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ON THE APPROXIMATION OF STARSHAPED SETS IN HAUSDORFF DISTANCE

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We show that in a normed linear space, each closed bounded starshaped set can be approximated in Hausdorff distance by starshaped sets thin both in the sense of measure and the lack of interior points. When the space is a Banach space, most of the compact starshaped sets are thin, in the sense of Baire Category.

Let X be a normed linear space. For a point p and a subset E of X, we define the *star from p over E* to be the union star $(p; E)$ of all closed line segments $[p, x]$ joining p and x as x ranges over E. A subset S of X is *starshaped* if star $(p; S)=S$ for some $p\in S$, and S is then said to be *starshaped with respect to p* [6]. Limits of sequences of closed starshaped sets with respect to the Hausdorff metric δ determined by the norm of X have been recently considered in [1]. For general X, the δ-limit of a sequence of closed starshaped sets need not be starshaped, even if the terms of the sequence are bounded sets; such limits are characterized by Theorem 1.5 of [1]. By virtue of Theorem 2.7 of [1], the limit of each such sequence will be, without fail, starshaped if and only if the space is reflexive.

It is the purpose of this note to show in the context of an arbitrary normed linear space, that within the space of closed bounded starshaped sets the *thin* starshaped sets are dense, where thin is interpreted both in the sense of measure and the lack of interior points. It is rather obvious (see Theorem 2 below) that with respect to Lebesgue measure on Euclidean space, the nowhere dense starshaped sets of measure zero are δ-dense in the closed bounded starshaped sets. But this is not at all obvious when closed bounded sets are no longer totally bounded, and Lebesgue measure is replaced by an arbitrary regular Borel measure. The necessity of restricting our attention to bounded sets is made clear by the following observation: If S is closed and starshaped and $\delta(S, X)<\infty$, then $S=X$.

Before proceeding, we take care of some preliminaries and notation. By ***a regular Borel measure on X***, we mean a measure μ defined on the Borel subsets of X satisfying the following properties:

(i) $\mu(K)<\infty$ for each compact set K;

(ii) for each Borel set B, $\mu B)=\sup\{\mu(K): K \text{ compact}, K\subset B\}$;

(iii) for each Borel set B, $\mu(B)=\inf\{\mu(V): V \text{ open}, V\supset B\}$.

We refer to properties (ii) and (iii) as *inner regularity* and *outer regularity*, respectively. We note here that the standard definition [3] of regular Borel measure is weaker than our notion; in fact, our notion has been called strong regularity in the literature [5]. We need its full strength.

The origin and closed unit ball of our normed space X will be denoted by θ and U, respectively. If p and q are in X, then $(p, q]$ will denote the half-open line segment joining p and q. The space of closed bounded (resp. compact) starshaped subsets of X will be denoted by $BS(X)$ (resp. $KS(X)$). A subset E of X is called ***λ-uniformly dis-***

crete if there exists $\lambda>0$ such that $\|x-y\|\geqq\lambda$ for each $\{x, y\}\subset E$. By Zorn's Lemma we have for any nonempty subset A of X and for each positive λ, a λ-uniformly discrete set E contained in A with $A\subset E+\lambda U$.

The following obvious facts used in the sequel still deserve mention: (a) given $\varepsilon>0$ and points p, q, p', and q' of X with $\|p-p'\|<\varepsilon$ and $\|q-q'\|<\varepsilon$, then $\delta([p, q], [p', q'])<\varepsilon$; (b) if μ is a regular Borel measure on X and p, q are in X, then by outer regularity and the compactness of $[v, q]$, there exists $\varepsilon>0$ for which $\mu([p, q]+\varepsilon U)<\infty$.

Lemma 1. *Let μ be a regular Borel measure on a normed linear space X, and let p and q be in X. Then for each positive ε, there exists some y in X with $\|p-y\|<\varepsilon$ and $\mu((q, y])=0$.*

Proof. Suppose there exists an $\varepsilon>0$ such that for each y in X with $\|p-y\|<\varepsilon$ we have $\mu((q, y])>0$. As mentioned above, there exist a positive λ for which $\mu([p, q]+\lambda U)<\infty$. We can assume without loss of generality that $\varepsilon<\lambda$. Pick a point y with $\|p-y\|<\varepsilon$ such that y is not colinear with p and q. We then have $(p, x]\cap(p, v]=\emptyset$ for each $\{x, v\}\subset[p, y]$. Let $\Omega=\{(q, x]: x\in(p, y]\}$. Observe that Ω is an uncountable set of disjoint segments each of whose measures is positive. Thus $\mu(\cup\Omega)=\infty$. Now $\cup\Omega\subset[p, q]+\lambda U$; so, by the monotonicity of the measure, we obtain $\mu([p, q]+\lambda U)=\infty$. This contradicts $\mu([p, q]+\lambda U)<\infty$.

