)|\geqq K_8(1+|\mu'|),\quad k=1,\ldots,n$$

for any $\mu'\in Z$ with $\operatorname{Re}\mu'\leqq 0$, where $K_8>0$ is a constant depending only on K_4 and K_6. Hence, by (2.7) and (2.6), we have

$$|\det(\lambda I-A(t,\omega))|\geqq K_9(1+|\lambda|)(c(t)a(\omega)+|\lambda|)^{n-1}$$

for any $\lambda\in Z$ with $\operatorname{Re}\lambda\leqq 0$, where $K_9=K_8^n\min\{1, K_7\}$. The above inequality implies the existence of the inverse matrix

$$(\lambda I-A(t,\omega))^{-1}=[d_{ik}(\lambda,t,\omega)]_{n\times n}$$

for any $\lambda\in Z$ with $\operatorname{Re}\lambda\leqq 0$, $t\in[0,t_0]$, $\omega\in\Omega_0$ and the inequality

$$|d_{ik}(\lambda,t,\omega)|\leqq K_{10}(1+|\lambda|)^{-1},\quad i,k=1,\ldots,n,$$

$K_{10}>0$ being a constant. Thus assumption (2.II) is satisfied. The existence of the matrix

$$A^{-1}(t,\omega),\quad t\in[0,t_0],\quad \omega\in\Omega_0$$

yields the existence of the matrix $B^{-1}(\omega)$, $\omega\in\Omega_0$ and the inequality

$$A^{-1}(t,\omega)=[c(t)a(\omega)]^{-1}B^{-1}(\omega),\quad t\in[0,t_0],\quad \omega\in\Omega_0.$$

Hence we have

$$[A(s,\omega)-A(t,\omega)]A^{-1}(\tau,\omega)=[c(s)-c(t)][c(\tau)]^{-1}I$$

for any $s, t, \tau\in[0,t_0]$, $\omega\in\Omega_0$. Consequently, assumption (2.III) is satisfied as well. This completes the proof.

Now we consider the operators $A(t)$ appearing in (1.1) in the complex case, i. e. in the space Y^n. We make the following assumption.

(2.VII) $a_{ik}:[0,t_0]\times\Omega\to Z$, $i,k=1,\ldots,n$ are such random functions that $a_{ik}(t)=a_{ik}(t,\cdot)\in Y$ $t\in[0,t_0]$. Moreover, the set

$$D=\{y=(y_1,\ldots,y_n)\in Y^n:\ \sum_{k=1}^n a_{ik}(t)y_k\in Y,\ i=1,\ldots,n\}$$

is independent of $t\in[0,t_0]$.

Then the matrix $[a_{ik}(t)]_{n\times n}$, $t\in[0,t_0]$ defines the operator $A(t):D\to Y^n$ by the formula

$$A(t)y=(\sum_{k=1}^n a_{1k}(t)y_k,\ldots,\sum_{k=1}^n a_{nk}(t)y_k),\ y=(y_1,\ldots,y_n)\in D.$$

Arguing as in the proof of Lemma 2.1, we obtain the following lemma.

Lemma 2.3. *If assumption (2.VII) is satisfied, then D is a linear dense set of the space Y^n and $A(t)$ are linear closed operators.*

Now let us introduce the following assumption

(2.VIII) For any $t \in [0, t_0]$ the resolvent $R(\lambda, A(t))$ exists for all $\lambda \in Z$ with Re $\lambda \leqq 0$ and there holds the inequality

$$|\!|\!| R(\lambda, A(t)) |\!|\!|_n \leqq K_{11}(1+|\lambda|)^{-1},$$

$K_{11}>0$ being a constant.

Assumptions (2.VII), (2.VIII) and Lemma 2.3 imply that for any $s \in [0, t_0]$ there exists in Y^n a strongly continuous semigroup of operators $e^{-tA(s)}$, $t \geqq 0$ (see [2], Section 2.3).

Finally, we make the following assumption.

