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DETECTING THE SIGNAL APPEARING TIME

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In a white Gaussian noise a signal appears at a random time θ whose distribution function is unknown. The linear minimax estimator of θ and the linear minimax risk have been found and investigated.

1. Formulation of the problem. Let $\eta(t)$ be a random process defined on $[0, T]$ by

$$d\eta(t)=\theta(t, \omega)\,dt + \varepsilon dW(t),$$

$$\eta(0)=0. \tag{1.1}$$

Here
$$\theta(t, \omega)=I\{\theta(\omega)\leq t\}(\omega)=\begin{cases}1, & \text{if } t\geq\theta(\omega),\\ 0, & \text{if } t<\theta(\omega)\end{cases}$$

and $\theta(\omega)$ is a random variable which takes values in $[0, T]$. The process $W(t)$ is a standard Wiener one and it is independent of θ, ε is a given positive constant.

Suppose we don't know the distribution function of θ. The problem is to estimate the moment $\theta(\omega)$ by using the trajectory $\eta(t)$, $0\leq t\leq T$.

This model describes the following real situation. In a white Gaussian noise with an intensity ε^2 a unit signal appears at a random moment. The additive sum (noise+signal) is integrated by a linear filter and the result obtained is observed. Using the observation (a continuous trajectory $\eta(t, \omega)$, $0\leq t\leq T$), we have to estimate the moment when the signal appears.

A great number of papers deal with problems close to the one formulated here (see, e. g., [1]—[3] and [4], Chapter 7) and they differ in the information about θ, given a priori. It seems quite natural to apply the minimax approach in our case. We are interested in the class M of all linear estimators of the form

$$\hat{\theta}=\int_0^T l(t)d\eta(t)-\alpha, \tag{1.2}$$

where the weght function $l(t)$ belongs to the Hilbert space $L_2[0, T]$ and α is a real constant.

Definition 1. *The linear quadratic minimax risk is defined by*

$$\Delta^2=\inf_{\hat{\theta}\in M}\sup_{\theta} E|\hat{\theta}-\theta|^2. \tag{1.3}$$

Here sup *is taken over all the random variables* θ *with values in* $[0, T]$.

Definition 2. *The linear estimator* θ^* *is called minimax in* M, *if for every linear estimator* $\hat{\theta}$

$$\sup_{\theta} E|\hat{\theta}-\theta|^2\geq\sup_{\theta} E|\theta^*-\theta|^2$$

holds, i. e. θ^* *reaches the infimum in* (1.3)

In the present paper our purpose is to find θ^* and Δ^2 and to investigate their properties. The results obtained here are given in [5].

2. Main results

Theorem 2.1. *The linear minimax estimator and the linear minimax risk in problem* (1.1) *are respectively*

$$\theta^* = -\frac{T}{T+4\varepsilon^2}\eta(T) + \frac{T^2+2T\varepsilon^2}{T+4\varepsilon^2}, \quad \Delta^2 = \frac{\varepsilon^2 T^2}{T+4\varepsilon^2}, \tag{2.1}$$

where $\eta(t)$ is the random process defined by (1.1).

Theorem 2.2. *Let the finite moment T be fixed and let the intensity of the noise ε^2 tend to zero. Then the linear minimax estimator $\theta^*=\theta^*(\varepsilon,T)$ is strong consistent and asymptotically unbiased. If $T\to\infty$ and $\varepsilon=T^{-a}$, $a>1/2$, then $\Delta^2(\varepsilon, T)_{T\to\infty}\to 0$ and θ^* is strong consistent and asymptotically unbiased, too.*

Theorem 2.1 shows, that the linear minimax estimator depends only on the final value $\eta(T)$. This fact seems quite natural. Indeed it is clear from (1.1) that the "weights" of the trajectory $\eta(t)$ observed, which are given by the weight function of the "best" linear estimator of θ, should be one and the same at every moment t, $0\le t<\theta$. That is valid for $\theta\le t\le T$ as well. Since the point of signal appearing is in general supposed to be a random one, one can expect that the weight function would be constant all over the period of observation. The Lemma in Section 3 confirms this assumption and then θ^* depends only on the final value $\eta(T)$.

