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STABILITY IN OPTIMIZATION*

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Continuity of the marginal value function and of the solution set of a parameterized family of minimum problems is investigated. Both the classical approach, like Berge's theorem, as well as the more recent ones involving variational convergences are considered. Special attention is paid to the convex case and the role of well posedness in order to strength stability results is put in evidence. Some applications to mathematical programming and results on the approximate solution sets end the paper.

Introduction. The paper deals with a famlly of problems Py concerning the minimization of a given function $f(.,y)\colon X\to\mathrm{R}$ over a subset Ky of X. This family of parameterized problems is generally originated by an initial unperturbed problem Py_0, because of errors on the observation of a given phenomenon and /or of numerical approximations while solving the problem itself. Here the interest is focused in the stability of the initial problem Py_0, that means, roughly speaking, continuity of the performance or marginal value function $v(y)=\inf\{f(x,y)\colon x\in Ky\}$ and continuities to be better specified of the solution set $My=\{x\in Ky\colon f(x,y)=v(y)\}$. I do not even touch other parallel theories, as sensitivity analysis or dual theories, even if they are of the greatest importance in the same context. There is a huge number of papers concerning the stability problem in optimization; here, rather arbitrarily perhaps, I recall between them for instance [2, 9—15, 18, 19, 24—28, 40—43]; see moreover the books [1, 5, 23] and the references therein. At a first sight it seems to be natural to distinguish between two different approaches to the stability problem; the first one, perhaps more classical, deals in general with a regular function and looks for appropriate continuity and compactness conditions on the constraint set multifunction in order to get the required stability of the problem. The second approach deals instead in general with a sequence of extended real valued functions (absorbing in this way the constraints on the function itself) approximating a limit function, that represents the initial problem; next right convergences are studied in order to get a minimal set of stability properties; these are shown to be strengthened if there is some compactness of minimizing sequences. The aims of this paper are several: at first to review quickly the properties of multifunctions that are needed treating with stability, then to give a survey of the results in the two approaches, to show next that both have essentially the same underlying ideas, to throw lights on the fact that it is rather well posedness than compactness the suitable property in order to get upper semicontinuity of the solution set (mainly in the convex case). Furthermore a section is dedicated to extensions of known results in mathematical programming: this kind of extension can be justified by the reason that it shows in a simple way what are the essential properties of the ordering and of monotone multifunctions to get their lower and upper semicontinuity. The paper is organized as follows: a first section is dedicated to preliminaries, notations and general assumptions, while section two reviews the main pro-

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perties of multifunctions we are interested in. The section three is dedicated to the value function v and the section four deals with the multifunction M; furthermore at its end some consideration is presented in order to get a synthesis between the two recalled approaches. Next section deals with the mathematical programming case; then we conclude with a short presentation of some results on the approximate solution sets, that generalize some known ones. This paper is an updating, a generalization and a deepening of my previous paper [32].

1. Preliminaries. Let (X, τ) be a topological space and $x \in X$. By $I(x)$, $V(x)$ and $W(x)$ we shall denote open neighborhoods of x. The empty set is denoted by $\emptyset$. If (X, d) is a metric space, then $S(x, r)$ is the (closed) sphere centered at x and with radius r. Given a non empty set C, its closure will be denoted by $\bar{C}$, its interior by $\mathring{C}$, its boundary by ∂C, while d_C is the indicator function of C, namely $d_C(x)=0$ if $x \in C$, $+\infty$ otherwise; $d(x, C)$ is defined as $\inf\{d(x, c): c \in C\}$ and C^a is the set: $C^a=\{x \in X: d(x, C)<a\}$. The measure of non compactness c of C is $c(C)=\inf\{\operatorname{diam} C_i: \bigcup C_i=C$ and the union is made on an arbitrary but finite number of indices$\}$. For a function $g: X \to (-\infty, +\infty]$ let epig be its epigraph: namely $\operatorname{epig}=\{(x, r) \in X \times \mathbb{R}: g(x) \leq r\}$. In the sequel we shall deal with a parameter space Y, which is a topological space, and with multifunctions $K: Y \rightrightarrows X$. Here multifunction means that, for every y, Ky is a closed not necessarily nonempty subset of X. Of course K can be also viewed as a function taking values on the set 2^X of the subsets of X; it follows then that continuity definitions for K can be given (suitably) topologizing 2^X. However, here we do not follow at all this approach and we give some standard definitions of semicontinuity without relating them to topologies in 2^X. The interested reader can be profitably consult [29], for instance, but with a caveat: in that book multifunctions are always nonempty valued. By graph of the multifunction K we shall indicate the subset of $Y \times X$: $\operatorname{gph} K=\{(y, x): x \in Ky\}$. For a (nonempty) subset A of Y, $K(A)$ is the subset of X: $K(A)=\{x \in X: x \in Ky$ for some $y \in A\}$ and for a subset C of X, $K^-(C)$ will indicate the subset of Y: $K^-(C)=\{y \in Y: Ky \cap C \neq \emptyset\}$. The following definitions of continuity for multifunctions and of convergence of sequences of sets are standard, even though some of them are often differently named: see for instance [3, 16] and [29].

Definition 1.1. *K is upper semicontinuous at a point $y_0 \in Y$ (u.s.c. for short) if for every open set A in X such that $A \supset Ky_0$, there is $V(y_0)$ such that $K(V) \subset A$.*

Equivalently, if for every closed subset C of X, $y_0 \in \overline{K^-(C)}$ implies $y_0 \in K^-(C)$.

Remark 1.2. Observe that $Ky_0=\emptyset$ and K u.s.c. at y_0 imply that there is $V(y_0)$ such that $KV=\emptyset$.

Definition 1.3. *K is lower semicontious at a point $y_0 \in Y$ (l.s.c. for short) if $Ky_0=\emptyset$ or if for every A open set in X such that $A \cap Ky_0 \neq \emptyset$, there is $V(y_0)$ such that, $\forall y \in V$, $Ky \cap A \neq \emptyset$.*

Definition 1.4. *K is closed at a point $y_0 \in Y$ if for every $x \notin Ky_0$ there are $I(x)$ and $V(y_0)$ such that $K(V) \cap I=\emptyset$.*

Remark 1.5. It is easy to check that K is closed at every point y if and only if gph K is a closed set.

Let now X be a metric space.

Definition 1.6. *K is upper Hausdorff semicontinuous at a point $y_0 \in Y$ (u.H.s.c. for short) if for every $a>0$ there is $V(y_0)$ such that $KV \subset Ky_0^a$.*

Definition 1.7. *K is lower Hausdorff semicontinuous at a point $y_0 \in Y$ (l.H.s.c. for short) if for every $a>0$ there is $V(y_0)$ such that $Ky_0 \subset Ky^a \forall y \in V$.*

The following facts are very easy to verity: u.(H.) and l.(H.) semicontinuities are not (directly) related, u.s.c. implies u.H.s.c. while l.H.s.c. implies l.s.c.; other relationships between them will be recalled in the next section. Furthermore if Y satisfies the first numerability axiom closedness at y_0 can be equivalently stated in the following way:

(1.1) $$\forall y_n \to y_0,\ \forall x_n \in Ky_n \text{ with } x_n \to x_0, \text{ than } x_0 \in Ky_0.$$

Finally observe that closedness property is preserved if we refine topologies either in X or in Y (or in both), while u. and l. semicontinuities are preserved if we refine the topology on Y or if we substitute topology on X with a coarser one (or both). Now we consider some convergence of sets: we bound ourself to the sequential case, although if much can be done also in the general case. Let $A_n \subset X$, which is supposed here metric.

