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GENERALIZED B-MANIFOLDS WITH BOUNDED HOLOMORPHIC 4-CURVATURE

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Lately, pseudo-Riemannian manifolds with a complex structure and a bounded holomorphic sectional curvature are examining. In this paper we study some properties of the generalized B-manifolds with a bounded holomorphic 4-curvature. We prove, that in such manifolds the holomorphic sectional curvature vanishes. Some relations among sectional curvatures in arbitrary linear non-degenerate holomorphic 4-dimensional subspace in every tangent space are obtained.

Let M be a pseudo-Riemannian manifold with a metric tensor field g and an almost complex structure J. If

$$g(JX, JY) = -g(X, Y), \quad X, Y \in \mathfrak{X}M,$$

then M is called [1] generalised B-manifold. Let us denote by GB the class of generalized B-manifolds. A subspace E^m in T_pM is said to be non-degenerate holomorphic one, if $JE^m = E^m$ and the dimension of the restriction of g on E^m is m. Evidently, E^m is even-dimensional. In [1], it is proved, that E^m admits orthonormal J-base:

$$\{x_1, x_2, \dots, x_q, Jx_1, Jx_2, \dots, Jx_q\}, \; g(x_i, x_j) = \delta_{ij}, \; g(x_i, Jx_j) = 0.$$

For example, every non-degenerate holomorphic 2-plane E^2 admits such a base

$$\{x, Jx\}, \; (g(x, x) = 1, \; g(x, Jx) = 0),$$

as well as every linear non-degenerate holomorphic 4-dimensional subspace E^4 admits a base

$$\{x, y, Jx, Jy\}, \; (g(x, x) = g(y, y) = 1, \; g(x, y) = g(x, Jy) = 0).$$

Let ∇ be the Levi — Civita connection generated by g and R be the curvature tensor field of ∇, i. e

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla H_Y \nabla_X Z - \nabla_{[X,Y]} X, \quad X, Y, Z \in \mathfrak{X}M.$$

We shall also denote by R the curvature tensor field of type (0, 4) such that

$$R(X, Y, Z, U) = g(R(X, Y)Z, U).$$

Let $\{x, y\}$ be a base of a non-degenerated section E^2 in T_pM. The value

$$K(x, y) = \frac{R(x, y, y, x)}{g(x, x)g(y, y) - g^2(x, y)} \tag{1}$$

is the sectional curvature of the E^2.

In particular the holomorphic sectional curvature $K(x, Jx) = H(x)$ of a non-degenerate holomorphic section in T_pM, with an orthonormal J-base $\{x, Jx\}$ is given by the formula

SERDICA Bulgaricae mathematicae publicationes. Vol. 13, 1987, p. 375—378.

(2) $$H(x) = -R(x, Jx, Jx, x).$$

We denote, that the tensor R^*, defined by

(3) $$R^*(x, y, z, u) = R(x, y, Jz, Ju)$$

is a Riemannian tensor, so we can also calculate a sectional curvature with respect to R^*. Let us denote this curvature by K^*.

The value

(4) $$h(x, y) = -\frac{R(x, Jx, Jy, y)}{\sqrt{g^2(x, x) + g^2(x, Jx)}\sqrt{g^2(y, y) + g^2(y, Jy)}}$$

is called [2] holomorphic bisectional curvature of the non-degenerate holomorphic sections with bases $\{x, Jx\}$ and $\{y, Jy\}$, respectively.

Let $M \in GB$ and $p \in M$. The manifold M is said to be of pointwise constant holomorphic sectional curvature, if the holomorphic sectional curvatures of all non-degenerate holomorphic sections in T_pM are the same. The manifold M is said to be with constant holomorphic sectional curvature, if the holomorphic sectional curvature does not depend on p.

Now, let E^4 be an arbitrary linear non-degenerate holomorphic subspace in T_pM and let $\{v_\lambda, v_\mu, v_\nu, v_\tau\}$ be an arbitrary base in E^4. Let $v_{ik} = g(v_i, v_k)$ be the Gram matrix and $V_{\lambda\nu}$ be the opposite matrix. The value

(5) $$K(E^4) = \frac{1}{2} \sum_{\lambda,\mu,\nu,\tau=1}^{4} V_{\lambda\tau} V_{\mu\tau} R(v_\lambda, v_\mu, v_\nu, v_\tau),$$

is the holomorphic 4-curvature [3] of E^4. Especially, if $\{x, y, Jx, Jy\}$ is an orthonormal J-base in E^4, then (5) implies

(6) $$K(E^4) = K(x, y) + K(x, Jx) + K(x, Jy) + K(y, Jx) + K(y, Jy) + K(Jx, Jy).$$

Theorem 1. *Let $M \in GB$, $\dim M \geq 6$, $p \in M$ and let E^4 be an arbitrary non-degenerate holomorphic subspace in T_pM. If there exists a constant $c(p)$, such that*

$$|K(E^4)| \leq c(p),$$

then M is with zero holomorphic sectional curvature.