Note that if $\varepsilon=0$ (no noise), then it follows from (2.1) that $\theta^*=\theta$. Theorem 2.2 shows that if $\varepsilon\to 0$ then $\theta^*\to\theta$ a. s. and $E\theta^*\to E\theta$. Meanwhile the finite moment is either fixed or it increases as follows: $T=\varepsilon^{-1/a}$, $a>1/2$. In both cases Δ^2 tends to zero.

3. One subsidiary lemma. Let $\widehat{\theta}$ be an estimator of the form (1.2). Denote by $F_\theta(t)$ the (unknown) distribution function of the random variable θ. After some calculations we get the following formula for the quadratic distance between $\widehat{\theta}$ and θ:

$$E|\widehat{\theta}-\theta|^2 = \int_{0-}^{T}\Big(\int_t^T l(s)ds - t - \alpha\Big)^2 dF_\theta(t) + \varepsilon^2\|l\|^2,$$

where $\|\cdot\|$ denotes the norm in the space $L_2[0, T]$. Then (see 1.3)

$$\Delta^2 = \inf_{\widehat{\theta}\in M}\sup_{\theta}\Big\{\int_{-0}^{T}\Big(\int_t^T l(s)ds - t-\alpha\Big)^2 dF_\theta(t)+\varepsilon^2\|l\|^2\Big\},$$

$$\Delta^2 = \inf_{l,\alpha}\sup_{\theta}\Big\{\int_{-0}^{T}\Big(\int_t^T l(s)ds - t-\alpha\Big)^2 dF_\theta(t)+\varepsilon^2\|l\|^2\Big\}. \tag{3.1}$$

On the right hand side of (3.1) sup is taken over all the possible distribution functions which concentrate the unit mass on the interval $[0, T]$ and inf is taken over all the functions $l(t)$, $l\in L_2[0, T]$ and over all the constans α, $\alpha\in R$.

Denote for brevity $f(t; l, \alpha)=\int_t^T l(s)ds-t-\alpha$ and let $t=t(l, \alpha)$ be any point of the maximum value of f^2 on $[0, T]$. Then it is clear that

$$\sup_F \int_{0-}^{T} [f(t; l, \alpha)]^2 dF(t) = [f(t(l, \alpha); l, \alpha)]^2$$

and this sup is reached by the distribution function

$$F(t)=\begin{cases}1, & \text{if } t \geq t(l, \alpha),\\ 0, & \text{if } t < t(l, \alpha).\end{cases}$$

Now from (3.1) we get

$$\Delta^2=\inf_{l,\,\alpha} \{[f(t(l, \alpha);\ l, \alpha]^2+\varepsilon^2 \| l \|^2\}. \tag{3.2}$$

L e m m a. *The solution of the problem*

$$[f(t(l, \alpha);\ l, \alpha)]^2+\varepsilon^2 \| l \|^2 \to \text{infimum},\ l \in L_2[0, T],\quad \alpha \in R, \tag{3.3}$$

is given by

$$l^*=-\frac{T}{T+4\varepsilon^2},\quad \alpha^*=-\frac{T^2+2T\varepsilon^2}{T+4\varepsilon^2}$$

and the extremum value in (3.3) *is* $\varepsilon^2 T^2/(T+4\varepsilon^2)$.