Definition 1.8. *The superior limit of A_n is the subset of X:*

(1.2) $$\text{ls } A_n = \{x \in X:\ x = \lim x_k,\ x_k \in A_{n_k},\ n_k \text{ a selection from the integers}\},$$

the inferior limit of A_n is the subset of X:

(1.3) $$\text{li } A_n = \{x \in X:\ x = \lim x_n,\ x_n \in A_n \text{ for all (large) } n\}.$$

Observe that (in the sequential case) K is closed at a point $y_0 \in Y$ if and only if

(1.4) $$\forall y_n \to y_0 \quad \text{li } Ky_n \subset Ky_0 \text{ or also}$$

(1.5) $$\forall y_n \to y_0 \ \text{ls } Ky_n \subset Ky_0.$$

Definition 1.9. *We say that A_n converges to A in Hausdorff's sense ($A_n \xrightarrow{H} A$) if:*

(1.6) $$\forall a > 0 \exists \bar{n}:\ \forall n > \bar{n} A_n \subset A^a$$

(1.7) $$\forall a > 0 \exists \bar{n}:\ \forall n > \bar{n} A \subset A_n^a \text{ (see [30]).}$$

Definition 1.10. *In a Banach space X we say that A_n converges to A in Mosco's sense ($A_n \xrightarrow{M} A$) if the following conditions hold:*

(1.8) $$\forall x \in A \exists x_n \in A_n:\ x_n \to x$$

(1.9) $$\forall x_k \xrightarrow{w} x_0,\ x_k \in A_{n_k} \Rightarrow x_0 \in A \ (n_k \text{ is a selection from the integers}).$$

Here $\xrightarrow{w}$ means weak convergence (see [37]). Observe that Mosco's conver-

(1.10) gence can be also stated with the formula: $A_n \xrightarrow{M} A$ if and only if $\text{wls } A_n \subset A \subset \text{sli } A_n$ where w and s indicate weak and strong convergence respectively.

We shall deal in some parts of the paper with well posed problems: this concept is introduced in:

Definition 1.11. *Given a topological space X, a function $f:\ X \to \mathbb{R}$ and a subset C of X, the problem (f, C) of minimizing f on C is called: well posed (w.p. for short) if it has solution c such that if $c_n \in C$ and $f(c_n) \to f(c)$, then $c_n \to c$; well posed in generalized sense (g.w.p. for short) if every minimizing sequence c_n as before has cluster points that are solutions of the problem (See [39] and [33]).*

Given a multifunction $K:\ Y \rightrightarrows X$ and a function $f:\ X \times Y \to \mathbb{R}$ we call Py the problem:

(1.11) $$\text{to minimize } f(\cdot, y) \text{ on } Ky.$$

Our goal is to study continuity properties of $v:\ Y \to [-\infty, +\infty]$:

(1.12) $$v(y)=\inf\{f(x, y): x\in Ky\}$$

the value of the problem Py, and of $M: Y\rightrightarrows X$:

(1.13) $$My=\{x\in Ky: f(x, y)=v(y)\}$$

the solution set of the problem Py.

We say that $v(y)=+\infty$ if $Ky=\emptyset$ and we observe that M can be empty: this is the reason why we consider multifunctions that can be empty valued.

2. Some facts about multifunctions. We shall review in this section some properties of multifunctions that will be used in the sequel or that show relationships between the introduced continuity concepts.

Proposition 2.1. *Let X be metrizable, $K: Y\rightrightarrows X$ u.s.c. at $y_0\in Y$. Then K is u.H.s.c. at y_0. Conversely if K is u. H.s.c. at y_0 for every metric compatible with the topology, then K is u.s.c. at y_0.*

Proof. The first statement is, as already mentioned, obvious. The proof of the second one can be found in [22], p. 147.

It is easy to show that in general u. H.s.c., for a fixed metric, does not imply u.s.c. But if K is u.H.s.c. at y_0 and Ky_0 is a compact set, then K is u.s.c. at y_0. This is due to the fact that if an open set A contains the compact set Ky_0, then $A\supset Ky_0^a$ for $0<a<\inf\{d(x, y), y\in Ky_0, x\in X\setminus A\}$. Last result can be generalized:

Proposition 2.2. *Let X be a complete metric space and let Y satisfy the first numerability axiom. Then the following are equivalent:*

K is u. s. c. at y_0.

K is u.H.s.c. at y_0 and $c(KV_n\setminus Ky_0)\underset{n}{\to}0$, where V_n is a neighborhood basis of y_0.

Proof. See [22] theorem 4, where closedness of Ky_0 is weakened introducing the concept of active boundary.

For other generalizations, see [20].

From proposition 2.2 we can derive next corollary, that can be however easily proved independently:

Corollary 2.3. *With the same assumptions of proposition 2.2 let K be u. s. c. at y_0. Then there exists $V(y_0)$ such that KV/Ky_0 is a bounded set. Moreover if $y_n\to y_0$ and $x_n\in Ky_n\setminus Ky_0$ then x_n has cluster points (belonging to Ky_0).*

For what l.s.c.'s are concerned it is easy to see that l.H.s.c. is a stronger property than l.s.c.: actually it is strictly stronger, as it is shown in the following:

Example 2.4. $X=\mathbb{R}^2$, $Y=[-1, 1]$, $Ky=\{(x_1, x_2): x_1\geqq 0, x_2\geqq yx_1\}$.

But if Ky_0 is a compact set, the two properties are equivalent.

Proposition 2.5. *Let $K: Y\rightrightarrows X$ be l. s. c. at $y_0\in Y$ and let Ky_0 be a compact set. Then K is l.H.s.c. at $y_0\in Y$.*

Proof. Let $a>0$. There are $x_1,\dots, x_n\in Ky_0$ such that $Ky_0\subset\bigcup_i S(x_i, a/2)$ and $V_i(y_0)(i=1,\dots. n)$ such that $Ky\cap S(x_i, a/2)\neq\emptyset\,\forall y\in V_i$. Let $V=V_1\cap\dots\cap V_n$. If $y\in V$, then $Ky\cap S(x_j; a/2)\neq\emptyset\,\forall y\Rightarrow Ky^a\supset S(x_i, a/2)\forall i\Rightarrow Ky^a\supset Ky_0$.

For extension, see [6] theorem 1.

Proposition 2.6. *Let $K: Y\rightrightarrows X$ be u.s.c. at $y_0\in Y$ and let X be a T_3 space, Then K is closed at $y_0\in Y$.*

Proof. Let $x\notin Ky_0$. Then there are open sets A and B such that $x\in A$, $B\supset Ky_0$. By u.s.c. $B\supset KV$ for some $V(y_0)$. Hence $KV\cap A=\emptyset$.

Proposition 2.7. *Let K_1, K_2 be multifunctions from Y to X, K_1 closed at $y_0\in Y$ and K_2 u.s.c. at $y_0\in Y$. Moreover let K_2y_0 be a compact set. Then $K=K_1\cap K_2$ is u. s. c. at the point y_0.*

Proof. See for instance [3] proposition 2 for the case $K_1y_0\cap K_2y_0\neq\emptyset$. Consider now the case $K_1y_0\cap K_2y_0=\emptyset$. If $K_2y_0=\emptyset$ we are done. Otherwise let $x\in K_2y_0$. Then there are open sets $A(x)$ and $B_x(y_0)$ such that $A(x)\cap K_1B_x=\emptyset$. Consider the set $A=\cup_{x\in K_2y_0}A(x)$. Then $A\supset K_2y_0$ which is a compact set. Hence there is a finite number of open sets, say $A(x_1)\dots A(x_n)$ such that $\cup_1^n A(x_i)\supset K_2y_0$. By u.s.c. of K_2 there is $I(y_0)$ such that $K_2(I)\subset\cup_1^n A(x_i)=\bar{A}$. Consider the set $W(y_0)=\cap_1^n B_{x_i}$ and finally $V(y_0)=I\cap W$. Then $K_2V\subset\bar{A}$ and $K_1V\cup\bar{A}=\emptyset$. This concludes the proof. The u.s.c. of the intersection of two u.s.c. multifunctions is easier to prove and it does not require, in normal spaces, compactness of one image set, see f. i. [32] prop. 8.6. On the other hand, here the assumption is crucial, for if we have $K_1y=[1/y, +\infty)$, $K_2y=[1/y, +\infty)$ for $y\neq 0\, K_10=0$, $K_20=[0, +\infty)$, then $K=K_1$ is not u.s.c. at 0.