(2.IX) There are constants $K_{12}>0$ and $\alpha \in (0, 1)$ such that

$$|\!|\!| [A(t)-A(s)]A^{-1}(\tau) |\!|\!|_n \leqq K_{12}|t-s|^{\alpha}, \quad t, s, \tau \in [0, t_0].$$

Arguing as in the proof of Lemma 2.2, one can prove that assumptions (2.VII)—(2.IX) are satisfied in the particular case (2.2) under assumptions (2.IV)—(2.VI) provided that

$$b_{ik} \in L^{\infty}(\Omega, Z), \quad i, k=1, \ldots, n. \tag{2.8}$$

In the case (2.8) the condition (2.4) with $b_{ii}(\omega)$ replaced by Re $b_{ii}(\omega)$ is a sufficient one for (2.3).

3. Existence theorems. First let us introduce some notation. For any interval $T \subset R$ and any nonnegative integer k by $C^k(T, X)$ we denote the vector space of all random functions $u: T \to X$ possessing continuous derivatives $u^{(i)}$, $i=0, 1, \ldots, k$. We abbreviate $C^0(T, X, X)=C(T, X)$. In the sequel we put $T=[0, t_0]$ or $T=(0, t_0]$. The vector space $C([0, t_0], X)$ provided with norm

$$\|u\|_{[0, t_0]}=\sup\{\|u(t)\|: t \in [0, t]\}$$

is a Banach space. We shall also use the Banach space $C^{(\varepsilon)}([0, t_0], X)$ ($\varepsilon \in (0, 1)$ being a constant) consisting of all functions $u \in C([0, t_0], X)$ with finite norm

$$\|u\|^{(\varepsilon)}_{[0, t_0]}=\|u\|_{[0, t_0]}+\sup\{\|u(t)-u(s)\|\,|t-s|^{-\varepsilon}: t, s \in [0, t_0]\}.$$

We extend the above notation to the case where X is replaced by X^n. Then we have

$$C^k(T, X^n)=\{u=(u_1, \ldots, u_n): u_i \in C^k(T, X),\ i=1, \ldots, n\},$$

$$C^{(\varepsilon)}([0, t_0], X^n)=\{u=(u_1, \ldots, u_n): u_i \in C^{(\varepsilon)}([0, t_0], X),\ i=1, \ldots, n\}.$$

Norms in the spaces $C([0, t_0], X^n)$ and $C^{(\varepsilon)}([0, t_0], X^n)$ are given by the formulas

$$\|u\|_{n, [0, t_0]}=\sup\{\|u(t)\|_n: t \in [0, t_0]\} \tag{3.1}$$

and

$$\|u\|^{(\varepsilon)}_{n, [0, t_0]}=\|u\|_{n, [0, t_0]}+\sup\{\|u(t)-u(s)\|_n|t-s|^{-\varepsilon}: t, s \in [0, t_0]\},$$

respectively.

All the above definitions can be extended to the complex case, i. e. to the case where X is replaced by Y.

Lemma 2.1 and Theorem 4 of [9] imply the following theorem.

Theorem 3.1. *If assumptions (2.I)—(2.III) are satisfied and* $u_0 \in X^n$, $f \in C^{(\beta)}([0, t_0], X^n)$ ($\beta \in (0, 1)$ *being a constant*), *then there exists a unique solution of the problem (1.1), (1.2). The solution is given by the formula*

$$u(t)=U(t, 0)u_0+\int_0^t U(t, s)f(s)ds,$$

where $U(t, s)$ *is the fundamental solution of the equation*

$$z'(t)+A(t)z(t)=0.$$

Taking advantage of Lemma 2.3 and Theorem 2.3.2 of [2], we can extend **Theorem** 3.1 to the complex case. Then, of course, assumptions of Theorem 3.1 have to be replaced by the following ones: (2.VII)—(2.IX), $u_0 \in Y^n$, $f \in C^{(\beta)}([0, t_0], Y^n)$.

