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A FORMULA FOR THE K-th COVARIANT DERIVATIVE

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The aim of the present paper is to give a formula for the k-th covariant derivative of tensor field along a given curve. In order to do that, first the symbols $P_j^{i[k]}$ and $Q_j^{i[k]}$ which depend on the Christoffel symbols are introduced. Some properties of them are also given. The main result is given by (3.1) and further it is generalized for $k \in R$.

1. Introduction of the symbols $P_j^{i[k]}$ and $Q_j^{i[k]}$. Let $x(s)$ be a given curve in a differentiable manifold M_n which is endowed with affine connection Γ. The k-th covariant derivative $\overset{k}{\nabla}_{\dot{x}}$ along the curve $x(s)$ will be denoted by ∇^k. We shall deal further in a local coordinate system.

The next three equalities $P_j^{i[0]}=\delta_j^i$, $p_j^{i[1]}=\Gamma_{jl}^i\,(dx^l/ds)$,

$$p_j^{i[k+1]}=dP_j^{i[k]}/ds+P_j^{l[k]}P_l^{i[1]}, \quad (k\in\{0,\ 1,\ 2, \dots\}), \tag{1.1}$$

determine all symbols $p_j^{i[k]}$ $(i,\ jl\in\{1, \dots,\ n\}$, uniquely. Analogously, we introduce the symbols $Q_j^{i[k]}$ by the next three equalities $Q_j^{i[0]}=\delta_j^i$, $Q_j^{i[1]}=-\Gamma_{jl}^i(dx^l/ds)$,

$$Q_j^{i[k+1]}=dQ_j^{i[k]}/ds+Q_l^{i[k]}Q_j^{l[1]}. \tag{1.2}$$

2. Some properties for the symbols $P_j^{i[k]}$ and $Q_j^{i[k]}$.

(i) *The transformation law for the components $P_j^{i[k]}$ from one coordinate system to another is given by the following formula*

$$P_r^{j[l]}=\sum_{p=0}^{l}\bar{P}_q^{i[p]}(\partial x^j/\partial\bar{x}^i)\,(\partial\bar{x}^q/\partial x^r)^{(l-p)}C_l^p, \tag{2.1}$$

where $(l-p)$ denotes $(l-p)$-th derivative by s.

Proof. We will prove (2.1) by induction of the index l. It is satisfied for $l=0$. Assume that (2.1) holds for l, then by differentiating of (2.1) we obtain

$$P_r^{j[l+1]}-P_r^{k[l]}P_k^{j[1]}=\sum_{p=0}^{l}(\bar{P}_q^{i[p+1]}-\bar{P}_q^{a[p]}\bar{P}_a^{i[1]})(\partial x^j/\partial\bar{x}^i)(\partial\bar{x}^q/\partial x^r)^{(l-p)}C_l^p$$

$$+\sum_{p=0}^{l}\bar{P}_q^{i[p]}\Big(\frac{\partial}{\partial s}(\partial x^j/\partial\bar{x}^i)\Big)(\partial\bar{x}^q/\partial x^r)^{(l-p)}C_l^p+\sum_{p=0}^{l}\bar{P}_q^{i[p]}(\partial x^j/\partial\bar{x}^i)(\partial\bar{x}^2/\partial x^r)^{(l+1-p)}C_l^p,$$

$$P_r^{j[l+1]}=\sum_{p=1}^{l+1}\bar{P}_q^{i[p]}(\partial x^j/\partial\bar{x}^i)(\partial\bar{x}^q/\partial x^r)^{(l+1-p)}C_l^{p-1}-\sum_{p=0}^{l}\bar{P}_q^{a[p]}\bar{P}_a^{i[1]}(\partial x^j/\partial\bar{x}^i)\,(\partial\bar{x}^q/\partial x^r(^{(l-p)}C_l^p$$

$$+\sum_{p=0}^{l}\bar{P}_q^{i[p]}\Big(\frac{d}{ds}(\partial x^j/\partial\bar{x}^i)\Big)(\partial\bar{x}^q/\partial x^r)^{(l-p)}C_l^p+\sum_{p=0}^{l}\bar{P}_q^{i[p]}(\partial x^j/\partial\bar{x}^i)\,(\partial\bar{x}^q/\partial x^r)^{(l+1-p)}C_l^p$$

