Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

ON EXTRINSIC SPHERES IN SASAKIAN MANIFOLDS

CZESŁAW KONOPKA

1. Introduction. An m-dimensional submanifold, $m\geqq 2$, of a Riemannian manifold is called an extrinsic sphere if it is umbilical and has nonzero parallel mean curvature vector field [2]. Recently, extrinsic spheres in Sasakian manifolds were studied by Yamaguchi, Nemoto and Kawabata [3]. The present paper also concerns properties of extrinsic spheres in such manifolds. More precisely, we show that an anti-invariant extrinsic sphere of dimension $\geqq 3$ is of constant curvature if its curvature tensor of the connection induced in the normal bundle satisfies the condition

$$\nabla_n \nabla_m R_{cb\alpha}{}^{\gamma} - \nabla_m \nabla_n R_{cb\alpha}{}^{\gamma} = 0.$$

Moreover, we give some sufficient conditions for an extrinsic sphere to be homothetic to a Sasakian manifold. Our theorems are strictly related to results of Yamaguchi, Nemoto and Kawabata in [3].

2. Preliminaries. Let $\widetilde{M}$ be an n-dimensional Sasakian manifold. Thus, $\widetilde{M}$ is a Riemannian manifold with metric tensor G together with (1.1) — tensor field F and a vector field v which satisfy (cf. [1]):

(2.1) $$F^i_r F^r_j = -\delta^i_j + v^i v_j, \quad v_r v^r = 1, \quad F^i_r v^r = 0,$$

(2.2) $$G_{rs} F^r_i F^s_j = G_{ij} - v_i v_j,$$

(2.3) $$\nabla_k F^i_j = -G_{kj} v^i + \delta^i_k v_j,$$

(2.3) $$\nabla_k v_j = F_{kj} (= -F_{jk}),$$

where ∇ denotes the covariant differentiation with respect to G, $v_i = G_{ir} v^r$ and $F_{ij} = G_{jr} F^r_i$.

Let M ($\dim M = m$, $m<n$) be a submanifold of $\widetilde{M}$. Denote by (x^i) and (u^a) local coordinates of $\widetilde{M}$ and M, respectively. The indices a, b, c, d, e, m, n run over the range $\{1, 2, \dots, m\}$ and the indices i, j, k, r, s over the range $\{1, 2, \dots, n\}$. Let $x^i = x^i(u^a)$ be a parametric representation of M in $\widetilde{M}$. Set $B^i_a = \partial x^i / \partial u^a$. Let g be the Riemannian metric induced on $\widetilde{M}$ of components $g_{ab} = G_{rs} B^r_a B^s_b$.

Next we take $(n-m)$-mutually orthogonal unit local vector fields N_α normal to M and denote their components by N^i_α. The Greek indices run over the range $\{m+1, m+2, \dots, n\}$. Denote by $g_{\alpha\gamma}$ the components of the metric tensor induced on the normal bundle $T^\perp M$ of M from the metric tensor G of $\widetilde{M}$, that is, $g_{\alpha\gamma} = G_{rs} N^r_\alpha N^s_\gamma$.

Let us express FB_a and FN_α as linear combinations of B_a and N_α as follows

(2.5) $$F^i_r B^r_a = f^e_a B^i_e + f^\gamma_a N^i_\gamma,$$

(2.6) $$F^i_r N^r_\alpha = f^a_\alpha B^i_a + f^\gamma_\alpha N^i_\gamma,$$

and let

$$v^i = v^a B_a^i + v^\gamma N_\gamma^i. \tag{2.7}$$

Then f_α^γ are components of an f-structure in the normal bundle $T^\perp M$.

It can be easily noted that $f_{ab} = -f_{ba}$, $f_{\alpha a} = -f_{a\alpha}$, $f_{\alpha\gamma} = -f_{\gamma\alpha}$, where $f_{ab} = f_a^c g_{cb}$, $f_{\alpha a} = f_\alpha^c g_{ca}$, $f_{a\alpha} = f_a^\gamma g_{\alpha\gamma}$ and $f_{\alpha\gamma} = f_\alpha^\gamma g_{\gamma\alpha}$.