Proof. Let α be an arbitrary non-degenerate holomorphic section in T_pM and let $\{x, Jx\}$ be an orthonormal J-base in α. There exists linear non-degenerate holomorphic 4-dimensional subspace E^4 with an orthonormal J-base $\{x, y, Jx, Jy\}$ [1]. It is clear that E^4 contains α. Then (6) is true for E^4. We state that for $|t| < 1$ the vectors

$$\frac{x + tJy}{\sqrt{1-t^2}}, \quad \frac{tJx - y}{\sqrt{1-t^2}}, \quad \frac{Jx - ty}{\sqrt{1-t^2}}, \quad \frac{-tx - Jy}{\sqrt{1-t^2}}$$

form an orthonormal J-base in E^4, too. Then according to (7) we have

(8) $$\left|K\left(\frac{x + tJy}{\sqrt{1-t^2}}, \frac{tJx - y}{\sqrt{1-t^2}}, \frac{Jx - ty}{\sqrt{1-t^2}}, \frac{-tx - Jy}{\sqrt{1-t^2}}\right)\right| \leq c(p).$$

Now, using (6) and (8), we obtain

(9) $$|K(E^4) - 2t^2 K(\dot{x}, \dot{y}) + t^4 K(E^4)| \leq c(p)(1-t^2)^2,$$

where $K(\dot{x}, \dot{y}) = K(x, y) + K(x, Jy) + K(y, Jx) + K(Jx, Jy)$.

From (9) by continuity we get

$$K(E^4)-K(\overset{*}{x}, \overset{*}{y})=0,$$

for $t=\pm 1$, which implies

$$H(x)+H(y)=0, \tag{10}$$

where $H(x)=K(x, Jx)$ is the holomorphic sectional curvature of α.

Since $\dim M\geqq 6$, then there exists a linear non-degenerate holomorphic subspace E^6 in T_pM, with an J-base $\{x, y, z, Jx, Jy, Jz\}$. Evidently E^6 contains E^4. Thank's to (10), the following equalities

$$H(z)+H(y)=0, \quad H(z)+H(x)=0 \tag{11}$$

are valid.

Using (10) and (11), we get

$$H(x)=0.$$

So the theorem is proved.

For brevity, every manifold satisfying the conditions in theorem 1 will be called manifold with bounded holomorphic 4-curvature.

From (2) and theorem 1 we get immediatelly the following assertion.

Corollary 2. *If $M\in GB$, $\dim M\geqq 6$ and M is manifold with a bounded holomorphic 4-curvature then*

$$R(x, Jx, Jx, x)=0, \tag{12}$$

where x is a vector in an arbitrary non-degenerate holomorphic section.

Let M be manifold with bounded holomorphic 4-curvature, $M\in GB$, $\dim M\geqq 6$. If $\{x, y, Jx, Jy\}$ is an orthonormal J-base in E^4, then $x+y$ is not null-vector and consequently by corollary 2 implies

$$R(x+y, Jx+Jy, Jx+Jy, x+y)=0.$$

Now using (12), we find

$$2R(x, Jx, Jy, y)+2R(x, Jy, Jx, x)+R(x, Jy, Jy, x)+R(y, Jx, Jx, y)=0. \tag{13}$$

Applying Bianchi's first identity in (13), we get

$$4R(x, Jx, Jy, y)-2R(x, y, Jy, Jx)+R(x, Jy, Jy, x)+R(y, Jx, Jx, y)=0. \tag{13'}$$

Now using (1), (3), (4) and (13'), we establish

Theorem 3. *Let $M\in GB$, $\dim M\geqq 6$ and let M be a manifold with a bounded holomorphic 4-curvature. Then*

$$4h(x, y)+2K^*(x, y)+K(x, Jy)+K(y, Jx)=0.$$

Let $M\in GB$. Then M is in the subclass L_3 of GB [2], if

$$R(JX, JY, JZ, JU)=R(X, Y, Z, U), \quad X, Y, Z, U\in \mathfrak{X}M. \tag{L_3}$$

Theorem 4. *Let M be a manifold with a bounded holomorphic 4-curvature in L_3. If $\dim M\geqq 6$ and $\{x, y, Jx, Jy\}$ is an orthonormal J-base of an arbitrary linear non-degenerate holomorphic subspace in T_pM, then*

$$2h(x, y)+K(x, Jy)+K^*(x, y)=0. \tag{14}$$

Proof. From (13') and (L_3) we have $2R(x, Jx, Jy, y)-R(x, y, Jy, Jx)+R(x, Jy, Jy, x)=0$. By virtue of the above relation, (1), (3) and (4) we get (14).

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Received 7. 7. 1986

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