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$$+[\sum_{p=0}^{l}\bar{P}_q^{c[p]}\,\partial x^k/\partial\bar{x}^c)(\partial\bar{x}^q/\partial x^r)^{(l-p)}C_l^p]\,[(\partial x^j/\partial\bar{x}^q)(\frac{d}{ds}(\partial\bar{x}^q/\partial x^k))+\bar{P}_a^{i[1]}(x^j/\partial\bar{x}^i)(\partial\bar{x}^a/\partial x^k)].$$

We notice that

$$\bar{P}_q^{i[p]}(\frac{d}{ds}(\partial x^j/\partial\bar{x}^i))(\partial\bar{x}^q/\partial x^r)^{(l-p)}C_l^p+\bar{P}_q^{c[p]}(\partial x^k/\partial\bar{x}^c)(\partial\bar{x}^q/\partial x^r)^{(l-p)}C_l^p(\partial x^i/\partial\bar{x}^q)(\frac{d}{ds}(\partial\bar{x}^q/\partial x^k))$$

$$=\bar{P}_q^{c[p]}(\partial\bar{x}^q/\partial x^r)^{(l-p)}C_b^p\,\partial x^k(\partial\bar{x}^c)\,[\partial\bar{x}^i/\partial x^k)(\frac{d}{ds}(\partial x^j/\partial\bar{x}^i))+(\partial x^j/\partial\bar{x}^i)(\frac{d}{ds}(\partial\bar{x}^i/\partial x^k))]=0.$$

Further we obtain

$$P_r^{j[l+1]}=\sum_{p=1}^{l+1}\bar{P}_q^{i[p]}(\partial x^j/\partial\bar{x}^i)(\partial\bar{x}^q/\partial x^r)^{(l+1-p)}C_l^{p-1}$$

$$+\sum_{p=0}^{l}\bar{P}_q^{i[p]}(\partial x^j/\partial\bar{x}^i)(\partial\bar{x}^q/\partial x^r)^{(l+1-p)}C_l^p=\sum_{p=0}^{l+1}\bar{P}_q^{i[p]}(\partial x^j/\partial\bar{x}^i)(\partial\bar{x}^q/\partial x^r)^{(l+1-p)}C_{l+1}^p.$$

(ii) *Analogously as the previous proof, one can prove the following* ***transformation*** *formula for the symbols* $Q_j^{i[k]}$

$$Q_r^{j[l]}=\sum_{p=0}^{l}\bar{Q}_b^{a[p]}(\partial\bar{x}^b/\partial x^r)(\partial x^j/\partial\bar{x}^a)^{(l-p)}C_l^p. \tag{2.2}$$

(iii) *The k-th derivative of* $P_r^{i(l)}$ *can be expressed in the following form*

$$(P_r^{j[l]})^{(k)}=\sum_{i=0}^{k}P_r^{a[l+k-i]}\,Q_a^{j[i]}\,C_k^i. \tag{2.3}$$

P r o o f. We will prove (2.3) by induction of k. (2.3) is satisfied for $k=0$. If it holds for the numbers $\{1,\dots,k\}$, then by differentiation of (2.3) we obtain

$$(P_r^{j[l]})^{(k+1)}=\sum_{i=0}^{k}[P_r^{a[l+1+k-i]}-P_r^{\lambda[l+k-i]}P_\lambda^{a[1]}]Q_a^{j[i]}C_k^i$$

$$+\sum_{i=0}^{k}P_r^{a[l+k-i]}[Q_a^{j[i+1]}-Q_\lambda^{j[i]}Q_a^{\lambda[1]}]\,C_k^i.$$

Using that $P_\lambda^{a[1]}=-Q_\lambda^{a[1]}$, we obtain

$$(P_r^{j[l]})^{(k+1)}=\sum_{i=0}^{k}P_r^{a[l+k+1-i]}Q_a^{j[i]}C_k^i+\sum_{i=1}^{k+1}P_r^{a[l+k+1-i]}Q_a^{j[i]}C_k^{i-1}=\sum_{i=0}^{k+1}P_r^{a[l+(k+1)-i]}Q_a^{j[i]}C_{k+1}^i.$$

(iv) *Analogously to the previous proof, one can vrove the* ***following formula***

$$(Q_r^{j[l]})^{(k)}=\sum_{i=0}^{k}Q_a^{j[l+k-i]}\,P_r^{a[i]}C_k^i. \tag{2.4}$$

(v) *The following formula*

$$\sum_{r=0}^{k}P_u^{j[k-r]}\,Q_t^{u[r]}\,C_k^r=\begin{cases}\delta_t^j & \text{if } k=0\\ 0 & \text{if } k\geq 1\end{cases} \tag{2.5}$$

holds.