We shall use this direct consequence of Lemma 1: if $p\in X$ and ε is any positive real number, then there exists $y\in p+\varepsilon U$ with $\mu(\{y\})=0$.

Lemma 2. *Let X be a normed linear space, and E a λ-uniformly discrete bounded subset of X not containing θ. Let μ be a regular Borel measure on X such that $\mu([\theta, x])=0$ for every $x\in E$, and $[\theta, x]\cap[\theta, y]=\{\theta\}$ for each $\{x, y\}\subset E$. Then star $(\theta; E)$ is a closed set of measure zero.*

Proof. Let $A=\text{star}(\theta; E)$. For each $x\in E$ we call the segment $(\theta, x]$ a stalk. We claim that no sequence in A whose terms lie on distinct stalks can have a cluster point other than θ. Otherwise, there would exist a sequence $\langle x_n\rangle$ in E with distinct terms such that $\langle x_n/\|x_n\|\rangle$ is a convergent sequence with distinct terms. As a bounded sequence of reals, $\langle\|x_n\|\rangle$ has a cluster point, which implies $\langle x_n\rangle$ has a cluster point, contradicting the uniform discreteness of E. Immediately, we conclude that A is closed. Also, each compact subset K of A can meet at most countably many stalks, for otherwise, infinitely many stalks contain points p of K away from θ, that is, $\|p\|\geqq\varepsilon$ for some $\varepsilon>0$. In this case, by the compactness of K, we could produce a sequence in A whose terms lie in distinct stalks convergent to some point of norm at least ε. Thus, $\mu(K)=0$, and by inner regularity, $\mu A=(0)$.

We remark that the boundedness assumption in Lemma 2 is not needed to conclude $\mu(\text{star}(\theta; E))=0$, but is needed for closedness of the set.

Lemma 3. *Let μ be a regular Borel measure on a normed linear space X, and let $E\subset\{x:\|x\|=r\}$ be a λ-uniformly discrete set. Then for each $\varepsilon>0$, there exists $p\in X$, dépendent on ε but not on E, and a closed set S starshaped with respect to p with $\mu(S)=0$ and $\delta(\text{star}(\theta; E), S)<\varepsilon$.*

Proof. Let $\varepsilon>0$. Without loss of generality, we may assume that $\varepsilon<\lambda$. Pick a point p with $\|p\|<\varepsilon/6$ and $\mu(\{p\})=0$. By Lemma 1, there exists for every $x\in E$ a point y_x with $\|x-y_x\|<\varepsilon/6$ and $\mu([p, y_x])=0$. Let $F=\{y_x: x\in E\}$, and set $S=\text{star}(p; F)$. We claim that S is the desired starshaped set. Observe that for each $x\in E$, we have both

$$[\theta, x]\subset[p, y_x]+\frac{\varepsilon}{6}u \text{ and } [p, y_x]\subset[\theta, x]+\frac{\varepsilon}{6}U$$

whence $\delta(\text{star}(\theta; E), S)<\varepsilon$. Clearly, $F\subset S\subset(r+\lambda)U$ and F is $\frac{2}{3}$-λ uniformly discrete. To invoke Lemma 2, it remains to show $[p, y_x]\cap[p, y_z]=\{p\}$ whenever $\{x, z\}\subset E$.

To the contrary, suppose there exists $\{x, z\}\subset E$ for which $(p, y_x]\cap(p, y_z]\neq\emptyset$. This means that p, y_x and y_z are colinear. We may assume that $\|y_z-p\|<\|y_x-p\|$, so that we have $\|y_x-p\|-\|y_z-p\|=\|y_x-y_z\|$. Now

$$\|y_x-p\|\leqq\|y_x-x\|+\|x\|+\|p\|<\frac{\varepsilon}{6}+r+\frac{\varepsilon}{6}$$

and

$$\|y_z-p\|\geqq\|z\|-\|z-y_z\|-\|p\|>r-\frac{\varepsilon}{6}-\frac{\varepsilon}{6}.$$

This gives us $\|y_x-y_z\|<\frac{2}{3}\varepsilon$ and, hence, $\|x-z\|<\varepsilon<\lambda$, contradicting the uniform discreteness of E.

It is easy to see that the starshaped set S constructed in the lemma above contains no spheroids, and is, hence, a nowhere dense set. We now present our main result.