Using these equalities and taking into account (2.5)~(2.7) and (2.1) we find

$$f_a^e f_{eb} - f_a^\alpha f_{b\alpha} = -g_{ab} + v_a v_b, \tag{2.8}$$

$$f_\alpha^e f_{ea} + f_{\gamma a} f_\alpha^\gamma = v_a v_\alpha, \tag{2.9}$$

$$f_\alpha^\gamma f_{\gamma\beta} - f_\alpha^a f_{\beta a} = -g_{\alpha\beta} + v_\alpha v_\beta, \tag{2.10}$$

$$f_{ea} v^e = f_{\alpha a} v^\alpha, \tag{2.11}$$

$$f_{a\alpha} v^a = -f_{\gamma\alpha} v^\gamma, \tag{2.12}$$

$$v_a v^a + v_\alpha a^\alpha = 1, \tag{2.13}$$

where $v_a = g_{ae} v^e$ and $v_\alpha = g_{\alpha\gamma} v^\gamma$. The equality (2.13) is a consequence of $v_r v^r = 1$.

If $FT_x M \subset T_x^\perp M$ for any $x \in M$ (i. e. $f_a^b = 0$), then M is said to be an anti-invariant submanifold of M (cf. [5]).

Denoting by h_{ab}^α the components of the second fundamental tensor of the submanifold M, we have the following equations of Gauss

$$\nabla_b B_a^i = h_{ab}^\alpha N_\alpha^i \tag{2.14}$$

and of Weingarten

$$\nabla_b N_\alpha^i = -h_b^{e}{}_\alpha B_e^i, \tag{2.15}$$

where ∇ denotes the Van der Waerden—Bortolotti covariant differentiation (see e. g. K. Yano and S. Ishihara [4]) and $h_b^e{}_\alpha = h_{ba}^\gamma g^{ae} g_{\alpha\gamma}$.

The mean curvature vector field of the submanifold M has local components $h^\alpha = 1/m g^{ab} h_{ab}^\alpha$ and the function h, such that $h^2 = g_{\alpha\gamma} h^\alpha h^\gamma$ is the mean curvature.

The submanifold M is said to be totally umbilical if $h_{ab}^\alpha = g_{ab} h^\alpha$, and totally geodesic if $h_{ab}^\alpha = 0$. If the mean curvature vector field of a totally umbilical submanifold is nonzero and parallel ($\nabla_a h^\alpha = 0$), then M is called an extrinsic sphere in $\widetilde{M}$ (cf. [2]) The mean curvature h of an extrinsic sphere is a nonzero constant.

Let M be an extrinsic sphere in $\widetilde{M}$. Differentiating (2.5)~(2.7) covariantly along the submanifold M we obtain, by virtue of (2.2)~(2.7), (2.14) and (2.15),

$$\nabla_c f_{ab} = v_a g_{bc} - v_b g_{ac} + u_a g_{bc} - u_b g_{ac}, \tag{2.16}$$

$$\nabla_b f_{a\alpha} = -v_\alpha g_{ab} - h_\alpha f_{ab} + f_{\gamma\alpha} h^\gamma g_{ab}, \tag{2.17}$$

$$\nabla_b f_{\alpha\gamma} = f_{b\alpha} h_\gamma - f_{b\gamma} h_\alpha. \tag{2.18}$$

$$\nabla_b v_a = P g_{ab} + f_{ba}, \tag{2.19}$$

$$\nabla_b v_\alpha = f_{b\alpha} - v_b h_\alpha, \tag{2.20}$$

where $u_a = f_{a\alpha} h^\alpha$ and $P = v_\alpha h^\alpha = v_i N_\alpha^i h^\alpha = v_i h^i$.

We remark also that from (2.20) it follows that

(2.21) $$\nabla_b P = u_b - h^2 v_b,$$

and from (2.17) it follows that

(2.22) $$\nabla_b u_a = -h^2 f_{ab} - P g_{ab},$$

3. Properties of extrinsic spheres. In this section we assume that M is an extrinsic sphere in a Sasakian manifold $\tilde{M}$.