Proposition 2.7 has some nice consequences. For example:

P r o p o s i t i o n 2.8. *Let $K: Y\rightrightarrows X$ be closed at $y_0\in Y$ and let $V(y_0)$ be such that $KV\subset C$ where C is a compact set. Then K is u.s.c. at y_0.*

P r o p o s i t i o n 2.9. *Let $K: Y\rightrightarrows X$ be u.s.c. and compact valued. Then for every compact set H, $K(H)$ is a compact set.*

P r o o f. See [3] prop. 5, p. 72.

Lower semicontinuity of the intersection is a much harder property to deal with. The paper [4] surveys and improves recent results, all concerning convex valued multifunctions.

A simple and manageable result is the following:

P r o p o s i t i o n 2.10. *Let Y be first countable and let $K, H: Y\rightrightarrows \mathrm{R}^n$ be convex valued. If $Ky_0\cap \mathring{H}y_0\neq\emptyset$ and H, K are l.s.c. at y_0, then $H\cap K$ is l.s.c. at y_0.*

P r o o f. See [36] theorem 3.1.

Observe that the previous result is not true if H and K take values in an infinite dimensional linear space: let X be a Hilbert space and e_n an orthonormal basis. Consider $Ky_n=Ky_0=\Sigma_{j=0}^{\infty}1/j\,e_j$ and $Hy_n=\mathrm{span}\{e_1,\dots, e_n\}Hy_0=X$.

3. The value function v. This section is devoted to the investigation of continuity properties of the value function v.

3.1 general results. We begin with some general and classical results.

P r o p o s i t i o n 3.1.1. *Let $K: Y\rightrightarrows X$ be l.s.c. at a point y_0. Let $f: X\times Y\to R$ be u.s.c. at the points of the set Ky_0xy_0. Then v is u.s.c. at y_0.*

P r o o f. If for every $V(y_0)$ there is y_v such that $Ky_v=\emptyset$, then $v(y_0)=+\infty$, due to l.s.c. of K. Then we can suppose that $v(y)<+\infty\,\forall y$. Suppose next $v(y_0)=-\infty$. For every fixed $\varepsilon>0$, let $x_0\in Ky_0$ such that $f(x_0, y_0)\leq-\varepsilon^{-1}$. By u.s.c. of f at (x_0, y_0) there are $V_1(x_0)$, $V_2(y_0)$ such that $\forall(x, y)\in V_1\times V_2\, f(x, y)\leq f(x_0, y_0)+\varepsilon$. By l.s.c. of K there is $V(y_0)$ such that $Ky\cap V_1\neq\emptyset\,\forall y\in V$. Let $W=V\cap V_2$. For every $y\in W$ there is $x\in Ky\cap V_1$. Hence $v(y)\leq f(x, y)\leq f(x_0, y_0)+\varepsilon\leq\varepsilon^{-1}+\varepsilon$. Summarizing, for every $\varepsilon>0$ there is $W(y_0)$ such that $\forall y\in W v(y)\leq-\varepsilon^{-1}+\varepsilon$. This means that $\lim_{y\to y_0}v(y)=-\infty$. An analogous argument for the case $v(y_0)>-\infty$ finishes the proof.

P r o p o s i t i o n 3.1.2. *Let $K: Y\rightrightarrows X$ be u.s.c. at a point y_0 and Ky_0 a compact set. Let $f: X\times Y\to\mathrm{R}$ be l.s.c. at every point of the set Ky_0xy_0. Then v is l.s.c. at y_0.*

P r o o f. If $Ky_0=\emptyset$ by u.s.c. of $Kv(y)=+\infty$ on a neighborhood of y_0. Then we can suppose $v(y_0)\in\mathrm{R}$. Let $x\in Ky_0$; due to l.s.c. of f at (x, y_0) there are $V(x)$ and $V_x(y_0)$ such that $f(x, y_0)-\varepsilon\leq f(z, y)\,\forall z\in V$, $\forall y\in V_x$. As Ky_0 is a compact set, then Ky_0 is contained in a finite number of sets V_x, say $Ky_0\subset\cup_1^n V_{x_i}$. By u.s.c. of K there is $I(y_0)$ such that $K(I)\subset\cup_1^n V_{x_i}$. Now consider $W(y_0)$: $W=I\cap V_{x_1}\cap\dots\cap V_{x_n}$. Let $y\in W$ and $x\in Ky$. Then $x\in V_{x_i}$, for some i, that implies $f(x, y)\geq f(x_i, y_0)-\varepsilon$. Hence $v(y_0)-\varepsilon\leq\min_i f(x_i, y_0)-\varepsilon\leq f(x, y)$. We conclude that $v(y_0)-\varepsilon\leq f(x, y)\,\forall y\in W\forall x\in Ky$, then $v(y_0)-\varepsilon\leq v(y)\forall y\in W$.

The previous propositions are really classical and then many references can be quoted; look for instance at [11] theorems 1 and 2, [3] theorems 2 and 1, and [27] theorems 6 and 5.

Remark 3.1.3. We want here observe that u.s.c. of v can be obtained in a much more general setting: compactness of Ky_0 is no not needed. On the other hand consider $f(x, y)=xy$ on $Ky=\mathrm{R}$. Then $v(y)=-\infty$ if $y\neq 0$ $v(0)=0$. Then l.s.c. of v is completely lost at $y_0=0$.

Previous propositions can be summarized in the following elegant theorem, which is due to C. Berge [11], to my knowledge, in its first version.

Theorem 3.1.4. *Let $K\colon Y\rightrightarrows X$ be l and u.s.c. at a point y_0 and let Ky_0 be a compact set. Let $f: X\times Y\to\mathrm{R}$ be continuous at every point of the set $Ky_0 x y_0$. Then v is continuous at y_0.*

Theorem 3.1.4 can be greatly improved if f does not depend on the parameter y, namely if we face a family of problems in which perturbations are allowed only on the constraint set. As a matter of fact, we have:

Theorem 3.1.5. *Let $K\colon Y\rightrightarrows X$ be u. and l.s.c. at a point y_0. Let $f\colon X\to\mathrm{R}$ be continuous at every point of the set Ky_0. Then v is continuous at y_0.*

Proof. If $v(y_0)=+\infty$, by u.s.c. of K we get $v(y)=+\infty$ on a neighborhood of y_0. Suppose now $v(y_0)>-\infty$ and let $x_0\in Ky_0$ such that $f(x_0)\leqq v(y)+\varepsilon/2$. Continuity of f entails the existence, for every $x\in Ky_0$, of $V(x)$ such that $|f(x)-f(z)|<\varepsilon/2\,\forall z\in V$. Let $A=\bigcup_{x\in Ky_0} V(x)$, $B=V_{x_0}$, both open sets such that $A\supset Ky_0$ and $B\cap Ky_0\neq\emptyset$. Then there is $I(y_0)$ such that $K(I)\subset A$ and such that $\forall y\in I\,Ky\cap B\neq\emptyset$. Therefore $\forall y\in I\,v(y)\geqq\inf_A f\geqq\inf_{Ky_0} f-\varepsilon/2$. Moreover $\forall y\in I\,v(y)\leqq\inf_{Ky\cap B} f\leqq f(x_0)+\varepsilon/2\leqq v(y_0)+\varepsilon$. Then $|v(y)-v(y_0)|<\varepsilon$. If $v(y_0)=-\infty$ it is enough to repeat the argument on the set B, or to return to proposition 3.1.1. Observe that the part involving the set B gives a simplified proof of u.s.c. of v, which was however already stated in proposition 3.1.1. This result is proved in its first version, to my knowledge, in [9], theorem 1.

3.2. Results with convexity and well posedness. All the following theorems use convexity as a fundamental tool. For this class of functions, they can be viewed as generalizations of Berge's theorem.

Theorem 3.2.1: *Let $f\colon X\times Y\to(-\infty, +\infty]$ be a convex function in the two variables jointly and suppose that there is $x_0\in X$ such that $f(x_0, .)$ is continuous at a point y_0. Let $v(y_0)\in\mathrm{R}$ (Let $K\colon Y\rightrightarrows X$ such that $Ky=X\,\forall y\in Y$). Then v is continuous at y_0.*

Proof. See [31] theorem 7.6.1.