Theorem 1. *Let μ be a regular Borel measure on a normed linear space X. Then the nowhere dense closed bounded starshaped sets of measure zero are δ-dense in the closed bounded starshaped sets.*

Proof. Let A be a closed bounded starshaped set in X. Without loss of generality we assume that A is starshaped with respect to θ. Let $\varepsilon>0$ and set $r=\sup\{\|x\|:x\in A\}$. Choose $n\in Z^+$ such that $r/n<\varepsilon/2$, and let $F_j=A\cap\{x:\|x\|=\frac{j}{n}r\}$ for $j=1,\dots,n-1$. Fix j, and choose an $\varepsilon/4$-uniformly discrete set $F'_j\subset F_j$ such that $F_j\subset F'_j+\frac{\varepsilon}{4}U$. Let $E_j=\mathrm{star}(\theta; F_j)$ and $E'_j=\mathrm{star}(\theta; F'_j)$. It is clear that $E_j\subset E'_j+\frac{\varepsilon}{4}U$. By Lemma 3 we can find $p\in X$ independent of j and a closed nowhere dense set S starshaped with respect to p satisfying $\mu(S_j)=0$ and $\delta(E'_j, S_j)<\varepsilon/4$, for $j=1,\dots,n-1$ We have

$$S_j\subset E'_j+\frac{\varepsilon}{4}U\subset E_j+\frac{\varepsilon}{4}U$$

and

$$E_j\subset E'_j+\frac{\varepsilon}{4}U\subset S_j+\frac{\varepsilon}{2}U.$$

We now let $S=\bigcup_{j=1}^{n-1}S_j$. By construction, the set S is a nowhere dense closed bounded starshaped set of measure zero. It remains to show that $\delta(A, S)\leqq\varepsilon$. We have $S\subset A+\varepsilon U$ because for each index j

$$S_j\subset E_j+\frac{\varepsilon}{4}U\subset A+\frac{\varepsilon}{4}U.$$

For each $a\in A$, we have $\frac{j}{n}r\leqq\|a\|\leqq(\frac{j+1}{n})r$ for some $j\in\{1,\dots,n-1\}$. The point $y=\frac{jr}{n}\cdot\frac{a}{\|a\|}$ is clearly an element of F_j, and moreover, we have $\|a-y\|\leqq r/n<\varepsilon/2$. Therefore,

$$a\in F_j+\frac{\varepsilon}{2}U\subset E_j+\frac{\varepsilon}{2}U\subset S_j+\varepsilon U\subset S+\varepsilon U.$$

Thus, $\delta(A, S)<\varepsilon$.

We conclude this paper with some results on the space of compact starshaped sets. First of all, we note from the recent monograph of E. Klein and A. Thompson [4] that the hyperspace of compact subsets of a completely metrizable space with the Hausdorff metric is itself completely metrizable. Since a compact δ-limit of a sequence of starshaped sets is again starshaped (see, e. g., Theorem 1.3 of [1]), we conclude that for a Banach space X, the hyperspace $\langle KS(X), \delta\rangle$, is also completely metrizable. As we shall see from Theorem 3 below, most starshaped sets are thin in such a space.

Theorem 2. *Let μ be a regular Borel measure on a normed linear space X. Then the nowhere dense compact starshaped sets of measure zero are δ-dense in the compact starshaped sets.*

Proof. Let A be a compact starshaped set. Without loss of generality, we may assume that A is starshaped with respect to θ. Let ε be positive. Since A is totally bounded, there is a finite set $E\subset A$ such that $A\subset E+\frac{\varepsilon}{4}U$. Take a point p with $\|p\|<\varepsilon/4$ and $\mu\{(p)\}=0$. By Lemma 1 there exists for each $x\in E$ a $y_x\in X$ with $\|x-y_x\|<\varepsilon/4$ and $\mu([p, y_x])=0$. The set $S=\text{star}(p;\{y_x : x\in E\})$ is a nowhere dense compact starshaped set with $\mu(S)=0$, and it is a routine matter to check that $\delta(A, S)<\varepsilon$.

We note that the approximating sets in Theorem 2, as stars over finite sets, are the analogues of polytopes with respect to convex sets.

