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VAGUE CONVERGENCE FOR SUMS OF INDEPENDENT RANDOM VARIABLES IN A TRIANGULAR ARRAY

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This paper deals with the vague convergence of sums of independent random variables in a triangular array provided that the summands are infinitely small. It is proved that the limit function is an infinitely divisible function. The main results concern necessary and sufficient conditions for the vague convergence of sums to a special limit function. Such assertions generalize the classical results of weak convergence of sums of independent random variables to a non-defective infinitely divisible distribution function. Essential tools are quoted from G. Siegel [6] and H.-J. Rossberg, B. Jesiak, G. Siegel [4].

1. Introduction. For each $n=1, 2, \dots, X_{n1}, X_{n2}, \dots, X_{nk_n}$ denote independent real-valued random variables. Let A_n be some sequence of real numbers and put

$$S_n = X_{n1} + X_{n2} + \cdots + X_{nk_n} - A_n, \quad n=1, 2, \dots, \tag{1.1}$$

where k_n is a sequence of integers tending to infinity. Introducing, further, the distribution functions (d. f.)

$$F_{nk}(x) = P(X_{nk} < x), \quad F_n(x) = P(S_n < x), \quad x \in R,$$

and supposing that the condition of uniform asymptotic negligibility is satisfied we have

$$\max_{1 \leq k \leq k_n} P(|X_{nk}| \geq \varepsilon) \to 0, \ n \to \infty, \text{ for all } \varepsilon > 0. \tag{1.2}$$

Then the classical theory of summation of independent random variables provides necessary and sufficient conditions for the weak convergence of F_n to a certain infinitely divisible (inf. div.) distribution function. The present paper aims at deriving necessary and sufficient conditions for the vague convergence of F_n to a certain limit function F which may be defective, i. e., $0 \leq F(\infty) \leq 1$. Vague convergence of F_n means that $F_n(I) \to F(I)$ only for all bounded continuity intervals I of F. A detailed study of this convergence can be found in [6]. So far vague limit theorems for sums of independent random variables are derived only under the assumption of left-sided tightness defined by

$$\lim_{T \to \infty} \sup_n F_n(-T) = 0 \tag{1.3}$$

(see [4, 5] and [7]). In this case probability mass of S_n may escape only to $+\infty$. In the present paper, probability mass may escape in both directions; our main result is formulated in theorem 3.5.

2. Notations and preliminary results. Let **M** be the set of all non-decreasing left-continuous and bounded functions F with $F(-\infty)=0$, and denote by **C** the set of all continuous bounded functions. If the sequence $F_n \in \mathbf{M}$ is uniformly bounded, then we say that F_n weakly converges to $F \in \mathbf{M}$ ($F_n \Rightarrow F$) if and only if

$$(2.1) \qquad \int_{-\infty}^{\infty} \varphi(x)\, dF_n(x) \to \int_{-\infty}^{\infty} \varphi(x)\, dF(x), \quad \text{for all} \quad \varphi \in \mathbf{C}.$$

We say that F_n vaguely converges to $F \in \mathbf{M}$ ($F_n \rightharpoonup F$) if and only if (2.1) is guaranteed for all $\varphi \in \mathbf{C}$ with $\varphi(x) \to 0$, as $|x| \to \infty$. Sometimes this convergence is called "weak convergence but for an additive constant" (see [2, 4]). Note that the convergence $F_n(x) \to G(x)$ for all continuity points x of an increasing function G implies $F_n \rightharpoonup F = G - G(-\infty) \in \mathbf{M}$; however the converse assertion does not follow in general. Assuming $F_n \rightharpoonup F$ we get $\lim_{n\to\infty} F_n(\infty) \geqq F(\infty)$. In this case $F_n \Rightarrow F$ holds if and only if $\lim_{n\to\infty} F_n(\infty) = F(\infty)$. Next we introduce the Fourier-Stieltjes transform (characteristic function) of $F \in \mathbf{M}$ by writing

$$f(t) = \int_{-\infty}^{\infty} e^{itx} dF(x), \quad t \in R.$$

Introduce, further, the integral characteristic $\hat{f}$ of F by

$$\hat{f}(t) = \int_0^t f(u)\, du, \quad t \in R.$$

The correspondence between $\hat{f}$, f, and F is one-to-one and we have $f(0) = F(\infty) = \frac{d}{dt}\hat{f}(0)$.