Observe that continuity of v is easily checked by remarking that v is a convex function, as it is not difficult to see. Moreover one can argue that the absence of constraints here is compensated by the fact that f is extended real valued. But convexity of f in the two variables jointly does not allow for absorbing general constraints: consider the following example: $f(x, y)=x+y$, $Ky=[-y^2, +\infty)$. Here v is not convex and previous proof does not apply. Lack of convexity of v depends essentially on the fact that gph K is not a convex set. (Continuity of v in the previous example can be seen as going back to an easy variant of theorem 3.1.5).

Now let X be a reflexive Banach space and Y with the first numerability axiom. Then we have:

Theorem 3.2.2. *Let $K\colon Y\rightrightarrows X$ be l. and u.s.c. at a point y_0 and convex valued. Let $f\colon X\times Y\to\mathrm{R}$ be convex and continuous at every point of the set $Ky_0 x y_0$. Then v is continuous at y_0.*

Proof. (Outline) U.s.c. follows from proposition 3.1.1 and does not use convexity. So l.s.c. must be checked, the only non trivial case being when $v(y)\in\mathrm{R}$. Let us suppose that there are sequences $y_n \to y_0$ and $x_n \in Ky_n$ and a real number $a>0$ such that $\lim f(x_n, y_n) < v(y_0) - a$. If $x_n \notin Ky_0$, for a subsequence, then by corollary 2.3 x_n has a cluster point belonging to Ky_0. This fact and continuity of f lead easily to a contradiction. Suppose then that the sequence x_n belongs to Ky_0, at least for large n. Then we can apply theorem 3.2.1 to the function $g(x, y) = f(x, y) + d_{Ky_0}(x)$. A simple calculation show that we get again a contradiction. Then v is continuous.

Remark 3.2.3. The example of remark 3.1.3 shows that convexity of $f(\cdot, y)$ for every value of the parameter y does not provide the same result. Hence both theorems above cannot avoid the very unattractive assumption of convexity of f in both variables jointly. A way of taking off this unpleasant assumption is suggested by the following:

Theorem 3.2.4. *Let $K: Y \rightrightarrows X$ and v be u.s.c. at a point y_0. Let K be convex valued. Let $f: X \times Y \to \mathbb{R}$ be uniformly continuous on bounded subsets of $X \times Y$ and $f(\cdot, y)$ a convex function for every y in a neighborhood of y_0. Suppose moreover that $f(\cdot, y_0)$ is g.w.p. on Ky_0 and that the problem of minimizing $f(\cdot, y)$ on Ky has a solution for every y in a neighborhood of y_0. Then v is continuous at y_0.*

Last theorem, and the previous one, are stated in [33]. However, theorem 3.2.4 there is claimed without the assumption that the problems Py have a solution in a neighborhood of y_0: I do not know if the statement is true without this assumption, that makes correct the proof in [33]. Still, convexity here is used for proving u.s.c. of the solution multifunction, from which we get l.s.c. of v. So it is used in some sense indirectly. However obseve, that here convexity appears in a much more reasonable way with respect to the previous theorems. Moreover, consider the following

Example 3.2.5. $f = \mathbb{R}^2 \times [0, +\infty) \to \mathbb{R}: f(x_1, x_2, y) = x_1^2 - x_1^6 x_2^2 y$

$$Ky = \{(x_1, 0) \cup (1/y, y): x_1 \in \mathbb{R}\} \text{ if } y \neq 0$$

$$K0 = \{(x_1, 0): 0 \leq x_1 \leq 1, \cup (x_1, x_2): x_1 \geq 1, 0 \leq x_2 \leq 1\}.$$

Then we have that $f(x_1, x_2, 0)$ is w. p. on $K0$ and $v(0)=0$, K is u.s.c. at 0 and f is uniformly continuous on every bounded subset of $\mathbb{R}^2 \times \mathbb{R}$. Moreover, $v(y) = 1/y^2 - 1/y^3 = f(1/y, y, y)$ (at least for small y) and v is u.s.c. at 0. With respect to the assumptions of theorem 3.2.4 only convexity is missing. And v is not (l.s.) continuous at 0.

3.3. Results in sequential setting. Often people like better deal with sequences rather than to establish results in general topological spaces. Therefore suppose to have a sequence of problems P_n of the kind: to minimize f_n on a set C_n, with f_n and C_n converging in some sense to f_0 and C_0 respectively. How can we translate previous theorems in this setting? One can think to put $Y = \{\cup_n 1/n \cup 0\}$, to define $f(x, 0) = f_0(x)$, $f(x 1/n) = f_n(x)$, $K1/n = C_n$ and $K0 = C_0$. Then we have, from theorem 3.1.4:

Proposition 3.3.1. *Let $f_0: X \to \mathbb{R}$ be a continuous function; suppose that f_n converge to f_0 locally uniformly and that $C_n \xrightarrow{H} C_0$, C_0 a compact set. Then $v_n \to v_0$.*

Proof. All is almost obvious, remembering that u.s.c. of the constraint set follows from the sentence after proposition 2.1.

Observe that if $f_n = f \forall n$ one can use theorem 3.1.5 rather than theorem 3.1.4 in order to avoid compactness assumptions. But observe that in such a case Hausdorff convergence does not provide the desired result. Namely consider $C_n = \{(x_1, x_2): x_1 \geq 0 x_2 \geq -1/n\}$, $C_0 = \{(x_1, x_2): x_i \geq 0\}$ and $f(x_1, x_2) = x_1 x_2$. Then l.s.c. of v at 0 is lost. Let us tackle the problem in a different way. Firstly, we shall allow functions to take value $+\infty$, avoiding in such a way the explicit mention of constraint sets. It is obvious indeed that, a part trivial cases, in the points where a function takes value $+\infty$, there it has no minimum. As it can be suspected at this time, u.s.c. of v is not hard to get. Look at the following condition:

(3.1) $\forall x \in X$ there is a sequence $x_n \in X$ such that $\limsup f_n(x_n) \leq f(x)$.

It is easy to show that:

Proposition 3.3.2. *If condition* (3.1) *holds, then* $\limsup v_n \leq v$.

On the other hand, l.s.c. of v is in general much harder to get. (One exception is theorem 3.1.5): we saw that some compactness on the problem is always needed in order to get it. But rather than requiring compactness of all the constraint set, in a sequential formulation it seems to be more appropriate to look directly at the (quasi) minima points. Then we have:

Proposition 3.3.3. *Suppose that* (3.1) *holds and moreover that*:

(3.2) *For every* $x_n \to x$ $\liminf f_n(x_n) \geq f(x)$.

Suppose that $v_n > -\infty$, *that there is* z_n *such that* $f_n(z_n) - v_n \to 0$ *and that* z_n *is relatively compact.*

Then $\lim v_n = v$.

Proof. The proof can be nowadays considered as standard (see f. i. [43] theorem 1). The convergence introduced in proposition 3.3.3 is the so-called variational convergence [43]. I want only to mention that a wide theory was recently developed to study Γ convergences, which are variational convergences in the sense that they pay attention to properties of the value function and of the solution set (for optimization problems). And some of them are the weakest ones fulfilling some stability properties with respect to minimum problems (see [8] and [34]), fully justifying in such a way their introduction in the variational context. Generalization of the above mentioned proposition can be found in [18].

There is at least another way to connect the sequential approach to classical theorems: this is shown in section 4.3 to which we address the reader.

3.4. Necessary conditions. We end the section quickly collecting some necessary conditions in order to achieve continuities of v.

Theorem 3.4.1. *Let* X *be a normal space,* K: $Y \rightrightarrows X$. *If for every continuous function* f: $X \to \mathrm{R}$ v *is l.s.c. at a point* y_0, *then* K *is u.s.c. at* y_0.

Theorem 3.4.2. *Let* X *be a topological space,* K: $Y \rightrightarrows X$. *If for every l.s.c. function* f: $X \to \mathrm{R}$ v *is l.s.c. at a point* y_0, *then* K *is u.s.c. at* y_0.