P r o o f. Evidently $[f(t(l, \alpha); l, \alpha)]^2=[\max_{0\leq t\leq T} | \int_t^T l(s)ds-t-\alpha |]^2$. We have

$$\max_{0\leq t\leq T} | \int_t^T l(s)ds-t-\alpha | = \max \{ \max_{0\leq t\leq T} (\int_t^T l(s)ds-t)-\alpha,$$

$$\alpha-\min_{0\leq t\leq T} (\int_t^T l(s)ds-t)\} = \max \{ h_1(l, \alpha),\ h_2(l, \alpha)\},$$

where

$$\begin{aligned} h_1(l, \alpha) &= \max_{0\leq t\leq T} (\int_t^T l(s)ds-t)-\alpha,\\ h_2(l, \alpha) &= \alpha-\min_{0\leq t\leq T} (\int_t^T l(s)ds-t). \end{aligned} \tag{3.4}$$

When $l(t)$ is fixed, then

$$h_1+h_2=\max_{0\leq t\leq T} (\int_t^T l(s)ds-t)-\min_{0\leq t\leq T} (\int_t^T l(s)ds-t) = \text{constant}=C.$$

Hence $\max(h_1, h_2)=\max(h_1,\ C-h_1)\geq C/2$ and the equality is reached when $h_1=h_2=C/2$. In this case from (3.4) we get

$$\alpha(l)=\frac{1}{2}\{ \max_{0\leq t\leq T} (\int_t^T l(s)ds-t)+\min_{0\leq t\leq T} (\int_t^T l(s)ds-t)\}, \tag{3.5}$$

$$h_2(l, \alpha(l))=h_1(l, \alpha(l))=\max_{0\leq t\leq T} (\int_t^T l(s)ds-t)-\alpha(l) \tag{3.6}$$

$$=\frac{1}{2}\{ \max_{0\leq t\leq T} (\int_t^T l(s)ds-t)-\min_{0\leq t\leq T} (\int_t^T l(s)ds-t)\}.$$

So we obtained that for every pair (l, α)

$$[f(t(l, \alpha); l, \alpha]^2 = \{\max_{0 \leq t \leq T} |\int_t^T l(s)ds - t - \alpha|\}^2 \geq [\max_{0 \leq t \leq T} |\int_t^T l(s)ds - t - \alpha(l)|]^2$$
$$= [f(t(l, \alpha(l)); l, \alpha(l)]^2 = [h_1(l, \alpha(l)]^2.$$

Hence

$$(3.7) \qquad \inf_{l, \alpha} \{[f(t(l, \alpha); l, \alpha]^2 + \varepsilon^2 \|l\|^2\} = \inf_l \{[f(t(l, \alpha(l)); l, \alpha(l)]^2 + \varepsilon^2 \|l\|^2\}$$
$$= \inf_l \{[h_1(l, \alpha(l))]^2 + \varepsilon^2 \|l\|^2\}.$$

Taking into consideration (3.7) and the expression for $h_1(l, \alpha(l))$ in (3.6) we get that the problem (3.3) is equivalent to the following problem:

$$(3.8) \quad \frac{1}{4}[\max_{0 \leq t \leq T} (\int_t^T l(s)ds - t) - \min_{0 \leq t \leq T} (\int_t^T l(s)ds - t)]^2 + \varepsilon^2 \|l\|^2 \to \text{infimum}, \ l \in L_2[0, T].$$

Denote by Φ the functional which is to be minimized in (3.8), and by L_β the hyper-plane $L_\beta = \{l: l \in \mathbf{L}_2[0, T], \int_0^T l(t)dt = \beta\}$, $\beta \in R$. Then

$$(3.9) \qquad \inf_{l \in L_2[0, T]} \Phi(l) = \inf_{\beta \in R} \inf_{l \in L_\beta} \Phi(l).$$

Let $l \in L_\beta$. We have

$$\Phi(l) = \frac{1}{4}[\max_{0 \leq t \leq T} (\beta - \int_0^t l(s)ds - t) - \min (\beta - \int_0^t l(s)ds - t)]^2 + \varepsilon^2 \|l\|^2$$

$$(3.10) \qquad = \frac{1}{4}[\max_{0 \leq t \leq T} (\int_0^t l(s)ds + t) - \min_{0 \leq t \leq T} (\int_0^t l(s)ds + t)]^2 + \varepsilon^2 \|l\|^2.$$