Now we state a criterion for vague convergence.

L e m m a 2.1 ([1]). *Let $F_n \in \mathbf{M}$. Then $F_n \rightharpoonup F$ for some $F \in \mathbf{M}$ if and only if $\hat{f}_n \to \hat{f}$ for some function $\hat{f}$. In either case, $\hat{f}$ is the integral characteristic of F.*

For later use we state also a condition sharper than $\hat{f}_n \to \hat{f}$.

L e m m a 2.2 ([6]). *Let f, f_n be the characteristic functions of F, $F_n \in \mathbf{M}$. Then*

$$(2.2) \qquad \lim_{n\to\infty} \int_0^t |f_n(u) - f(u)|\, du = 0, \quad \textit{for all } t > 0,$$

if and only if $F_n \rightharpoonup F$ and the sequence F_n satisfies

$$(2.3) \qquad \lim_{T\to\infty} \sup_n \sup_{|x| \geqq T} (F_n(x+h) - F_n(x-h)) = 0, \quad \textit{for some } h > 0.$$

Simple examples show that (2.2) is indeed sharper than $f_n \to f$; on the other hand, (2.2) is weaker than weak convergence since (2.3) does not imply relative compactness in the weak topology (for details see [6]).

Next we are going to consider infinitely divisible distribution functions. We begin with a generalization of this notion. A function $F \in \mathbf{M}$ is called infinitely divisible (inf. div.), if for every $k = 2, 3, \ldots$ there is a function $F_{(k)} \in \mathbf{M}$ such that $F = F_{(k)}^{*k}$ (convolution power). For every non-vanishing inf. div. function $F \in \mathbf{M}$ (i. e., if $F(\infty) > 0$) we can define an inf. div. distribution function $\tilde{F} = F / F(\infty)$. Using the well-known canonical representation of inf. div. distribution functions (Lévy-Hinčin-canonical representation) we may write

$$(2.4) \qquad \log f(t) = \rho + iat + \int_{-\infty}^{\infty} k(t, x)\, dG(x), \quad t \in R.$$

Here $k(t, x) = (e^{itx} - 1 - \frac{itx}{1+x^2}) \frac{1+x^2}{x^2}$;
moreover, $G \in \mathbf{M}$ is some function (spectral function), and ρ, a are certain real numbers. The quantities ρ, a, and the function G are uniquely defined by F; in particular we have $\log F(\infty) = \rho$. Thus we can briefly write $F = \mathscr{H}[\rho;\, a,\, G]$.

Next we state a vague limit theorem for inf. div. functions.

Lemma 2.3 ([6]). *Let $F_n=\mathscr{H}[\rho_n;\ a_n,\ G_n]$ be a sequence of inf. div. functions. Then $\{F_n\}$ is uniformly bounded and (2.2) holds for some non-vanishing function $F\in\mathbf{M}$ if and only if the following assertions are true:*

(i) $\{G_n\}$ is uniformly bounded and satisfies (2.3);

(ii) There exist real numbers ρ, a, and a function $G\in\mathbf{M}$ such that

$$a_n\to a,\quad G_n\to G,\quad G_n(\infty)-\rho_n\to G(\infty)-\rho.$$

In either case, F is inf. div. and admits the representation $F=\mathscr{H}[\rho;\ a,\ G]$.