P r o o f. We will prove (2.5) by induction of the index k. It is trivially satisfied for $k=0$ and $k=1$. Assume that (2.5) holds for the number $k\geq 1$. Then by differentiation of (2.5) we obtain

$$\sum_{r=0}^{k} (P_u^{j[k-r+1]} - P_u^{\lambda[k-r]} P_\lambda^{j[1]}) Q_t^{u[r]} C_k^r + \sum_{r=0}^{k} P_u^{j[k-r]} (Q_t^{u[r+1]} - Q_\lambda^{u[r]} Q_t^{\lambda[1]}) C_k^r = 0.$$

If we put $r-1$ instead of r in the sum $\Sigma_{r=0}^k P_u^{j[k-r]} Q_t^{u[r+1]}$ and use that $P_j^{i[1]} = -Q_j^{i[1]}$, we obtain that (2.5) holds for $k+1$ too.

3. Formula for the k-th covariant derivative.

Theorem 1. *The k-th covariant derivative of a tensor field $A \in C^k$ of type (p, q) can be expressed in the following form*

$$\nabla^k A_{i_1 \dots i_q}^{j_1 \dots j_p} = \sum_{a=0}^{k} \sum_{\substack{m_1 + \dots m_p + l_1 + \dots + l_q = a \\ m_1, \dots, m_p, l_1, \dots, l_q \in \{0, 1, 2 \dots\}}} P_{u_1}^{j_1[m_1]} \dots P_{u_p}^{j_p[m_p]} \tag{3.1}$$

$$\times Q_{i_1}^{v_1[l_1]} \dots Q_{i_q}^{v_q[l_q]} \frac{d^{k-a} A_{v_1 \dots v_q}^{u_1 \dots u_p}}{ds^{k-a}} \frac{k!}{(k-a)!\, m_1! \dots m_p!\, l_1! \dots l_2!}$$

Remark. We concert to write Σ_m instead of $m_1 + \dots + m_p$ and Σl instead of $l_1 + \dots + l_q$. If we use that $(-l)! = \pm\infty$ for $l \in \{1, 2, \dots\}$ then (3.1) can be written in the form

$$\nabla^k A_{i_1 \dots i_q}^{j_1 \dots j_p} = P_{u_1}^{j_1[m_1]} \dots P_{u_p}^{j_p[m_p]} Q_{i_1}^{v_1[l_1]} \dots Q_{i_q}^{v_q[l_q]} \tag{3.2}$$

$$\times \frac{d^{k-\Sigma m - \Sigma l} A_{v_1 \dots v_q}^{u_1 \dots u_p}}{ds^{k-\Sigma m - \Sigma l}} \frac{k!}{(k-\Sigma m - \Sigma l)!\, m! \dots m_p!\, l_1! \dots l_q!},$$

where it is understood that the symbols of summation over the indices $m_1, \dots, m_p$ $l_1, \dots, l_q$ in the set $\{0, 1, 2, \dots\}$ are omitted and the symbols of summation over the indices $u_1, \dots u_p$, $v_1, \dots, v_q$ in the set $\{1, \dots, n\}$ are also omitted.