For the proofs, see [21] propositions 7 and 6.

Theorem 3.4.3. *Let* X *be a metric space. If for every continuous function* f: $X \to \mathrm{R}$ v *is u.s.c. at a point* y_0, *then* K *is l.s.c. at* y_0.

Proof. See [32] theorem 3.14. Metrizability can be weakened to normality, as shown in [23].

As already mentioned, necessary conditions in the sequential case can be found in [8, 34] and references quoted therein. Observe moreover that other results of this type can be obtained by noticing that $\inf f = -f^*(0)$, where f^* is the Young — Fenchel conjugate of f then studying pointwise convergence of f_n^* to f^*.

The final remark of the section is the following: closedness of K at a point y_0 does not entail in general any semicontinuity property for v, as it is easily seen.

4. The solution multifunction M.

4.1. Upper semicontinuity. From theorem 3.1.4 we easily get:

Theorem 4.1.1. *Let* K: $Y \rightrightarrows X$ *be u. and l. s. c. at a point* y_0 *and* Ky_0 *a compact set. Let* f: $X \times Y \to \mathrm{R}$ *be continuous at the points of the set* $Ky_0 \times y_0$. *Then* M *is u.s.c. at* y_0.

Proof. Let $H(y)=\{x\in X: f(x, y)-v(y)\leq 0\}$. Then $M=H\cap K$ and we can apply proposition 2.7 to conclude ([11], p. 122, [3], p. 70). A more general way to get theorem 4.1.1 is given in [29] where it is clear that only u.s.c. of the value function v plays a role. And this is actually the key assumption, rather than l.s.c. of K. On the other hand, theorem 3.4.3 shows that l.s.c. of K is strongly related to u.s.c. of v.

Remark 4.1.2. Another nice generalization can be checked by observing that the topologies on the space X involved in propositions 3.1.1 and 3.1.2 can be independent at all: we can get still continuity of v and u.s.c. of M, this last one with respect to the topology that makes u.s.c. the constraint set K. (See [23]).

Remark 4.1.3. L.s.c. of M is a much stronger property to achieve. Consider for instance $f(x, y)=xy$, $Ky=[y, 1]$ and $P0$. (Here we have much convexity) Even if f does not depend on y the situation does not improve: consider $f(x)=\max(0, 1-x^2)$, $Ky=[y-1, y+1]$ and $M0$. For results in this direction in the sequential case, see [17]. Recently S. Dolecki [19] characterized in an abstract way l.s.c. of M by means of the so-called "decisively growth" property. Several results about l.s.c. of M in interesting particular cases, mainly in mathematical programming, are presented in [5].

Theorem 4.1.4. *Let X be a complete metric space and let Y satisfy the first numerability axiom. Let K: $Y\to X$ be u. and l.s.c. at a point $y_0\in Y$ and let $f: X\to \mathbb{R}$ be continuous at every point of the set Ky_0. Let V_n is a neighborhood basis of y_0 and $B_n=\bigcup_{y\in V_n}(Ky_0\setminus Ky)$. If $c(B_n)\to 0$, then M is u.s.c. at y_0.*

Proof. See [6] theorem 8. Generalizations, mainly with the goal of weakening closedness of the constraints sets, are presented in [7].

Example 4.1.5. Here $c(B_n)$ "does not tend to" 0 and M is not u.s.c.: $f(x)=x^2$ if $x\leq 1$, $f(x)=1/x$ if $x>1$, $Ky=[y, 1/y^2]$ if $y\neq 0 K0=[0, +\infty)$. Lack of u.s.c. can be here attributed to the fact that the set $K0$ is too big with respect to the sets Ky, for y close to 0.

Theorem 4.1.6. *Let X be a normal space, K: $Y\rightrightarrows X$ u.s.c. at a point y_0, f: $X\times Y\to\mathbb{R}$ continuous at all points of the set Ky_0, xy_0, v continuous at y_0. Suppose that $Lr=\{x\in X: f(x, y_0)\leq r\}$ is u.s.c. at $v(y_0)$. Then M is u.s.c. at y_0.*

Proof. See [6] theorem 10.

The (easy) proof is based on the sentence after proposition 2.7. This explains the requirement of X being normal.

4.2 Results with convexity and generalized well posedness. In order to achieve u.s.c. of M, previous theorems use or compactness of the constraint set (theorem 4.1.1) or some assumptions about the measure of non compactness of sets related with the neighbouring constraint sets (theorem 4.1.4) or finally a suitable u.s.c. assumption on the level sets of the function $f(\cdot, y_0)$ on all the space X (theorem 4.1.6). Next theorems capture better, in my opinion, the real feature of the problem: they avoid compactness of the constraint sets using g.w.p., which is a (local) property relating the function with the constraint set. In the next theorems Y will always satisfy the first numerability axiom and X will be a reflexive Banach space.

Theorem 4.2.1. *Let K: $Y\rightrightarrows X$ be u.s.c. at a point y_0, let $f: X\to\mathbb{R}$ be continuous at every point of the set Ky_0 and g.w.p. on Ky_0. Moreover, suppose that v is u.s.c. at y_0. Then M is u.s.c. at y_0.*

Proof. See [33] where the theorem was stated with the additional assumption f being uniformly continuous on bounded sets. But it can be easily dropped out with the help of corollary 2.3.

Theorem 4.2.2. *In the same assumptions of theorem 4.2.1 on K and v, suppose that f: $X\times Y\to\mathbb{R}$ is uniformly continuous on bounded subsets of $X\times Y$, $f(\cdot, y)$ is a convex function for every y and $f(\cdot, y_0)$ g.w.p. on Ky_0. Then M is u.s.c. at y_0.*

Proof. See [33]. It is worthwhile to observe that both convexity and g.w.p. play a role in theorem 4.2.2 and that convexity cannot substitute g.w.p. in theorem 4.2.1. Consider at this end the following examples:

Examples 4.2.3. Let X be a Hilbert space, e_n an orthonormal basis on X. Let $Ky_0=X$ and $Ky_n=[e_{n4}, 1/n\, e_n]$. Let $f(x)=\Sigma_{k=1}^{\infty}(x, e_k)^2/k^2$. Then every assumption of theorem 4.2.1 but g. w. p. is fulfilled, and moreover f is strictly convex. Nevertheless $My_n=\{n^4/n^4+1\, e_{n4}\}$, while $M0=\{0\}$.

Example 4.2.4. $f(x, y)=\min(x^2, (x-1/y)^2-y)$ if $y\neq 0$, $f(x)=x^2$. Then $f(x, 0)$ is w.p., f is continuous, there are no constraints, v is continuous at 0: only convexity is dropped out. Here $My=\{1/y\}$, $y\neq 0$ and $M0=\{0\}$.

Next theorem has a rather different philosophy: we find in the assumption that much more regularity is requested for f, but on the other hand, a really mild condition, with respect to above mentioned theorems, must be fulfilled by the constraint set multifunction.

Theorem 4.2.5. *Let f: $X\to\mathrm{R}$ be convex and bounded and bounded sets. Let K: $Y\rightrightarrows X$ be convex valued and continuous in Mosco's sence (see* (1.8) (1.9) *and finally let f be g.w.p. on every closed convex set of X. Then M is u.s.c.*

Proof. See [35] where the theorem was stated with the additional assumption of uniqueness of minima; but only minor changes are needed to prove it in this form. Moreover, in [33] (theorem 3.6) another generalization is provided.

Finally, another nice result in the finite dimensional case is stated in the following theorem:

Theorem 4.2.6. *Let K: $Y\rightrightarrows\mathrm{R}^n$ be convex valued, u. and l.s.c. at a point y_0. Let f: $\mathrm{R}^n\to\mathrm{R}$ be quasi convex and continuous at every point of Ky_0. If My_0 is a nonempty bounded set, then M is u.s.c. at y_0.*

Proof. See [14], theorem 1.3.3.