Finally, we introduce the concentration function Q of $F\in\mathbf{M}$ by

$$Q(F,\ h)=\sup_x\,(F(x+h)-F(x-h)),\quad h\geqq 0.$$

It is obvious that $Q(F,.)\in\mathbf{M}$. Further we have

$$Q(F,\ 0)=0,\quad Q(F,\ +\infty)=F(+\infty).$$

3. Limit theorems for sums. It is noteworthy that the classical method of accompanying distributions due to Bavli and Gnedenko is useful also for proving vague limit theorems for sums. We begin with notations and list some well-known facts, see [2] and [4].

Corresponding to the triangular array $\{X_{nk}\}$, $k=1,2,\ldots,k_n$, we introduce another triangular array $\{\tilde{X}_{nk}\}$, $k=1,2,\ldots,k_n$, of independent random variables, where the random variables $\tilde{X}_{nk}$ are subject to the inf. div. distribution function $\tilde{F}_{nk}$ with characteristic function

$$\log\tilde{f}_{nk}(t)=f_{nk}(t)\,e^{-ia_{nk}t}-1+ia_{nk}t.$$

Here we set

$$a_{nk}=\int_{|x|<\tau}x\,dF_{nk}(x),\quad \text{for fixed } \tau>0. \tag{3.1}$$

The distribution function $\tilde{F}_n$ of $\tilde{S}_n=\sum_{k=1}^{k_n}\tilde{X}_{nk}-A_n$ is then called accompanying distribution function of F_n. $\tilde{F}_n$ is clearly inf. div. and admits representation $\tilde{F}_n=\mathscr{H}[0;\ a_n,\ H_n]$ with

$$a_n=\sum_{k=1}^{k_n}\left(a_{nk}+\int_{-\infty}^{\infty}\frac{x}{1+x^2}\,dF^0_{nk}(x)\right)-A_n, \tag{3.2}$$

$$H_n(x)=\sum_{k=1}^{k_n}\int_{-\infty}^{x}\frac{y^2}{1+y^2}\,dF^0_{nk}(y),\quad x\in R, \tag{3.3}$$

$$F^0_{nk}(x)=F_{nk}(x+a_{nk}),\quad x\in R. \tag{3.4}$$

Now we can establish relations between F_n and $\tilde{F}_n$. In this way the asymptotic behaviour of F_n can be studied by considering that of $\tilde{F}_n$. We begin with the following basic lemma:

Lemma 3.1 ([4], Lemma 11. 2.1). *Suppose the condition of uniform asymptotic negligibility of the triangular array $\{x_{nk}\}$ is satisfied. Then the following assertions are equivalent:*

(i) $$\inf_n Q(F_n,\ h)>0,\quad \textit{for some } h>0;$$

(ii) $$\inf_n Q(\tilde{F}_n, h) > 0, \quad \text{for some } h > 0;$$

(iii) $$\sup_n \sum_{k=1}^{k_n} \int_{-\infty}^{\infty} \min\{1, x^2\}\, dF^0_{nk}(x) < \infty$$

In all these cases the convergence

$$(f_n(t) - \tilde{f}_n(t)) \to 0, \quad \text{as } n \to \infty \tag{3.5}$$

is uniform on any finite interval; $|t| \leq T$.

We note that (i) and (ii) are satisfied, for instance, if $F_n \to F \neq 0$ and $\tilde{F}_n \to F \neq 0$, respectively. Using this fact and applying Lemma 3.1, the following result is easily seen:

Theorem 3.2 ([4], proposition 11. 3.2). *Suppose the condition of uniform asymptotic negligibility of the triangular array $\{X_{nk}\}$ is satisfied, and let $F \neq 0$ be some unction belonging to* **M**. *Then*

$$F_n \to F, \; n \to \infty \text{ if and only if } \tilde{F}_n \to F, \; n \to \infty.$$

Unfortunately, it may happens that the limit function occuring in Theorem 3.2 is not inf. div.; for an example see [7]. Hence we need additional conditions guaranteeing that F is inf. div.

Before giving our main result (Theorem 3.5) we state and prove an improved version of Theorem 3.2. To this end we make use of the Levy metric ρ_L between two distribution functions.