Proof. One can easily verify that ∇^0 is the identity operator and ∇^1 is the well-known covariant differentiation along the curve $x(s)$. In order to prove the theorem 1 it is sufficient to prove that

$$\nabla(\nabla^k A) = \nabla^{k+1} A. \tag{3.3}$$

Using (1.1), (1.2) and the definition of ∇, we obtain

$$\nabla(\nabla^k A_{i_1 \dots i_q}^{j_1 \dots j_p}) = \{P_{u_1}^{j_1[m_1+1]} \dots P_{u_p}^{j_p[m_p]} Q_{i_1}^{v_1[l_1]} \dots Q_{i_q}^{v_q[l_q]}$$

$$+ \dots + P_{u_1}^{j_1[m_1]} \dots P_{u_p}^{j_p[m_p+1]} Q_{i_1}^{v_1[l_1]} \dots Q_{i_q}^{v_q[l_q]} + P_{u_1}^{j_1[m_1]} \dots P_{u_p}^{j_p[m_p]} Q_{i_1}^{v_1[l_1+1]} \dots Q_{i_q}^{v_q[l_q]}$$

$$+ \dots + P_{u_1}^{j_1[m_1]} \dots P_{u_p}^{j_p[m_p]} Q_{i_1}^{v_1[l_1]} \dots Q_{i_q}^{v_q[l_q+1]}\}$$

$$\times \frac{d^{k-\Sigma m - \Sigma l} A_{v_1 \dots v_q}^{u_1 \dots u_p}}{ds^{k-\Sigma m - \Sigma l}} \frac{k!}{(k-\Sigma m - \Sigma l)!\, m_1! \dots m_p!\, l_1! \dots l_q!} + P_{u_1}^{j_1[m_1]} \dots P_{u_p}^{j_p[m_p]} Q_{i_1}^{v_1[l_1]} \dots Q_{i_q}^{v_q[l_q]}$$

$$\frac{d^{k+1-\Sigma m - \Sigma l} A_{v_1 \dots v_q}^{u_1 \dots u_p}}{ds^{k+1-\Sigma m - \Sigma l}} \frac{k!}{(k-\Sigma m - \Sigma l)!\, m_1! \dots m_p!\, l_1! \dots l_q!}.$$

If we put m_1 instead of m_1+1 in the first summand, m_2 instead of m_2+1 in the second summand and so on, after summing up we obtain (3.3).

4. Fractional covariant derivatives. Analogously to the formula (3.1) now we can introduce an operator ∇^α for $\alpha \in R$ by the following formula

$$\nabla^\alpha A^{j_1 \dots j_p}_{i_1 \dots i_q} = \sum_{a=0}^{\infty} \sum_{\substack{m_1+\dots+m_p+l_1+\dots l_q=a \\ m_1,\dots m_p, l_1,\dots,l_q \in \{0,1,2,\dots\}}} P^{j_1[m_1]}_{u_1} \dots P^{j_p[m_p]}_{u_p} \times Q^{v_1[l_1]}_{i_1} \dots Q^{v_q[l_q]}_{i_q} \frac{d^{\alpha-a} A^{u_1 \dots u_p}_{v_1 \dots v_q}}{ds^{\alpha-a}} \frac{\alpha(\alpha-1)\dots(\alpha-a+1)}{m_1! \dots m_p! \, l_1! \dots l_q!}. \tag{4.1}$$

If $a=0$ in (4.1), then we define $\alpha(\alpha-1)\dots(\alpha-a+1)=1$. If the Christoffel symbols are identically equally equal to zero, then

$$\nabla^\alpha A^{j_1 \dots j_p}_{i_1 \dots i_q} = d^\alpha A^{j_1 \dots j_p}_{i_1 \dots i_q} / ds^\alpha$$

and so (4.1) is a generalization of the ordinary fractional derivative [3]. In [1] it is given another formula for fractional covariant derivatives.

We notice that the right side of (4.1) is a series. If that series diverges then we should apply a corresponding method for summing up of divergent series.

It is obvious that ∇^α coincides with the α-th covariant derivative if $\alpha \in \{0, 1, 2 \dots\}$. In order to prove some other properties for the operator ∇^α we shall use the following two properties for the ordinary fractional derivatives

$$\frac{d^\alpha}{ds^\alpha} \circ \frac{d^\beta}{ds^\beta} = \frac{d^{\alpha+\beta}}{ds^{\alpha+\beta}}, \tag{4.2}$$

$$\frac{d^\alpha}{ds^\alpha}(f_1 \dots f_k) = \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} \dots \sum_{l_{k-1}=0}^{\infty} \frac{d^{l_1} f_1}{ds^{l_1}} \dots \frac{d^{l_{k-1}} f_{k-1}}{ds^{l_{k-1}}} \times \frac{d^{\alpha-l_1-\dots-l_{k-1}} f_k}{ds^{\alpha-l_1-\dots-l_{k-1}}} \frac{\alpha(\alpha-1)\dots(\alpha-l_1-l_2-\dots-l_{k-1}+1)}{l_1! \dots l_{k-1}!} \tag{4.3}$$

and we shall assume everywhere that the ordinary fractional derivatives are defined. We shall also assume in the next proofs that the corresponding series are convergent, and we concert to omit the symbols of summation.