Definition. *Let f be an extended real-valued function on a metric space X We say that f is* upper semiconditions *at a point p in X if whenever r is real and $r>f(p)$, there is a neighbourhood V of p such that $f(x)< r$ for each $x\in V$.*

A regular Borel measure μ is, of course, a function from the Borel subsets of X to $[0, \infty]$. We claim here that its restriction to the compact subsets of X is upper semicontinuous with respect to Hausdorff distance (more general results can be found in [2]). To this end, fix a compact set K and let r be real with $\mu(K)<r$. By outer regularity of μ, there is an open set V with $V\supset K$ and $\mu(V)<r$. Since K is compact, there exists a positive ε such that $K+\varepsilon U\subset V$. Clearly, $\delta(A, K)<\varepsilon$ implies $A\subset K+\varepsilon U\subset V$, whence $\mu(A)<r$.

It is not true, however, that a regular Borel measure μ must be upper semicontinuous at an arbitrary closed bounded starshaped set. In fact, given an arbitrary noncompact closed bounded starshaped set S in X, we can exhibit a regular Borel measure on X whose restriction to $BS(X)$ is not upper semicontinuous at S. Since S is noncompact, it has a sequence $\langle x_n\rangle$ with no cluster point. We can assume without loss of generality that S is starshaped with respect to θ, each x_n is not θ, and $x_n/\|x_n\|\neq x_j/\|x_j\|$ whenever $n\neq j$. Choose $r>0$ such that $S\subset rU$. For each $n\in Z^+$, take y_n to be the farthest point of S from θ along the ray from θ through x_n. It is easy to see that $\langle y_n\rangle$ has no cluster point as well. Also, for each $n\in Z^+$, let $z_n=(\frac{n+1}{n})y_n$. By the choice of y_n, no z_n lies in S. Moreover, $\langle z_n\rangle$ can have no cluster point, for if it had one, then so will $\langle y_n\rangle$. As a result, the counting measure μ_S for $\{z_n : n\in Z^+\}$ (cf. problem 9.2 of [5]) is regular. Given $\varepsilon>0$, choose $k\in Z^+$ such that $r/k<\varepsilon/2$, and set $S_k=\text{star}(\theta;\{z_n : n\geq k\})$. Let $A=S\cup S_k$. For each $n\geq k$, we have

$$[0, z_n]\subset[0, y_n]+\frac{r}{n}U\subset S+\frac{r}{n}U\subset S+\frac{\varepsilon}{2}U.$$

Hence, $A\subset S+\frac{\varepsilon}{2}U$. It is now clear that $\delta(A, S)<\varepsilon$, $\mu_S(A)=\infty$, and $\mu_S(S)=0$. Thus μ_S is not upper semicontinuous at S.

We now use the upper semicontinuity of μ restricted to compacta to present a Baire category result involving the space of compact starshaped sets, sharpening and extending a result of T. Zamfirescu [7] (see also [8]).

Theorem 3. *Let μ be a regular Borel measure on a normed linear space X. Then the compact nowhere dense starshaped sets of measure zero form a dense G_δ subset of the compact starshaped sets.*

Proof. Let $\Omega=\{S: S\in KS(X)$ and $\mu(S)=0\}$ and let $\Sigma=\{S: S\in KS(X)$ and S is nowhere dense$\}$. In our usage, the thin sets are just $\Omega\cap\Sigma$. By virtue of Theorem 2 it suffices to show that both Ω and Σ are G_δ subsets of the hyperspace. Clearly $\Omega=\bigcap_{n=1}^{\infty}\Omega_n$ and $\Sigma=\bigcap_{n=1}^{\infty}\Sigma_n$, where for each n,

$$\Omega_n=\{S: S\in KS(X) \text{ and } \mu(S)<1/n\}$$

$$\Sigma_n=\{S: S\in KS(X) \text{ and } S \text{ contains no closed ball of radius } 1/n\}.$$

By the upper semicontinuity of μ on compacta, each set Ω_n is open. To show each Σ_n is open, we show Σ_n^c is closed in $KS(X)$. Suppose $\langle S_j\rangle$ is a sequence in Σ_u^c δ-convergent to $S\in KS(X)$. For each $j\in Z^+$ choose $a_j\in S_j$ with $a_j+\frac{1}{n}U\subset S_j$. Since S is compact and

$$\lim_{j\to\infty}\inf_{x\in S}\|x-a_j\|=0,$$

by passing to a subsequence, we may assume that $\langle a_j\rangle$ converges to some point p in S, whence $p+\frac{1}{n}U\subset S$ (see, e. g., Theorem 4.3.5 of [4]). Thus, Σ_n^c is δ-closed in $KS(X)$, and we are done.

We mention in closing that Theorems 2 and 3 (and Lemma 1, on which they depend) remain true with much less than regularity: it is enough to require that the measure is finite on some neighbourhood of each point of X. From this it follows that compact sets have finite measure and the measure is outer regular at each compact set.

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