Theorem 3.3. *Suppose that the uniform asymptotic negligibility of the triangular array $\{X_{nk}\}$ is satisfied as well as the condition (iii) of Lemma 3.1 is true. Then*

$$\rho_L(F_n, \tilde{F}_n) \to 0, \quad \text{as } n \to \infty.$$

For the proof we need

Lemma 3.4 ([8]). *Let F, G be distribution functions having the characteristic functions f, g. Then*

$$\lambda(f, g) = \min_{T>0} \max\left\{\frac{1}{2} \max_{|t| \leq T} |f(t) - g(t)|, \; \frac{1}{T}\right\}$$

is a metric. Further, λ and ρ_L are equivalent, i. e., $\rho_L(F_n, G_n) \to 0$ is equivalent to $\lambda(f_n, g_n) \to 0$.

Proof of Theorem 3.3. Condition (iii) of Lemma 3.1 implies $|f_n(t) - \tilde{f}_n(t)| \to 0$ uniformly on $-T \leq t \leq T$, $T > 0$ fixed. Thus there exists a sequence T_n tending to infinity such that

$$\max_{|t| \leq T_n} |f_n(t) - \tilde{f}_n(t)| \to 0,$$

and, as a consequence, $\lambda(f_n, \tilde{f}_n) \to 0$. Now Lemma 3.4 yields $\rho_L(F_n, \tilde{F}_n) \to 0$ as asserted.

Before formulating the announced result we remind the reader of the notation a_{nk}, F^0_{nk}, a_n, H_n given in (3.1)—(3.4).

Theorem 3.5. *Suppose the condition of uniform asymptotic negligibility is satisfied for the triangular array $\{X_{nk}\}$. Then the characteristic functions f_n of $S_n = \sum_{k=1}^{k_n} X_{nk} - A_n$ with d. f. F_n satisfy (2.2) for some function $F \neq 0$ belonging to* **M**

with characteristic function f, if and only if the following assertions are true:
(i) The sequence $\{H_n\}$ is uniformly bounded and satisfies (2.3);
(ii) There exist real numbers ρ, a, *and a function* $H \in \mathbf{M}$ *such that*

$$a_n \to a, \quad H_n \to H, \quad H_n(\infty) \to H(\infty) - \rho.$$

In either case, F is inf. div. and admits the representation $F = \mathscr{H}[\rho;\ a, H]$, $-\infty < \rho = \log F(\infty) \leq 0$. *In particular, we have* $F_n \to F$.

Proof. The *if-part:* By Lemma 2.3, (i) and (ii) imply

$$\int_0^t |\tilde{f}_n(u) - f(u)|\, du \to 0 \text{ as } n \to \infty, \text{ for all } t > 0,$$

and moreover, $F = \mathscr{H}[\rho;\ a, H] \neq 0$. Now it is easily seen by $F \neq 0$ that relation (ii) of Lemma 3.1 is valid. Thus we have (3.5) and obtain

$$\int_0^t |f_n(u) - \tilde{f}_n(u)|\, du \to 0 \text{ as } n \to \infty, \text{ for all } t > 0, \tag{3.6}$$

and accordingly

$$\int_0^t |f_n(u) - f(u)|\, du \to 0 \text{ as } n \to \infty, \text{ for all } t > 0$$

as asserted.

The only-if part can be carried out in a similar way where (3.6) again applies.

Now we are going to introduce Levy's canonical representation of inf. div. functions $F \in \mathbf{M}$, $F \neq 0$. We have

$$\log f(t) = \rho + \mathrm{i}at - \frac{1}{2}\sigma^2 t^2 + \int_{R\setminus\{0\}} \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) dL(x),$$

where ρ, a, $\sigma^2 \geqq 0$, are real numbers and $L\colon R\setminus\{0\} \to R$ is the Levy spectral function, i. e., L is non-decreasing and left-continuous in $(-\infty, 0)$ and $(0, \infty)$, respectively. Further, we have $L(-\infty) = L(\infty) = 0$ and