Now we consider the problem (1.3), (1.2) under the following assumptions.

(3.I) Assumptions (2.I)—(2.III).

(3.II) We have $B_i: S \to C([0, t_0], X)$, $i=1, \dots, n$, where S denotes the set of all functions $u \in C([0, t_0], X^n)$ such that u takes values in the domain of $A^{\alpha'}(0)$ ($\alpha' \in [0, 1)$ being a constant) and

$$A^{\alpha'}(0)u \in C([0, t_0]), X^n).$$

(3.III) For any $t \in [0, t_0]$, $u \in C([0, t_0]), X^n)$ we have

$$\| [B_i A^{-\alpha'}(0)u](t) \| \leq N_1(1+\| u \|_{n, [0, t]}), \ i=1, \dots, n,$$

where N_1 is a positive constant and $\| u \|_{n, [0, t]}$ is the norm defined by (3.1) with t_0 replaced by t (for $t=0$ this norm equals $\| u(0) \|_n$).

(3.IV) There is a constant $\beta \in (0, 1)$ and for any $a>0$ there is a positive constant $N_2=N_2(a)$ such that if

$$u, v \in C([0, t_0], X^n), \ \| u \|_{n, [0, t_0]}, \ \| v \|_{n, [0, t_0]} \leq a, \tag{3.2}$$

then

$$\| B_i A^{-\alpha'}(0)u](t)-[B_i A^{-\alpha'}(0)v](t) \| \leq N_2 d_\beta(\| u-v \|_{n, [0, t]}), \ t \in [0, t_0], \ i=1, \dots, n,$$

where

$$d_\beta(s)=\begin{cases} s^\beta, & 0 \leq s \leq 1, \\ s, & s>1. \end{cases}$$

(3.V) The element u_0 appearing in (1.2) belongs to the domain of $A^\gamma(0)$ for some $\gamma \in (\alpha', 1)$.

(3.VI) For any $a>0$ there is a constant $\beta'=\beta'(a) \in (0, 1)$ such that each operator $B_i A^{-\alpha'}(0)$ ($i=1, \dots, n$) maps the set

$$W(a, \delta)=\{u \in C^{(\delta)}([0, t_0], X^n): \| u \|^{(\delta)}_{n, [0, t_0]} \leq a\}, \ \delta=\gamma-\alpha'$$

into a bounded set of the space $C^{(\beta')}([0, t_0], X)$.

(3.VII) For any $a>0$ there is a constant $N_3=N_3(a)>0$ such that for any set $U \subset W(a, \gamma-\alpha')$ and any $t \in [0, t_0]$, $i=1, \dots, n$ we have

$$m([B_i A^{-\alpha'}(0)U](t)) \leq N_3 \sup\{m_n(U(\tau)): 0 \leq \tau \leq t\},$$

where $U(\tau)=\{n(\tau): u \in U\}$, m and m_n denote the Hausdorff's measures of noncompactness of the spaces $(X, \| \cdot \|)$ and $(X^n, \| \cdot \|_n)$, respectively.

One can easily find that assumptions (3.II)—(3.IV) and (3.VI) imply the following ones, respectively.

(3.II') The operator B appearing in (1.3) is defined on S and takes values in $C([0, t_0], X^n)$.

(3.III') For any $t \in [0, t_0]$, $u \in C([0, t_0], X^n)$ we have

$$\| [BA^{-\alpha'}(0)u] (t) \|_n \leq nN_1(1+\| u \|_{n, [0, t]}).$$

(3.IV′) For any $a>0$ and any u, v satisfying (3.2) we have

$$\| [BA^{-\alpha'}(0)u](t)-[BA^{-\alpha'}(0)v](t) \|_n \leq nN_2 d_\beta(\| u-v \|_{n,[0,t]}),\ t\in[0,t_0].$$

(3.VI′) For any $a>0$ the operator $BA^{-\alpha'}(0)$ maps the set $W(a, \gamma-\alpha')$ into a bounded set of the space $C^{(\beta')}[0, t_0], X^n)$.