4.3. The sequential case and closedness of *M*. Through this section Y will satisfy the first numerability axiom: hence we shall give every statement in terms of sequences. The most important and simple theorem on closedness of M can be considered the following:

Theorem 4.3.1. *Let K: $Y\rightrightarrows X$ be closed and l.s.c. at a point y_0, let f: $X\times Y\to\mathrm{R}$ be continuous at every point of the set Ky_0xy_0. Then M is closed at y_0.*

Proof. Suppose that $y_k\to y_0$, $x_k\in My_k$ and $x_k\to x_0$. Then, by closedness of K, $x_0\in Ky_0$. By proposition 3.1.1 v is u.s.c. at y_0 hence $f(x_0, y_0)=\lim f(x_k, y_k)=\lim v(y_k)\leqq v(y_0)$.

See f. i. [27] theorem 8. For a similar statement, involving sequences of functions, see theorem 1.2.2 of [14].

But it is again variational convergence that provides the simplest structure on the data guaranteeing closedness of M.

Theorem 4.3.2. *Let X be a convergence space, f_n: $X\to(-\infty, +\infty]$, $n\geqq 0$. Suppose that $f_n\xrightarrow{v}f_0$ (see* (3.1) *and* (3.2)). *Then M is closed.*

Proof. See [43] theorem 1, or [23].

Observe that the theorem holds a fortiori for every more restrictive convergence than the variational one as for instance epi or Mosco convergence. It is not difficult to see that theorem 4.3.1 and other similar statements can be viewed as a particular case of theorem 4.3.2; see [43] for a discussion and further references. But even u.s.c. results can be captured by means of theorem 4.3.2, if we have enough compactness on the constraint set multifunction. This is done using proposition 2.8. Theorem 4.1.1 for instance, in the first formulation given by Berge, is an example of how this device can be used. Suppose now to strengthen condition (3.1) by means of the additional requirement that the sequence x_n actually converges to x; we switch then from variational to epiconvergence. Last one, though too restrictive for our variotional purposes, is however more attractive from a topological point of view. Let us see now how from theorem 4.1.1 we can indeed get the standard properties of epiconvergence (and of variational convergence, stated in proposition 3.3.2 and theorem 4.3.2) about the function v and the multifunction M. Therefore, suppose we are given a sequence of extended real valued functions g_n epiconverging to a function g_0. Define the function $f: X\times R\to R$ as $f(x, z)=z$ and consider the problem of minimizing f on $Ky_n=\mathrm{epi}\, g_n$. As already remarked epiconvergence of g_n to g_0 is equivalent to say that K is l.s.c. and closed at y_0 (along the sequence y_n). Then from theorem 4.3.1 we are able to get u.s.c. of v and the inclusion $\mathrm{ls}\, My_n\subset My_0$. Now it is easy to see that $v_n=\inf g_n$ and $M_n=\{(\mathrm{MIN}\, g_n,\ \inf g_n)\}n\geqq 0$. Hence from theorem 4.3.1 we get $\limsup \inf g_n\leqq \inf g_0$. Moreover, to see closedness property, suppose there is $x\in \mathrm{ls}\,\mathrm{MIN}\, g_n$. Namely there are x_k and n_k, selection from the integers, such that $g_{n_k}(x_k)=v_{n_k}$ and $x_k\to x$. But $(x_k, v_0+\varepsilon)\in \mathrm{epi}\, g_{n_k}$, at leats for large k, hence $(x, v_0+\varepsilon)\in \mathrm{epi}\, g_0$, $\forall\varepsilon<0$. Then $g(x)=v_0=\inf g_0$ showing that $\mathrm{ls}\,\mathrm{MIN} g_n\subset \mathrm{MIN} g_0$. Still, by (3.2) we get also $\lim v_{n_k}=v_0$. The former result shows also that, if g_0 is lower bounded but it does not have minima, then either g_n do not eventually have minima, or every sequence of minima does not have any converging subsequence. Summarizing, in some sense we have closed a circle: from the "topological" version given by the theorem 4.3.1 we generalize to variational convergence, that is often replaced by the more attractive, in some sense, epiconvergence. But standard variational properties of epiconvergence can be easily proved by means of theorem 4.3.1. It is clear, at this point, that the two approaches are quite similar in their essence; theorems dealing with u.s.c., on the other hand, can be viewed as giving different results.

4.4 Necessary conditions.

Theorem 4.4.1. *Let X be a metric space. If for every $f: X\to R$ which is continuous M is closed at the point y_0, then for every $y_n\to y_0$ or $\mathrm{li}\, Ky_n=\emptyset$ or K is closed and l.s.c. at y_0.*

Proof. See [14] theorem 1.2.1.

Theorem 4.2.2. *Le X be a metric space and $Ky\neq\emptyset\,\forall y$. If for every $f: X\to R$ which is continuous M is u.s.c. at a point y_0, then K is u. and l.s.c. at y_0.*

Proof. See [6] theorems 2—6.

Next theorem has a different goal: it can be considered as related to theorem 4.2.5.

Theorem 4.4.3. *Let X be a reflexive Banach space and let f be strictly convex and continuous function with bounded level sets. If K is convex valued and continuous in Hausdorff's sense (see* (1.6) *and* (1.7)) *then f is w.p. on every closed convex subset of X.*

5. Some applications. This section is devoted to an application of the previous results, having in mind the paper [24]: the generalization here given, as already mentioned, can be justified by that fact that it is simple, hopefully elegant, and mainly it shows the relevant properties of the ordering for getting the desired results.

We shall work with the following set of assumptions: the parameter space Y satisfies the first numerability axiom. Moreover, it is defined on it a partial order $>$ (namely a relation with the properties of irreflexivity: $a\not> a$, antisymmetry: $a>b\Rightarrow b\not> a$, and transitivity $a>b,\ b>c\Rightarrow a>c$). Moreover, for every $y\in Y$ the sets $\{z\in Y:\ z>y\}$ and $\{z\in Y:\ z<y\}$ are nonempty open sets. Finally, for every $V(y)$ there are $y_1,\ y_2\in V$ such that $y_1<y<y_2$. Let Z be a topological space and let X be a normed space where a bounded set is relatively compact with respect to some topology t, not necessarily related to the norm. $C: Y\rightrightarrows Z$ is a nonempty valued closed and monotone multifunction. Here monotone means that $y_1>y_2$ implies $Cy_1\supset Cy_2$. Let $g: (X, t)\to Z$ be a continuous function. Define $K: Y\rightrightarrows X$ as $Ky=\{x\in X$ such that $g(x)\in Cy\}$ and observe that K is closed valued. The following propositions characterize continuities of K.

Proposition 5.1. *K is u.s.c. at a point y_0 if and only if there $y>y_0$ such that $Ky\setminus Ky_0$ is a bounded set.*

Proof. Necessity condition is easily seen taking into account corollary 2.3 and the existence, for any $V(y_0)$, of an element $y_1 \in V$ such that $y_1 > y_0$. To show the opposite implication suppose that there exist a closed set F and a sequence $y_n \to y_0$ such that $Ky_n \cap F \neq \emptyset$, $Ky_0 \cap F = \emptyset$. Let $x_n \in Ky_n \cap F$, then $x_n \in (Ky_n \setminus Ky_0) \cap F$ and as $y_n \to y_0$ then $y_n < \bar{y}$ for all large n: this implies boundedness of x_n. Then x_n has a cluster point, say x_0. By continuity of g, $g(x_n) \to g(x_0)$, by closedness of C at y_0, as $g(x_n) \in Cy_n$, then $g(x_0) \in Cy_0$. Then $x_0 \in Ky_0 \cap F$. The contradiction ends the proof.