$$\int_{R\setminus\{0\}} \min\{1, x^2\}\, dL(x) < \infty.$$

Since ρ, a, σ^2, L are uniquely defined by F we may briefly write $F = \mathfrak{L}[\rho;\ a, \rho^2, L]$. Given an inf div. function $F = \mathscr{H}[\rho;\ a, G] \in \mathbf{M}$ we have only to put $\sigma^2 = G(+0) - G(0)$ and

$$L(x) = \begin{cases} \displaystyle\int_{-\infty}^{x} \frac{1+y^2}{y^2}\, dG(y), & x < 0; \\ \displaystyle -\int_{x}^{\infty} \frac{1+y^2}{y^2}\, dG(y), & x > 0. \end{cases}$$

Before formulating next result we introduce the functions

$$L_n(x) = \begin{cases} \displaystyle\sum_1^{k_n} F_{nk}(x), & x < 0; \\ \displaystyle\sum_1^{k_n} (F_{nk}(x) - 1), & x > 0. \end{cases}$$

Further we remind the reader of the notation a_{nk}, a_n, see (3.1) and (3.2).

Theorem 3.6. *Suppose that the uniform asymptotic negligibility condition of the triangular array* $\{X_{nk}\}$ *is satisfied. Then the characteristic functions* f_n *of* $S_n=\sum_1^{k_n} X_{nk}-A_n\sim F_n$ *satisfy (2.2) for some function* $F\in\mathbf{M}$, $F\neq 0$, *having characteristic function f, if and only if the following assertions are satisfied:*
(i) *The sequence* L_n *satisfies (2.3);*
(ii) *There exist real numbers* ρ, a, σ^2, *and a Levy spectral function L such that* $a_n\to a$,

$$L_n(x_2)-L_n(x_1)\to L(x_2)-L(x_1) \tag{3.7}$$

for all continuity points x_1, x_2 *of* L *with* $-\infty<x_1<x_2<0$ *and* $0<x_1<x_2<\infty$, *respectively;*

$$\lim_{\varepsilon\to+0}\varliminf_{n\to\infty}\sum_{k=1}^{k_n}\Big[\int_{|x|<\varepsilon}x^2dF_{nk}(x)-\Big(\int_{|x|<\varepsilon}xdF_{nk}(x)\Big)^2\Big]$$

$$=\lim_{\varepsilon\to+0}\varlimsup_{n\to\infty}\sum_{k=1}^{k_n}\Big[\int_{|x|<\varepsilon}x^2dF_{nk}(x)-\Big(\int_{|x|<\varepsilon}xdF_{nk}(x)\Big)^2\Big]=\sigma^2;$$

$$\lim_{T\to\infty}\varliminf_{n\to\infty}\int_{|x|\geqq T}dL_n(x)=\lim_{T\to\infty}\varlimsup_{n\to\infty}\int_{|x|\geqq T}dL_n(x)=-\rho. \tag{3.8}$$

In either case, F is inf. div. and admits the representation $F=\mathfrak{L}[\rho;\ a,\ \sigma^2,\ L]$, $-\infty<\rho=\log F(\infty)\leqq 0$. *In particular, we have* $F_n\to F$.

The proof of this statement is like the corresponding proof of the classical result in the case of weak convergence, where Theorem 3.5 is useful (see [2, 4]).

It is clear that (2.3) stands for an essential additional assumption guaranteeing the representation of vague limits $F\in\mathbf{M}$ of $S_n\sim F_n$ by inf. div. functions. Further, we observe that (2.3) for L_n can be rewritten in terms of random variables, namely we have

$$\lim_{T\to\infty}\sup_{|x|\geqq T}\sup_{n}\sum_{k=1}^{k_n}P(|X_{nk}-x|\leqq h)=0 \quad \text{for some } h>0. \tag{3.9}$$

The condition (2.3) for L_n holds true if and only if the sequence H_n possesses the same property. It is also interesting to remark that (3.9) is significantly weaker than relative compactness. To see this we quote the following assertion (see [4], Proposition 11. 3. 1):

Assuming $F_n\to F\neq 0$ *for some* $F\in\mathbf{M}$ *we have* $F_n\Rightarrow F$ *and F is non-defective if and only if*

$$\lim_{T\to\infty}\sup_{n}\sum_{k=1}^{k_n}P(|X_{nk}|\geqq T)=0. \tag{3.10}$$

It is noteworthy that the uniform asymptotic negligibility is not mentioned in this case.