Taking into consideration the definition of the norm $\|\cdot\|_n$ and Lemma 4.7 of [7] (see also the proof of Lemma 3.1 of [8]), we find that assumption (3.VII) yields the following one.

(3.VII′) For any $a>0$ and any $U\subset W(a, \gamma-\alpha')$, $t\in[0, t_0]$ we have

$$m_n([BA^{-\alpha'}(0)U](t))\leq nN_3 \sup\{m_n(U(\tau)):\ 0\leq\tau\leq t\},$$

where

$$BA^{-\alpha'}(0)U=\{BA^{-\alpha'}(0)u:\ u\in U\}.$$

With the aid of paper [9] one can show that the results of Section 2 of [8] remain valid in the case of real Banach spaces. Consequently, the results of Sections 3 and 4 of [8] are valid in this case as well. Now observe that assumptions (3.II′)—(3.IV′) (3.VI′), (3.VII′) imply that the operator B satisfies assumptions (3.II)—(3.IV), (3.VI), (3.VII) of [8] with $N=1$ and with X replaced by X^n. Hence, using Theorem 3.1 of [8] (in the case of a real Banach space), we obtain the following theorem.

Theorem 3.2. *If assumptions (3.I)—(3.VII) are satisfied, then the problem (1.3), (1.2) has a solution*

$$u\in C^{(\delta)}([0, t_0], X^n)\cap C^1((0, t_0], X^n),\ \delta=\gamma-\alpha'$$

such that

$$A^{\alpha'}(0)u\in C^{(\delta)}([0,t_0], X^n).$$

Particular cases of the operators B_i similar to those of [8] can be considered Let us consider, for instance, operators B_i $(i=1, \ldots, n)$ given by the formula

$$(B_i u)(t)=f_i(\cdot, t, u(t), u(\varphi_i(t)), E_i(u(t)), E_i u(\psi_i(t))), G_i(u(t)), G_i(u(\chi_i(t)))), \tag{3.3}$$

where $u(\varphi_i(t))=(u_1(\varphi_{i1}(t)), \ldots, u_n(\varphi_{in}(t)))$, and $u(\psi_i(t))$, $u(\chi_i(t))$ are defined likewise.

We make the following assumptions.

(3.VIII) $f_i: \Omega\times[0, t_0]\times R^{6n}\to R$ $(i=1, \ldots, n)$ are random functions satisfying the inequality

$$|f_i(\omega, 0, \ldots, 0)|\leq\xi_0(\omega),\ \omega\in\Omega_0,$$

where $\xi_0\in X$ is some nonnegative random variable and $\Omega_0\in\Gamma$ is such a set that $P(\Omega_0=1)$.

(3.IX) There are constants ε, $\varepsilon_1\in(0, 1)$, $N_4>0$ such that for any $a>0$ we have

$$|f_i(\omega, t, u_1, \ldots, u_6)-f_i(\omega, u', u_1', \ldots, u_6')|\leq N_4[|t-t'|^{\varepsilon_1}+(1+a^{1-\varepsilon})(|u_3-u_3'|+|u_4-u_4'|)^{\varepsilon}$$

$$+|u_1-u_1'|+|u_2-u_2'|+|u_5-u_5'|+|u_6-u_6'|],\ i=1, \ldots, n$$

for any $\omega\in\Omega_0$, $t, t'\in[0, t_0]$, $u_3, u_3', u_4, u_4'\in R_a^n$, $u_k, u_k'\in R^n$, $k=1, 2, 5, 6$, where

$$|u_p-u_p'|=\sum_{r=1}^{n}|u_{pr}-u_{pr}'|,\ R_a^n=\{w\in R^n:\ |w|\leq a\}.$$

(3.X) The element u_0 appearing in (1.2) belongs to the domain of $A^{\gamma}(0)$, $\gamma \in (0, 1)$ being a constant.