(i) ∇^α *maps the tensors of type* (p, q) *into tensors of the same type.*

Proof. We should prove the transformation formula

$$\nabla^\alpha A^{j_1 \dots j_p}_{i_1 \dots i_q} = \frac{\partial x^{j_1}}{\partial \bar{x}^{r_1}} \dots \frac{\partial x^{j_p}}{\partial \bar{x}^{r_p}} \frac{\partial \bar{x}^{a_1}}{\partial x^{i_1}} \dots \frac{\partial \bar{x}^{a_q}}{\partial x^{i_q}} \nabla^\alpha \bar{A}^{r_1 \dots r_p}_{a_1 \dots a_q}. \tag{4.4}$$

Using the formulas (2.1), (2.2) and (4.3), we obtain

$$\nabla^\alpha A^{j_1 \dots j_p}_{i_1 \dots i_q} = \bar{P}^{a_1[c_1]}_{b_1} \frac{\partial x^{j_1}}{\partial \bar{x}^{a_1}} \left(\frac{\partial \bar{x}^{b_1}}{\partial x^{u_1}}\right)^{(k_1-c_1)} C^{c_1}_{k_1} \dots \bar{P}^{a_p[c_p]}_{b_p} \frac{\partial x^{j_p}}{\partial \bar{x}^{a_p}} \left(\frac{\partial \bar{x}^{b_p}}{\partial x^{u_p}}\right)^{(k_p-c_p)} C^{c_p}_{k_p} \bar{Q}^{m_1[n_1]}_{d_1} \frac{\partial \bar{x}^{d_1}}{d x^{i_1}} \left(\frac{\partial x^{v_1}}{\partial \bar{x}^{m_1}}\right)^{(l_1-n_1)} C^{n_1}_{l_1} \dots \bar{Q}^{m_q[n_q]}_{d_q} \frac{\partial \bar{x}^{d_q}}{\partial x^{i_q}} \left(\frac{\partial x^{v_q}}{\partial \bar{x}^{m_q}}\right)^{(l_q-n_q)} C^{n_q}_{l_q} \tag{4.5}$$

$$\times \left(\frac{\partial x^{u_1}}{\partial \bar{x}^{r_1}}\right)^{(t_1)} \cdots \left(\frac{\partial x^{u_p}}{\partial \bar{x}^{r_p}}\right)^{(t_p)} \left(\frac{\partial \bar{x}^{s_1}}{\partial x^{v_1}}\right)^{(w_1)} \cdots \left(\frac{\partial \bar{x}^{s_q}}{\partial x^{v_q}}\right)^{(w_q)}$$

$$\times \frac{d^{\alpha-\Sigma k-\Sigma l-\Sigma t-\Sigma w}}{ds^{\alpha-\Sigma k-\Sigma l-\Sigma t-\Sigma w}} \bar{A}^{r_1 \ldots r_p}_{s_1 \ldots s_p} \frac{(\alpha-\Sigma k-\Sigma l)\ldots(\alpha-\Sigma k-\Sigma l-\Sigma t-\Sigma w+1)}{t_1!\ldots t_p!\, w_1!\ldots w_q!}$$

$$\times \frac{\alpha(\alpha-1)\ldots(\alpha-\Sigma k-\Sigma l+1)}{k_1!\ldots k_p!\, l_1!\ldots l_q!}.$$