Applying this result, we get:

Corollary 3.7. *Under the conditions of Theorem (3.5) the following relations are equivalent:*

(i) $F_n\Rightarrow F$;
(ii) $F(\infty)=1$;
(iii) $H_n\Rightarrow H$;
(iv) $\rho=0$;
(v) H_n *is tight, i. e., we have*

$$\lim_{T\to\infty} \sup_n \int_{|x|\geq T} dH_n(x) = 0;$$

(vi) *L_n is tight, i. e., (3.10) is satisfied.*

Another method which leads to the infinite divisibility of a vague limit function $F \neq 0$ is based on the one-sided tightness of H_n and L_n, respectively (for details see [4], section 11.3).

4. Special cases and applications. In the present section we deal with the vague convergence of sums in a triangular array for a normal limit function. Further we give improved versions and generalizations of results derived in [4], where the so-called assumption of restricted convergence is applied. We adhere to the notation of the preceding section; in particular we need

$$S_n = \sum_{k=1}^{k_n} X_{nk} - A_n \sim F_n,\ a_{nk},\ a_n,\ F_{nk}^{\circ}(x) = F_{nk}(x + a_{nk}),\ L_n.$$

To begin with we state limit theorems for a normal function $\psi = cN(\mu, \sigma^2) \in \mathbf{M}$, $0 < c \leq 1$ that is, we have $\psi = \mathfrak{L}[\rho;\ \mu, \sigma^2, 0]$ with $\rho = \log c$. Denoting the characteristic function of F_n, ψ by f_n, ρ, and specifying Theorem 3.6, we get:

Theorem 4.1. *Suppose that the condition of infinite smallness of the triangular array $\{X_{nk}\}$ is satisfied, and let ψ be defined as above. Then*

$$\lim_{n\to\infty} \int_0^t |f_n(u) - \varphi(u)|\, du = 0, \quad \textit{for all } t > 0,$$

if and only if the following assertions are true:

$$\sum_{k=1}^{k_n} a_{nk} - A_n \to \mu, \quad n \to \infty; \tag{4.1}$$

$$\lim_{n\to\infty} \sup_{|x|\geq T} \sum_{k=1}^{k_n} P(|X_{nk} - x| \leq h) = 0, \tag{4.2}$$

for some $h > 0$ and all $T > h$;

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \Big[\int_{|x|<\varepsilon} x^2 dF_{nk}(x) - \Big(\int_{|x|<\varepsilon} x dF_{nk}(x)\Big)^2 \Big] = \sigma^2$$

for all $\varepsilon > 0$;

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} P(|X_{nk}| \geq T) = -\rho, \textit{for all } T > 0.$$

It is well known that (4.1) is equivalent to $a_n \to \mu$ provided that $L = 0$ is true. The relation (4.2) shows not only $L = 0$, but also the equivalence of the above conditions for σ^2 and ρ with the corresponding ones given in Theorem 3.6.

Let us now introduce the set $\bar{\mathbf{M}}$ of all non-decreasing left-continuous functions V with $0 \leq V(-\infty) \leq V(\infty) \leq 1$. Given a function $V \in \bar{\mathbf{M}}$, $V \neq 0$, we have the unique representation

$$V = cF + d \tag{4.3}$$

for some d. f. F and some constants c, d with

$$0 < c = V(\infty) - V(-\infty) \leq 1, \qquad 0 \leq d = V(-\infty) \leq 1 - c.$$