(3.XI) The operators E_i: $X^n \to X^n$ $(i=1, \ldots, n)$ are completely continuous and G_i: $D_i \to X^n$ $(i=1, \ldots, n)$ are linear closed operators, where $D_i \subset X^n$ include the domain of $A^{\alpha'}(0)$, $\alpha' \in (0, \gamma)$ being a constant.

(3.XII) The functions $\varphi_{ik}, \psi_{ik}, \chi_{ik}: [0, t_0] \to R$, $i, k=1, \ldots, n$ satisfy the inequalities

$$0 \leq \varphi_{ik}(t) \leq (t), \quad 0 \leq \psi_{ik} \leq (t) \leq t, \quad 0 \leq \chi_{ik}(t) \leq t$$

and are uniformly Hölder continuous.

Assumptions (3.VIII) and (3.IX) yield the inequality

$$|f_i(\omega, t, u_1, \ldots, u_6)| \leq \xi_1(\omega) + N_5(|u_1| + \ldots + |u_6|) \tag{3.4}$$

for any $(\omega, t, u_1, \ldots, u_6) \in \Omega_0 \times [0, t_0] \times R^{6n}$, $i=1, \ldots, n$ where N_5 is a positive constant random and $\xi_1 \in X$ is some nonnegative random variable. Arguing like in [8] (after the inequality (5.5)), we conclude that assumptions (3.VIII)—(3.XII) and the inequality (3.4) imply assumptions (3.II)—(3.VII) for the case (3.3). Therefore Theorem 3.2 holds true in this case.

In the considered particular case we may take, for instance,

$$G_i = A^{\alpha'}(0), \quad E_i u = h_i(\|u\|_n), \quad u \in X^n, \quad i=1, \ldots, n,$$

where h_i: $[0, \infty) \to R^n$ $(i=1, \ldots, n)$ are continuous functions.

Now we state an existence and uniqueness theorem for the problem (1.3), (1.2). For this purpose denote

$$F = \bigcup_{Q} C^{(q)}([0, t_0], X), \quad F_n = \bigcup_{Q} C^{(q)}([0, t_0], X^n), \quad Q = \{q \in (0, 1)\}.$$

Theorem 3.3. *Let assumptions (3.I)–(3.III), (3.V) be satisfied and suppose* $B_i A^{-\alpha'}(0)$ $(i=1, \ldots, n)$ *map* F_n *into* F. *Moreover, we assume that there are constants* $N_5 > 0$, $\gamma' \in (0, 1-\gamma)$ *such that for any* $a \geq e$ *and any* u, v *satisfying (3.2) we have*

$$\|[B_i A^{-\alpha'}(0)u](t) - [B_i A^{-\alpha'}(0)v](t)\| \leq N_5 (\ln a)^{\gamma'} \|u - v\|_{n, [0, t]}, \quad t \in [0, t_0], \quad i=1, \ldots, n.$$

Then the problem (1.3), (1.2) *has a unique solution* u *in the set* $F_n \cap C^1((0, t_0], X^n)$ *Moreover, we have*

$$u, \; A^{\alpha'}(0)u \in C^{(q)}([0, t_0], X^n), \quad q = \gamma - \alpha'.$$

To prove Theorem (3.3) we have to transform assumptions concerning the operators B_i $(i=1, \ldots, n)$ onto assumptions for B (see (3.II′), (3.III′)). Next we apply Theorem 3.2 of [8] (for a real Banach space) with $N=1$ and with X replaced by X^n.

The particular case (3.3) is an example illustrating Theorem 3.3 provided f_i are independent of the fifth and sixth arguments. Then, of course, it is necessary to modify assumptions (3.VIII), (3.IX) and (3.XII), to retain (3.X), and to omit (3.XI).

Taking advantage of paper [8], one can extend Theorems 3.2 and 3.3 to the complex case. For this purpose we have to make assumptions appropriate for that case (clearly, assumption (3.I) has to consist of (2.VII)—(2.IX)).

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Received 26.2.1986