Lemma 3.1. *For M we have*

(3.1) $$v^e T_{cbae} = 0,$$

(3.2) $$f_{e\alpha} T^e_{cba} + f^\gamma_a R_{cb\alpha\gamma} = 0,$$

(3.3) $$f^\alpha_a(\nabla_n \nabla_m R_{cb\gamma\alpha} - \nabla_m \nabla_n R_{cb\gamma\alpha}) - f^e_\gamma(\nabla_n \nabla_m R_{cbae} - \nabla_m \nabla_n R_{cbae})$$
$$= (1+h^2)(f_{n\gamma} T_{cbam} - f_{m\gamma} T_{cban} + f^e_\gamma T_{cbne} g_{am} - f^e_\gamma T_{cbme} g_{an}),$$

where

(3.4) $$T_{dcba} = R_{dcba} - (1+h^2)(g_{ad} g_{bc} - g_{ac} g_{bd}), \quad T^a_{dcb} = T_{dcbe} g^{ea},$$

R_{dcba} *and* $R_{cb\gamma\alpha} = R^\varkappa_{cb\gamma} g_{\alpha\varkappa}$ *are covariant components of the curvature tensor of* M *and the connection in the normal bundle of* M, *respectively*.

Proof. Differentiating (2.19) covariantly and applying the Ricci identity, (2.16), (2.21) and (3.4), we find (3.1). Differentiating now (2.17) covariantly and applying (2.16), (2.18), (2.20) and the Ricci identity, we obtain

$$f_{e\alpha} R^e_{cba} + f_{a\gamma} R^\gamma_{cb\alpha} = (1+h^2)(g_{ab} f_{c\alpha} - g_{ac} f_{b\alpha}).$$

This, by making use of (3.4) can be written in the form (3.2).

To prove (3.3) we transvect the Ricci identity

$$\nabla_n \nabla_m R_{cb\gamma\alpha} - \nabla_m \nabla_n R_{cb\gamma\alpha} = -R_{eb\gamma\alpha} R^e_{nmc} - R_{ce\gamma\alpha} R^e_{nmb} - R_{cb\varkappa\alpha} R^\varkappa_{nm\gamma} - R_{cb\gamma\varkappa} R^\varkappa_{nm\alpha}$$

with f^α_a and use (3.2), (3.4) and the known identity

$$\nabla_n \nabla_m R_{dcba} - \nabla_m \nabla_n R_{dcba} = -R_{dcbe} R^e_{nma} - R_{dcea} R^e_{nmb} - R_{deba} R^e_{nmc} - R_{ecba} R^e_{nmd}.$$

This completes the proof.

In Theorem 3.4 we consider anti-invariant extrinsic spheres. The following proposition gives a characterisation of the anti-invariance:

Proposition 3.2. *An extrinsic sphere M in a Sasakian manifold is anti-invariant if and only if the tensor field of components f_{ab} is parallel on M.*

Proof. The necessary condition is obvious. Suppose now that $\nabla_c f_{ab} = 0$. Then, in virtue of (2.16), we obtain $u_a = -v_a$. The last equation, by covariant differentiation an making use of (2.19) and (2.22) leads to $f_{ab} = 0$, which completes the proof.

Contrary to the above we have:

Proposition 3.3. *Let M be an extrinsic sphere in a Sasakian manifold. There are no open subset of M on which $\nabla_b f_{a\alpha} = 0$.*

Proof. Suppose that U is an open subset of M on which $\nabla_b f_{a\alpha} = 0$. Then, from this and (2.17) it follows that $f_{ab} = 0$ and $f_{\gamma\alpha} h^\gamma = v_\alpha$. From (2.16) and (2.12) we get $u_a = -v_a$ and $u_e v^e = v_\alpha v^\alpha$ respectively. Consequently, we have $v_e v^e + v_\alpha v^\alpha = 0$ on U. This contradicts the equality (2.13). The proof is complete.

Theorem 3.4. *Let M be an anti-invariant extrinsic sphere in a Sasakian manifold and $\dim M \geq 3$. If the curvature tensor in the normal bundle of M satisfies the condition*

(3.5) $$\nabla_n \nabla_m R_{cb\gamma\alpha} - \nabla_m \nabla_n R_{cb\gamma\alpha} = 0,$$

then M is of constant curvature.