We notice that for a fixed value of the index c_1, the last term can be written in the following form

$$\sum_{a=0}^{\infty} \sum_{k_1-c_1+t_1=a} \left(\frac{\partial \bar{x}^{b_1}}{\partial x^{u_1}}\right)^{(k_1-c)} \frac{1}{(k_1-c_1)!} \left(\frac{\partial x^{u_1}}{\partial \bar{x}^{r_1}}\right)^{(t_1)} \frac{1}{t_1!} M(a)$$

uch that for fixed value of $k_1+t_1=a+c_1$ th e term $M(a)$ do not depend on k_1 and t_1. So that term is equal to

$$\sum_{a=0}^{\infty} \left(\frac{\partial \bar{x}^{b_1}}{\partial \bar{x}^{r_1}}\right)^{(a)} \frac{1}{a!} M(a) = \delta^{b_1}_{r_1} M(0).$$

So in the right side of the equation (4.5) we should substitute $k_1=c_1$ and $t_1=0$ Analogously, we should substitute there $k_2=c_2$, $t_2=0, \ldots$, $k_p=c_p$, $t_p=0$, $l_1=n_1$, $w_1=0, \ldots$, $l_q=n_q$, $w_q=0$. Then (4.4) can easily be obtained.

(ii) ∇^α *commutes with the contraction.*

Proof. Let us put $j_p=i_q=t$ in (4.1). Using the property (2.3) for $l=0$, we obtain

$$\sum_{m_p=0}^{\infty} \sum_{l_q=0}^{\infty} P^{t[m_p]}_{u_p} Q^{v_q[l_q]} \frac{1}{m_p!} \frac{1}{l_q!} = \sum_{a=0}^{\infty} \sum_{m_p+l_q=a} P^{t[m_p[}_{u_p} Q^{v_q[l_q]}_{t} \frac{1}{m_p!} \frac{1}{l_q!} = \sum_{a=0}^{\infty} \frac{1}{a!} (\delta^{v_q}_{u_p})^{(a)} = \delta^{v_q}_{u_p}.$$

Substituting that in the right side of (4.1), we obtain the proof of this property.

(iii)

$$\nabla^\beta(\nabla^\alpha A^{j_1 \ldots j_p}_{i_1 \ldots i_q} = \nabla^{\alpha+\beta} A^{j_1 \ldots j_p}_{i_1 \ldots i_q}. \tag{4.6}$$

Proof. Using the definition (4.1) twice and the formulas (4.2), (4.3), (2.3) and (2.4), one obtains

$$\nabla^\beta(\nabla^\alpha A^{j_1 \ldots j_p}_{i_1 \ldots i_q}) = P^{j_1[k_1]}_{u_1} \ldots P^{j_p[k_p]}_{u_p} Q^{v_1[l_1]}_{i_1} \ldots Q^{v_q[l_q]}_{i_q} P^{d_1[a_1+c_1]}_{w_1} Q^{u_1[r_1-c_1]}_{d_1} C^{c_1}_{r_1}$$

$$\ldots P^{d_p[a_p+c_p]}_{w_p} Q^{u_p[r_p-c_p]}_{d_p} C^{c_p}_{r_p} Q^{t_1[b_1+n_1]}_{m_1} P^{m_1[s_1-n_1]}_{v_1} C^{n_1}_{s_1} \ldots Q^{t_q[b_q+n_q]}_{m_q} P^{m_q[s_q-n_q]}_{v_q} C^{n_q}_{s_q}$$

$$\frac{d^{(\alpha-\Sigma a-\Sigma b)+(\beta-\Sigma k-\Sigma b)-\Sigma r-\Sigma s}}{ds^{\alpha+\beta-\Sigma a-\Sigma b-\Sigma k-\Sigma l-\Sigma r-\Sigma s}} A^{w_1 \ldots w_p}_{t_1 \ldots t_q} \frac{\beta(\beta-1)\ldots(\beta-\Sigma k-\Sigma l-\Sigma r-\Sigma s+1)}{k_1!\ldots k_p!\, l_1!\ldots l_q!}$$

$$\times \frac{\alpha(\alpha-1)\ldots(\alpha-\Sigma a-\Sigma b+1)}{a_1!\ldots a_p!\, b_1!\ldots b_q!} \frac{1}{r_1!\ldots r_p!\, s_1!\ldots s_q!}.$$