Denote by $\mathbf{S}_V$, $\mathbf{S}_V \subseteq R$, the set of continuity points of V and let us consider the convergence

$$F_n(x) \to V(x), \quad n \to \infty, \quad \text{for all } x \in \mathbf{S}_V. \tag{4.4}$$

The connection between (4.4) and the vague convergence is very simple. Namely, we have (4.4) for some $V \in \overline{\mathbf{M}}$ if and only if $F_n \to \widetilde{F}$ for some $\widetilde{F} \in \mathbf{M}$ and, moreover, $F_n(x_0) \to \alpha$ for a certain real number α and some $x_0 \in \mathbf{S}_{\widetilde{F}}$. In either case (4.3) holds with $c = \widetilde{F}(\infty)$, $d = \alpha - \widetilde{F}(x_0)$, $F = \frac{1}{c}\widetilde{F}$.

Next we turn to the condition of restricted convergence:

$$F_n(x) \to V(x), \quad n \to \infty, \quad \text{for all } x \in \mathbf{S}. \tag{4.5}$$

Here $\mathbf{S}$ stands for a fixed set with a finite limit point. Obviously, (4.5) is distinctly weaker than (4.4). Thus it arises the following problem:
Find additional conditions such that (4.5) implies (4.4).

The rest of this section aims at deriving special results of this kind, where V is an inf. div. function.

Theorem 4.2. *Suppose that the condition of uniform asymptotic negligibility of the triangular array $\{X_{nk}\}$ is satisfied. Assume further that (2.3) is valid for the sequence L_n. Put $V = \mathfrak{L}[\rho;\ a, \sigma^2, L] + d \in \overline{\mathbf{M}}$ and let $\sigma^2 > 0$. Then we have*

$$\lim_{n\to\infty} F_n(x) = V(x), \quad \textit{for all } x \in R$$

if and only if the following two conditions are fulfilled:

(i) $$\delta = \lim_{\varepsilon\to+0} \varliminf_{n\to\infty} \sum_{k=1}^{k_n} \int_{|x|<\varepsilon} x^2 dF^0_{nk}(x) > 0;$$

(ii) $$\lim_{n\to\infty} F_n(x) = V(x), \quad \text{for all } x \in \mathbf{S},$$

where $\mathbf{S}$ is a set with a finite limit point.
In either case we have the convergence

$$\lim_{n\to\infty} \int_0^t |f_n(u) - v(u)|\, du = 0, \quad t > 0, \tag{4.6}$$

for the characteristic functions of F_n and V.

Next we state the special case of weak convergence.

Corollary (4.3). *Assume that the suppositions of Theorem 4.2 are satisfied with $V = \mathfrak{L}[0;\ a, \sigma^2, L]$ and let $\sigma^2 > 0$. Then (i) and (ii) imply $F_n \Rightarrow V$.*

This statement is like proposition 11. 6.1 in [4]; but in corollary 4.3 we only assume (2.3) for L_n instead of the sharper condition (3.10) used in [4].

Proof of Theorem 4.2. The only-if part is trivial to see since (i) is a well-known relation for the normal component $\delta = \sigma^2 > 0$.

The if-part: Let n' be a subsequence satisfying

$$\lim_{n'\to\infty} F_{n'}(x) = V_0(x), \quad x \in \mathbf{S}_{V_0},$$

for a certain $V_0 \in \overline{\mathbf{M}}$. By virtue of $\sigma^2 > 0$, V is strongly increasing and analytic in a domain containing R, (cf. [4], proposition 7. 5. 1; V is even an integral function). Now assumption (ii) leads us to $V_0 \neq$ const on $\mathbf{S}$. Thus we have $F_{n'} \to \widetilde{F} = V_0 - V_0(-\infty) \neq 0$

and the suppositions of Theorem 3.6 are fulfilled. Accordingly, $\tilde{F}=\mathfrak{L}[\rho_0;\ a_0,\ \sigma_0^2,\ L_0]$ for some real numbers $\rho_0\leqq 0$, a_0, $\sigma_0^2\geqq 0$, and a certain spectral function L_0. Further, our assumption (i) implies $\sigma=\sigma_0^2>0$ such that $\tilde{F}$ is analytic, too. Thus V_0 is analytic and we get