Proposition 5.2. *$I(y) = \{x \in X\colon g(x) \in Cu$ for some $u < y\}$. Then is l.s.c. at a point y_0 if and only if $Ky_0 = \overline{Iy_0}$.*

Proof. At first observe that $Ky_0 \supset \overline{Iy_0}$ for g is continuous, C is monotone and Cy_0 is a closed set. Now suppose there exists $\hat{x} \in Ky_0 \setminus \overline{Iy_0}$. Then there is $V(\hat{x})$ such that $\forall x \in V g(x) \notin$ Cu for any $u < y_0$, namely there is V such that $\forall x \in V x \notin Ku$ for any $u < y_0$. But the open set $\{u\colon u < y_0\}$ meets every neighborhood of y_0, and this is in contradiction with l.s.c. of K. Conversely let A be an open set such that $A \cap Ky_0 \neq \emptyset$ and let $x_0 \in A \cap Ky_0$. Then there is a sequence $x_k \in Iy_0$ such that $x_k \to x_0$, hence $x_{\hat{k}} \in A$ (for a certain $\hat{k}$) and $g(\hat{x}) \in C\hat{u}$ for some $\hat{u} < y_0$. Suppose there is $y_n \to y_0$ such that $A \cap Ky_n = \emptyset$. But $Cy_n \supset C\hat{u}$ eventually, then $g(x_{\hat{k}}) \in Cy_n \Rightarrow x_{\hat{k}} \in A \cap Ky_n$.

Remark 5.3. Suppose that Y is a normed linear space and R a closed convex acute cone in Y, with nonempty interior. If we define $y_1 > y_2$ if and only if $y_1 - y_2 \in \mathring{R}$, this partial order fulfills all the requested properties. The case $X = \mathbb{R}^n$, $Z = \mathbb{R}^k$, $Y = \mathbb{R}^k$ equipped with the cone $R = \mathbb{R}^k_+$, $Cy = y + \mathbb{R}^k_-$, generalizes theorems 1 and 2 in [24]. Suppose to have now a continuous function $f\colon X \to \mathbb{R}$ to be minimized on the set Ky. Then we have:

Theorem 5.4. *Suppose that $\overline{Iy_0} = Ky_0$ and there is $y_1 > y_0$ such that $Ky_1 \setminus Ky_0$ is a bounded set. Then v is continuous at y_0. If moreover Ky_0 is a compact set, then M is u.s.c. at y_0.*

Proof. Apply theorems 3.1.5 and 4.1.1. Observe that in the compact case continuity of v and u. s. c. of M still hold if f does depend on the parameter y.

Remark 5.5. Other results in the same spirit can be found in [5, 13, 26—28]. Also theorems 4.2.1 and 4.2.2 can provide further generalizations. A slight different approach, with appropriate perturbations on the function g, can be found in [36] and [43]. We observe explicitly that the previous approach cannot treat equality constraints, splitting them in two inequalities.

We end the section giving an example of applicability of theorem 5.3.

Suppose $X = \mathbb{R}^k$ and that g satisfies the following condition: $\forall y_1 < y_2$, $\forall x_1, x_2$ such that $g(x_1) \in Cy_1$, $g(x_2) \in Cy_2$, $\forall z \in (x_1, x_2)$ there is y_z such that $y_1 < y_z < y_2$ and

$$g(z) \in Cy_z. \tag{5.1}$$

Theorem 5.6. *Let $Iy_0 \neq \emptyset$ and Ky_0 a compact set. Then v is continuous and M is u.s.c. at y_0.*

Proof. Let $x_0 \in Iy_0$ and suppose there exists $x_1 \in Ky_0 \setminus \overline{Iy_0}$. Then $g(x_0) \in Cy_1$ for some $y_1 < y_0$ and $g(x_1) \in Cy_0$. Hence by condition (5.1) the segment $[x_0, x_1) \in Iy_0$ and this leads to a contradiction. Then K is l.s.c. at y_0 by proposition 6.2. Suppose now that for evey $y > y_0 Ky \setminus Kv_0$ is an unbounded set. Let S be a ball big enough to contain Ky_0 in its interior and let $x_0 \in Ky_0$. For $y_n \to y_0$, $y_n > y_0$ there is $x_n \in Ky_n$: $|x_n| \to +\infty$. Call $t_n = [x_0, x_n] \cap \partial S$. Then $g(t_n) \in Cy_n$ (by (5.1)). Moreover t_n has a cluster point $t_0 \in \partial S$. Then $g(t_n) \to g(t_0)$, $g(t_n) \in Cy_n$ and, by closedness of C, $g(t_0) \in Cy_0$, namely $t_0 \in Ky_0 \cap \partial S$. The contradiction ends the proof.

Remark 5.7. Condition (5.1) amounts to strict quasi convexity of g (componentwise) in the simplest case mentioned in remark 5.3. Observe that (5.1) forces l.s.c. of C (at these points belonging to the image of g).

6. Approximate solution sets. We state in this final section some results on the approximate solutions of the problems Py, related to the so-called asymptotically minimizing sequences. We define, for every $\varepsilon>0$

(6.1) $$M_\varepsilon(y)=\{x\in Ky: f(x, y)\leqq v(y)+\varepsilon\} \text{ if } v(y)>-\infty$$

(6.2) $$M_\varepsilon(y)=\{x\in Ky: f(x, y)\leqq -\varepsilon^{-1}\} \text{ if } v(y)=-\infty.$$

Observe that we have, for every y, $My=\bigcap_\varepsilon M_\varepsilon y$.

Theorem 6.1. *In the same assumptions of theorem* 4.1.1 M_ε *is u.s.c.* $\forall\varepsilon>0$.

Proof. As theorem 4.1.1.

Next proposition shows how variational convergence behaves with respect to approximate solutions sets.

Proposition 6.2. *In the same assumptions of theorem 4.3.2, then* M_ε *is closed* $\forall\varepsilon\geqq0$.

Proof. See [19, 34, 43].

Remark 6.3. Result of proposition 6.2 holds, a fortiori, for epiconvergence or Mosco convergence. However, rather surprisingly, it does not characterize variational convergence as the weakest one fulfilling that closedness property, see [34].

From now on let us suppose that Y satisfies the first numerability axiom; moreover, let us call asymptotically minimizing (for the problem Py_0) those sequences x_n such that $x_n\in Ky_n$, for some $y_n\to y_0$, moreover $v(y_n)>-\infty$ and $f(x_n, y_n)-v(y_n)\to0$. At first observe that proposition 6.2 has two immediate consequences. The first one is that, for $y_n\to y_0$, $My_0\supset\bigcap_{\varepsilon>0}\operatorname{ls} M_\varepsilon y_n$ and the second one is that if x_n is an asymptotically minimizing sequence for Py_0 such that $x_n\to x_0$ (along a subsequence), then $x_0\in M$.

Proposition 6.4. *Let K be u. and l.s.c. at a point y_0 and compact valued. Let $f: X\times Y\to\mathbb{R}$ be continuous at every point of the set Ky_0xy_0. Then there are sequences x_n asymptotically minimizing for Py_0, every such sequence has cluster point, every cluster point belongs to My_0.*

Proof. $\bigcup_n x_n\subset\bigcup_n Ky_n$ which is a compact set by proposition 2.9. Let $x_0=\lim x_n$ (for a subsequence). As $f(x_n, y_n)-v(y_n)\to0$, $\forall\varepsilon>0\exists\bar n: \forall n>\bar n x_n\in M_\varepsilon y_n$; then $x_0\in M_\varepsilon y_0$ (by theorem (6.1) $\forall\varepsilon>0$. This concludes the proof.

Remark 6.5. If K is not compact valued on a neighborhood of y_0, then the statement of proposition 6.3 can fail, even if Ky_0 is a compact set. Consider $f(x, y)=0$, $Ky=S(0, y)$ on an infinite dimensional Hilbert space.

Theorem 6.6. *Let $K: Y\rightrightarrows X$ be l.s.c. at a point y_0. Let $f: X\times Y\to\mathbb{R}$ be continuous at every point of the set Ky_0xy_0. Let $My_0\neq\emptyset$. If for every $y_n\to y_0$ we have $\bigcap_{\varepsilon>0}\operatorname{li} M_\varepsilon y_n=My_0$, then v is continuous at y_0.*

Proof. By proposition 3.1.1 $\limsup v(y_n)\leqq v(y_0)$. Let us show that $\liminf v(y_n)\geqq v(y_0)$. Let $x\in My_0$, then $\forall\varepsilon>0\exists x_n^\varepsilon\in M_\varepsilon(y_n)$ such that $x_n^\varepsilon\to x$. Then $v(y_0)=f(x, y_0)=\lim f(x_n^\varepsilon, y_n)\leqq\liminf v(y_n)+\varepsilon\forall\varepsilon>0$.