We state that the solution of the problem $\Phi(l) \to$ infimum, $l \in L_\beta$, is given by

$$(3.11) \qquad l_\beta(t) = \beta/T.$$

In fact if $l \in L_\beta$, then $\|l\|^2 \geq T^{-1}\beta^2 = \|l_\beta\|^2$. Using (3.10), we get

$$\Phi(l) \geq \frac{1}{4}[(\int_0^t l(s)ds + t)|_{t=0}^{t=T}]^2 + \varepsilon^2 \|l_\beta\|^2 = \frac{(\beta + T)^2}{4} + \frac{\varepsilon^2\beta^2}{T}.$$

One can easily verify that the equality in the above inequality is reached when $l(t) = l_\beta(t)$. Thus we have

$$\inf_{l \in L_\beta} \Phi(l) = \Phi(l_\beta) = \frac{(\beta+T)^2}{4} + \frac{\varepsilon^2\beta^2}{T}.$$

Further it is easy to get

$$\inf_{\beta \in R} \{\frac{(\beta+T)^2}{4} + \frac{\varepsilon^2\beta^2}{T}\} = \frac{\varepsilon^2 T^2}{T + 4\varepsilon^2}$$

and inf is established by $\beta^* = -T^2/(T+4\varepsilon^2)$. Hence (see (3.9) and (3.11).)

$$\inf_{l \in L_2[0, T]} \Phi(l) = \frac{\varepsilon^2 T^2}{T + 4\varepsilon^2} = \Phi(\frac{\beta^*}{T}) = \Phi(-\frac{T}{T + 4\varepsilon^2}) = \Phi(l^*).$$

Thus l^* solves (3.8) and consequently it solves (3.3) as well (see (3.7)). To complete the proof we have to calculate $\alpha(l^*)$. Using (3.5), we get $\alpha(l^*)=\alpha^*$. The Lemma is proved.

4. Proofs of the theorems. According to (3.2), Δ^2 equals to the minimum value obtained when solving the problem (3.3). This value appeared to be $\varepsilon^2 T(T+4\varepsilon^2)^{-1}$. Since pair (l^*, α^*) reaches this minimum value, then by the definition of θ^* and from the Lemma we get

$$\theta^* = \int_0^T l^*(t)d\eta(t) - \alpha^* = -\frac{T}{T+4\varepsilon^2}\eta(T) + \frac{T^2+2T\varepsilon^2}{T+4\varepsilon^2}.$$

Thus Theorem 2.1 is proved.

One can find that for every random variable θ

$$E|\theta^*-\theta|^2 = \frac{16\varepsilon^4}{(T+4\varepsilon^2)^2}\int_0^T (t^2-tT)dF_\theta(t) + \Delta^2,$$

where $F_\theta(t)$ is the distribution function of θ. Then the inequality $E|\theta^*-\theta|^2 \leqq \Delta^2$ becomes an equality if and only if $P\{\theta=0\}=1$ or $P\{\theta=T\}=1$, so that these two boundary cases appear to be the worst ones in process of the linear minimax estimation.

Turn to theorem 2.2. Since $\eta(T)=T-\theta+\varepsilon W(T)$ we get for θ^*

$$\theta^* = \frac{2T\varepsilon^2}{T+4\varepsilon^2} - \frac{T}{T+4\varepsilon^2}\theta - \frac{\varepsilon T}{T+4\varepsilon^2}W(T), \tag{4.1}$$

$$E\theta^* = \frac{T}{T+4\varepsilon^2}(E\theta+2\varepsilon^2).$$

When $\varepsilon\to 0$ (T is fixed), from (4.1), we get $\theta^*\to\theta$ a. s., $E\theta^*\to E\theta$. Let $T\to\infty$, $\varepsilon=T^{-a}$ $a>1/2$. Then

$$\Delta^2 = \frac{T^2}{T^{1+2a}} \underset{T\to\infty}{\longrightarrow} 0$$

and again from (4.1) we conclude $\theta^*\to\theta$ a. s., $E\theta^*\to E\theta$. So theorem 2.2 is proved

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