$$\lim_{n'\to\infty} F_{n'}(x)=V_0(x) \quad \text{for all } x\in R.$$

Now from (ii) we infer that $V_0(x)=V(x)$, $x\in \mathbf{S}$, and from the uniqueness theorem of analytic functions it follows that $V_0=V$. This tells us that V_0 does not depend on the special subsequence n' and we have $\lim_{n\to\infty} F_n(x)=V(x)$ for all $x\in R$ as asserted.

Next we deal with the vague convergence of F_n to a normal limit function $V=c\Phi+d$, $\Phi=N(\mu,\ \sigma^2)$, assuming the condition of restricted convergence. Then it suffices to consider a set $\mathbf{S}$ with four points only.

Theorem 4.4. *Suppose that the condition for uniform asymptotic negligibility of the triangular array $\{X_{nk}\}$ is satisfied. Let (2.3) be satisfied for the sequence L_n. Further, define V as above and let $\sigma^2>0$. Then we have*

$$\lim_{n\to\infty} F_n(x)=V(x), \quad \textit{for all } x\in R,$$

if and only if the following two assertions are fulfilled:

(i) $$L_n(x_2)-L_n(x_1)\to 0, \quad n\to\infty,$$

for all continuity points x_1, x_2 of L with $-\infty<x_1<x_2<0$ and $0<x_1<x_2<\infty$, respectively;

(ii) There is a set $\mathbf{S}$ containing at least four different real numbers such that

$$\lim_{n\to\infty} F_n(x)=V(x), \quad x\in \mathbf{S}.$$

In either case the relation (4.6) holds for the corresponding characteristic functionals f_n, v.

Before proving this result we state the special case of weak convergence; then it is enough to choose a set $\mathbf{S}'$ containing at least two different points.

Corollary 4.5. *Assume that the suppositions of Theorem 4.4 are satisfied with $V=N(\mu,\sigma^2)$ and let $\sigma^2>0$. Then all assertions of Theorem 4.4 remain true replacing $\mathbf{S}$ by $\mathbf{S}'$.*

Proof of Theorem 4.4. The only-if part is easy to see from Theorem 4.1.

The if-part: We choose a subsequence n' satisfying

$$\lim_{n'\to\infty} F_{n'}(x)=V_0(x), \quad X\in \mathbf{S}_{V_0},$$

for a certain $V_0\in\bar{\mathbf{M}}$. Moreover, since $c>0$, V is strongly increasing. In exactly the same way as in the proof of Theorem 4.2 we may apply Theorem 3.6 so that $F_n\to F=V_0-V_0(-\infty)\neq 0$, $F=\mathfrak{L}[\rho_0;\ \mu_0,\ \sigma_0^2,\ L_0]$, say. Our assumption (i) implies $L=0$, that is, F is a normal function and, as a consequence, so is V_0. Noting that V is normal by assumption and applying (ii) we get

$$V_0(x)=c_0N(\mu_0,\ \sigma_0^2)(x)+d_0=V(x)=cN(\mu,\ \sigma^2)(x)+d, \quad x\in\mathbf{S}. \tag{4.7}$$

Now it is easy to check the equations

$$c=c_0,\ d=d_0,\ \mu=\mu_0,\ \sigma=\sigma_0.$$

Thus $V=V_0$ and the limit function V does not depend on the choose of the special subsequence n'. This proves

$$\lim_{n\to\infty} F_n(x)=V(x), \quad x\in R,$$

as asserted.

Proof of Corollary 4.5. This assertion can be shown in the same way as Theorem 4.4; instead of (4.7) we then have

$$V_0(x)=N(\mu_0, \sigma_0^2)(x)=V(x)=N(\mu, \sigma^2)(x), \quad x\in \mathbf{S}',$$

so that even in this case $V=V_0$ follows.

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