Similarly as in the proof of (i) we assume that c_1 is fixed. The last term can be separated into sums for which $k_1+r_1-c_1=a(=\text{const})$ where a changes from 0 to infinity. From (2.5) we obtain

$$P^{j_1[k_1]}_{a_1} Q^{u_1[r_1-c_1]}_{d_1} \frac{1}{k_1!\,(r_1-c_1)!} = \begin{cases} \delta^{j_1}_{d_1} & \text{if } r_1=c_1 \text{ and } k_1=0. \\ 0 & \text{otherwise} \end{cases}$$

Using that fact in the above equality, we should substitute there $r_1=c_1$ and $k_1=0$. Analogously we should substitute there $r_2=c_2$, $k_2=0,\ldots$, $r_p=c_p$, $k_p=0$, $s_1=n_1$, $l_1=0,\ldots$, $s_q=n_q$, $l_q=0$. After these substitutions it is easy to see that if we prove the following formula

$$\text{(4.7)}\qquad \sum_{a_1+c_1=k_1}\cdots\sum_{a_p+c_p=k_p}\ \sum_{b_1+n_1=l_1}\cdots\sum_{b_q+n_q=l_q}$$

$$\times\frac{\beta(\beta-1)\ldots(\beta-\Sigma c-\Sigma n+1)}{c_1!\ldots c_p!\,n_1!\ldots n_q!}\ \frac{\alpha(\alpha-1)\ldots(\alpha-\Sigma a-\Sigma b+1)}{a_1!\ldots a_p!\,b_1!\ldots b_q!}=\frac{(\alpha+\beta)(\alpha+\beta-1)\ldots(\alpha+\beta-\Sigma k-\Sigma l+1)}{k_1!\ldots k_p!\,l_1!\ldots l_q!},$$

where $a_1,\ldots,a_p$, $c_1,\ldots,c_p$, $b_1,\ldots,b_q$, $n_1,\ldots,n_q\in\{0,1,2,\ldots\}$, then the proof of (4.6) will be finished. The formula (4.7) can be regarded as equality between two polynomials of α and β. So it is sufficient to prove (4.7) if α and β are natural numbers. Suppose that $f_1,\ldots,f_p$, $g_1,\ldots,g_q$, $h\in C^{\alpha+\beta}$ are arbitrary functions of t. The coefficients in front of

$$f_1^{(k_1)}\ldots f_p^{(k_p)}g_1^{(l_1)}\ldots g_q^{(l_q)}h^{(\alpha+\beta-k_1-\ldots-k_p-l_1-\ldots-l_q)}$$

from the left and the right side of the equality

$$[(f_1\ldots f_pg_1\ldots g_qh)^{(\alpha)}]^{(\beta)}=(f_1\ldots f_pg_1\ldots g_qh)^{(\alpha+\beta)}$$

should be equal, and as a consequence we obtain (4.7).

5. Applications. The formula (3.1) gives a simple method of calculation of the k-th covariant derivative along a given curve $x(s)$. Further, the formula (4.1) for $\alpha=-1$ gives a solution of covariant tensor integral. Our solution reduces to calculate all of the symbols $P_j^{i[k]}$ and $Q_j^{i[k]}$ $(i,\ j\in\{1,\ldots n\},\ k\in\{0,1,\ldots,\})$ and to calculate the successive integrals of the field A, which is easier than to solve a system of differential equations [1].

If we consider the following system of linear differential equations

$$\text{(5.1)}\qquad dY^i/ds+\sum_{j=1}^{n}f_j^iY^j=g^i,\quad i=1,\ldots,n,$$

where f_j^i, g^i are functions of s, then putting $P_j^{i[1]}=f_j^i$, we obtain the following tensor equation $\nabla^1Y^1=g^i$ and its solution is $Y^i=\nabla^{-1}g^i$.

Similarly, if we consider the following system of linear differential equations

$$\text{(5.2)}\qquad d^2Y^i/ds^2+2\sum_{j=1}^{n}f_j^i\,dY^j/ds+\sum_{r=1}^{n}[df_r^i/ds+\sum_{l=1}^{n}f_l^if_r^l]\,Y^r=g^i,\quad i=1,\ldots,n,$$

where f_j^i, g^i are functions of s, then putting $P_j^{i[1]}=f_j^i$, we obtain the following tensorial equation $\nabla^2Y^i=g^i$ and its solution is $Y^i=\nabla^{-2}g^i$.

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