Observe that the assumption $My_0\neq\emptyset$ cannot be omitted.

Theorem 6.7. *Let $K: Y\rightrightarrows X$ be l.s.c. and closed at a point y_0. Let $f: X\times Y\to\mathbb{R}$ be continuous at every point of the set Ky_0xy_0. If v is continuous at y_0, then $My_0=\bigcap_{\varepsilon>0}\operatorname{li} M_\varepsilon y_n$.*

Proof. It is easy to see that $My_0\supset\bigcap_{\varepsilon>0}\operatorname{li} M_\varepsilon y_n$. Let us show the opposite inclusion. Suppose $My_0\neq\emptyset$, the only relevant case. Suppose that $\bar x\in My_0$ and there is $\varepsilon>0$ such that $\bar x\notin\operatorname{li} M_\varepsilon y_n$. By l.s.c. of K there is $x_n\to\bar x$ such that $x_n\in Ky_n$. So for infinite indexes $jx_j\notin M_\varepsilon y_j$. This means that $f(x_j, y_j)>y(y_j)+\varepsilon_0$. Then $v(y_0)=\lim f(x_j, y_j)\geqq\liminf v(y_j)+\varepsilon_0=v(y_0)+\varepsilon_0$.

Remark 6.8. The condition $M_0 \supset \bigcap_{\varepsilon>0} \mathrm{li}\, M_\varepsilon y_n$ means that every limit of an asymptotically minimizing sequence is indeed a solution of the problem Py_0. The opposite inclusion, on the other side, means that every solution of the unperturbed problem can be approximated by means of an asymptotically minimizing sequence.

Theorem 6.9. *Let X be a metric space and let K: $Y \rightrightarrows X$ be u. and l.s.c. at a point y_0. Let f: $X \times Y \to \mathbb{R}$ be continuous at the set $Ky_0 x y_0$ and let $My_0 \neq \emptyset$. Then v is continuous at y_0 if and only if, for every $y_n \to y_0$, $My_0 = \bigcap_{\varepsilon>0} \mathrm{li}\, M_\varepsilon y_n$.*

Proof. Immediate from theorems 6.7 and 6.9.

This result generalized theorems 2 and 3 of [2].

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REFERENCES

1. H. Attouch. Variational convergence for functions and operators. Boston, 1984.
2. H. Attouch, R. Wets. Approximation and convergence in nonlinear optimization. — In: Nonlinear Programming. (O. Mangasarian, R. Meyer, S. Robinson eds.), 1981.
3. J. P. Aubin. Mathematical methods of game and economic theory. Amsterdam, 1979.
4. D. Aze, J. P. Penot. Operations on convergent sets and functions (to appear).
5. B. Banks et al. Nonlinear parametric optimization. Basel, 1983.
6. E. Bednarczuk. On upper semicontinuity of global minima in constrained optimization problems. *J. Math. Anal. Appl.*, **86**, 1982.
7. E. Bednarczuk. Characterization of semicontinuity of solutions to abstract optimization problems *Numer. Funct. Anal. Optimiz.*, **9**, 1987.
8. G. Beer, P. Kenderov. On the argmin multifunction for lower semicontinuous functions. *Proc. Amer. Math. Soc.*, **102**, 1988, 107—113.
9. V. I. Berdysev. Stability of a minimization problem under perturbation of the set of admissible elements. *Math. U. S. S. R. Sbornik*, **32**, 1977.
10. V. I. Berdysev. Continuity of a multivalued mapping connected with the problem of minimizing a functional. *Mdth. U. S. S. R. Izvestija*, **16**, 1981.
11. C. Betge. Topological spaces. New York, 1963.
12. E. Cavazzuti, N. Pacchiarotti. Condizioni sufficienti per la dipendenza continua dei minimi da un parametro. *Boll. Unione mat. Ital.*, **13B**, 1976, 261—279.
13. R. Colgen. On parametric optimization in Hilbert spaces. *Numer. Func. Anal. Optimiz.*, **4**, 1981-82.
14. G. Dantzig, J. Folkman, N. Shapiro. On the continuity of the minimum set of continuous functions. *J. Math. Anal. Appl.*, **17**, 1967.
15. E. De Giorgi. Convergence problems for functionals and operators. — In: Proc. Internal. Meeting on Recent Methods in Nonlinear Analysis, Roma, 1978.
16. S. P. Delahye, J. Denel. The continuities of the point to set maps; definitions and equivalences. *Math. Progr. Study*, **10**, 1979.
17. I. Del Prete, M. B. Lignola. On the variational properties of $\Gamma(d)$-convergence. *Ric. Mat.*, **31**, 1983.
18. S. Dolecki. Lower semicontinuity of marginal functions. — In: Proc. Symp. on Operations Research, 1983.
19. S. Dolecki. Convergence of minima in convergence spaces. *Optimization*, **17**, 1986.
20. S. Dolecki, A. Lechicki. On structure of upper semicontinuity. *J. Math. Anal. Appl.*, **88**, 1982.
21. S. Dolecki, S. Rolewicz. A characterization of semicontinuity-preserving multifunctions. *J. Math. Anal. Appl.*, **65**, 1978.
22. S. Dolecki, S. Rolewicz. Metric characterizations of upper semicontinuity. *J. Math. Anal. Appl.*, **69**, 1979.
23. A. Dontchev, T. Zolezzi (Book in preparation).
24. J. P. Evans, F. G. Gould. Stability in nonlinear programming. *Oper. Research*, **18**, 1970, 107—118.
25. M. Furi, M. Martelli, A. Vignoli. On minimum problems for families of functionals. *Annali Mat.*, **86**, 1970, 181—188.

26. M. G. Greenberg, W. P. Pierskalla. Extensions of the Evans-Gould stability theorems for mathematical programs. *Oper. Research*, **20**, 1972, 143—153.
27. W. Hogan. Point-to-set maps in mathematical programming. *SIAM Review*, **15**, 1973, 591—603.
28. A. Kirsh. Continuous perturbations of infinite optimization problems. *J. Optim. Theory Appl.*, 32, 1980, 171—182.
29. E. Klein, A. C. Thompson. Theory of correspondences. New York, 1984.
30. J. K. Kuratowski. Topology. New York, 1966.
31. F. J. Laurent. Approximation et Optimization. Paris, 1972.
32. R. Lucchetti. On the continuity of the optimal value and of the optimal set in minimum problems. (Pubblicazioni IMA 133), Genova, 1983.
33. R. Lucchetti. On the continuity of the minima for a family of constrained optimization problems. *Numer. Funct. Anal. Optimiz.*, **7**, 1984, 349—362.
34. R. Lucchetti, C. Malivert. Variational convergences and level sets multifunctions (to appear).
35. R. Luccetti, F. Patrone. Hadamard and Tyhonov well posedness of a certain class of convex functions. *J. Math. Anal.*, **88**, 1982.
36. R. Lucchetti, F. Patrone. Closure and upper semicontinuity results in mathematical programming, Nash and economic equilibria. *Optimization*, **17**, 1986.
37. U. Mosco. Convergence of convex sets and of solutions of variational inequalities. *Adv. in Math.*, **3**, 1969.
38. J. P. Penot. Continuity properties of performance functions. — In: Optimization Theory and Algorithms. New York, 1983.
39. A. Tyhonov. On the stability of the functional optimization problem. *Soviet Math. Doklady*, **6**, 1966.
40. R. Wets. On the continuity of the value of a linear program and of related polyhedral valued multifunctions. *Math. Progr. Study*, **24**, 1985, 14—29.
41. T. Zolezzi. On convergence of minima. *Boll. Unione mat. Ital.*, **8**, 1973.
42. T. Zolezzi. Problem di perturbazione in programmazione nonlineare. — In: Proc. Studio di problemi limite dell'analisi functionale, 1982.
43. T. Zolezzi. On stability analysis in mathematical programming. *Math. Progr. Study*, **21**, 1984, 227—242.

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