Proof. Suppose that (3.5) is satisfied. Then, permuting (3.3) cyclically with respect to indices (a, b, c), adding the resulting equations and using the relation $T_{cbae} + T_{bace} + T_{acbe} = 0$ and the first Bianchi identity, we find

$$f_\alpha^e(T_{cbne}g_{am} - T_{cbme}g_{an} + T_{bane}g_{cm} - T_{bame}g_{cn} + T_{acne}g_{bm} - T_{acme}g_{bn}) = 0.$$

Hence, by transvection with $f_d^\alpha g^{am}$ and making use of (2.8), (3.1) and our assumption $(f_{ab} = 0)$, we have

(3.6) $$(m-3)T_{cbnd} = T_{db}g_{cn} - T_{dc}g_{bn},$$

where $T_{ab} = T_{cbae}g^{ce}$.

Next, transvecting (3.6) with g^{bn}, we get $T_{dc} = 0$ (since $m \geqq 3$). If $m > 3$, then the relation $T_{dc} = 0$ used in (3.6) yields $T_{abcd} = 0$. As it is well known, if $m = 3$, then $T_{dc} = 0$ always implies $T_{abcd} = 0$. Thus, M is a space of constant curvature. Our Theorem is thus proved.

Corollary 3.5. *Let M be a complete, connected, simply connected, anti-invariant extrinsic sphere in a Sasakian manifold and* $\dim M \geqq 3$. *If the curvature tensor in the normal bundle of M satisfies* (3.5), *then M is isometric to an ordinary sphere.*

Theorem 3.6. *If the projection of the vector field v on M has nonzero constant length* (*i. e.,* $v_e v^e = \mathrm{const} \neq 0$), *then M is homothetic to a Sasakian manifold.*

Proof. Differentiating the relation $v_e v^e = \mathrm{const} \neq 0$ covariantly and taking into consideration (2.19), we obtain $f_{ea}v^e = Pv_a$. Hence we see that $P = 0$. It has been proved ([3], see the proof of Theorem) that if $P = 0$ and $v_e v^e = \mathrm{const} \neq 0$, then M is homothetic to a Sasakian manifold. This completes the proof.

If $\nabla_c f_\alpha^\gamma = 0$, then the f-structure in the normal bundle $T^\perp M$ of M is said to be parallel.

Theorem 3.7. *If the f-structure is parallel in the normal bundle and* $P = v_\alpha h^\alpha = 0$ *on M, then M is homothetic to a Sasakian manifold.*

Proof. Suppose that $\nabla_c f_\alpha^\gamma = 0$ holds. Then, from (2.18) it follows that $f_{\alpha\alpha}h_\gamma = f_{\alpha\gamma}h_\gamma$. Hence, by covariant differentiation and using of (2.17), we find

(3.7) $$v_\alpha h_\gamma - h_\alpha v_\varkappa = h_\alpha f_{\gamma\varkappa}h^\varkappa - f_{\alpha\varkappa}h^\varkappa h_\gamma.$$

Transvecting now (3.7) with v^γ and considering (2.12), we have $Pv_\alpha - h_\alpha v_\gamma v^\gamma = -h_\alpha u_e v^e - Pf_{\alpha\gamma}h^\varkappa$. If $P = 0$, then $u_e = h^2 v_e$ and the above equality can be rewritten in the form $h^2 v_e v^e - v_\alpha v^\alpha = 0$. Therefore, (2.13) gives $(1 + h^2)v_e v^e = 1$, which together with Theorem 3.6 completes the proof.

REFERENCES

1. D. E. Blair. Contact manifolds in Riemannian geometry. (*Lecture Notes math.*, *Vol. 509*). Berlin, 1976.
2. K. Nomizu, K. Yano. On circles and spheres in Riemannian geometry. *Math. Ann.*, **210**, 1974, 163-170.
3. S. Yamaguchi, H. Nemoto, N. Kawabata. Extrinsic spheres in a Sasakian manifold. *Tensor*, **40**, 1983, 184-188.
4. K. Yano, S. Ishihara. Submanifolds with parallel mean curvature vector. *J. Differential Geometry*, **6**, 1971, 95-118.
5. K. Yano, M. Kon. Structures on manifolds. (*Series in Pure Math.*, *Vol. 3*). Singapore, 1984.

Institute of Mathematics
Technical University
Wybrzeze Wyspianskiego 27,
50-370 Wroclaw
Poland

Received 2. 2. 1988