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ON THE EXISTENCE OF $\varnothing$-MINIMAL VIABLE SOLUTIONS FOR A CLASS OF DIFFERENTIAL INCLUSIONS

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In this paper we establish the existence of $\varnothing$-minimal viable solutions for a class of differential inclusions with a Hausdorff continuous orientor field defined on a general Banach space and satisfying a compactness hypothesis and a strong Nagumo type condition (Theorems 3.1 and 3.2). When the state space is finite dimensional, we show that the strong Nagumo type condition can be weakened to a regular Nagumo type (tangential) condition (Theorem 3.3).

1. Introduction. In a recent paper M. Falcone and P. Saint Pierre [7] established sufficient conditions for the existence of slow viable solutions for a class of differential inclusions defined on a finite dimensional Banach space. In this note we generalize the results of Falcone—Saint Pierre [7] by relaxing some of their hypotheses and by establishing an existence result for infinite dimensional differential inclusions.

Consider the following multivalued Cauchy problem on a Banach space X

$$(*)\qquad \left\{\begin{array}{l} \dot{x}(t)\in F(x(t)) \text{ a. e.} \\ x(0)=x_0\in K\subseteq X \\ x(t)\in K,\ t\in T=[0, b] \end{array}\right\}$$

In their recent works, K. Deimling [4] and the author [14], proved that under some compactness type hypothesis on the orientor field $F(x)$, a necessary and sufficient condition for the existence of solutions of (*) is that for all $x\in K$, $F(x)\cap T_K(x)\neq\varnothing$ (Nagumo type condition). Here $T_K(x)$ denotes the Bouligand tangent cone to K at x.

In this paper we will be looking for a special type of viable solutions, namely solutions with velocity which is minimal with respect to a certain criterion $\varnothing(\cdot)$ ($\varnothing(\cdot)$-minimal solutions). So let $\varnothing: X\to R$ be a continuous, convex function. We say that a trajectory $x(\cdot)$ of (*) is "$\varnothing$-minimal, viable" if and only if

$$\varnothing(\dot{x}(t))=\inf\{\varnothing(z):\ z\in F(x(t))\cap T_K(x(t))\} \text{ a. e.}$$

Note that if $\varnothing(x)=\|x\|$ (the norm function), we recover the notion of slow solution which is important in mathematical economics and control theory (see J.-P. Aubin [1] and C. Henry [8]). In this case $\varnothing(\cdot)$ is nothing else but the metric projection on the set $R(x)=F(x)\cap T_K(x)$. Recall that if the underlying state space X is a strictly convex, reflexive Banach space and $K\subseteq X$ is nonempty, closed, convex, then the metric projection function $x\to\text{proj}\,(x, K)$ is single valued.

2. Preliminaries. Let X be a Banach space. Throughout this paper we will be using the following notations:

$$P_{f(c)}(X)=\{A\subseteq X: \text{ nonempty, closed, convex)}\}$$

and
$$P_{k(c)}(X)=\{A\subseteq X: \text{ nonempty, compact, (convex)}\}.$$

On $P_f(X)$ we can define a generalized metric $h(\cdot,\cdot)$, known as the Hausdorff metric, by setting:

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$$h(A, B)=\max \{\sup_{a\in A} (\inf (\|a-b\|: b\in B)), \sup_{b\in B} (\inf (\|b-a\|: a\in A))\}.$$

Recall that $(P_f(X), h)$ is a complete metric space.

A multifunction $F: X\to P_f(X)$ is said to be Hausdorff continuous (h-continuous), if it is continuous as a function from X into the metric space $(P_f(X), h)$.

More generally, if Y, Z are Hausdorff topological spaces, a multifunction $F: Y\to 2^Z\setminus\{\emptyset\}$ is said to be lower semicontinuous (l.s.c.), if for all $U\subseteq Z$ open, $F^-(U)=\{y\in Y: F(y)\cap U\neq\emptyset\}$ is open in Y. If Y, Z are metric spaces, this definition is equivalent to saying that for any $y_n\to y$ in Y, we have $F(y)\subseteq \underline{\lim}\, F(y_n)=\{y\in Y: \lim d(y, F(y_n))=0\}$, where $d(y, F(y_n))=\inf\{\|y-z\|: z\in F(y_n)\}$. We will say that $F: Y\to 2^Z\setminus\{\emptyset\}$ is upper semicontinuous (u.s.c.), if for all $U\subseteq Z$ open $F^+(U)=\{y\in Y: F(y)\subseteq U\}$ is open in Y (see J. -P. Delahaye, J. Denel [5]).

Now let us return to X being a Banach space, let $K\subseteq X$ be nonempty and let $x\in K$ The "Bouligand or contingent cone" to K at x is defined by:

$$T_K(x)=\{h\in X: \underline{\lim}_{\lambda\downarrow 0} \frac{d_K(x+\lambda h)}{\lambda}=0\},$$

where for any $z\in X$, $d_K(z)=\inf\{\|z-x'\|: x'\in K\}$ (see J. -P. Aubin, A. Cellina [2]) It is clear that this cone is closed, $T_K(x)=T_{\overline{K}}(x)$ and if $x\in \operatorname{int} K$, then $T_K(x)=X$. Note that unfortunately $T_K(x)$ in general is not convex. However if K is convex (or more generally locally convex at x), then $T_K(x)$ is convex. Also note that if int $K\neq\emptyset$, then for all $x\in \overline{K}$, int $T_K(x)\neq\emptyset$ (see Aubin–Ekeland [13, p. 169]).

By $\alpha(\cdot)$ we will denote the "Kuratowski measure of noncompactness" which is defined on the nonempty, bounded subsets of X. So if A is such a set we have:

$$\alpha(A)=\inf\{d>0: A=\bigcup_{k=1}^{m} A_k, \text{ for some } m \text{ and } A_k\text{'s s.t. diam } (A_k)\leq d\}.$$

Finally given a multifunction $F: Y\to 2^Z\setminus\{\emptyset\}$, by "graph of F" we will mean the set Gr $F=\{(y, z)\in Y\times Z: z\in F(y)\}$.

3. Main results. Let X be a Banach space and $\emptyset: X\to \mathrm{R}$ a continuous, convex function. We will be looking for $\emptyset$-minimal viable trajectories of (*). Recall that $x: T=[0, b]\to X$ is an "$\emptyset$-minimal viable trajectory" if there exists $f\in S^1_{F(x(\cdot))}=\{g\in L^1(X): g(t)\in F(x(t))$ a.e.$\}$ s.t. $x(t)=x_0+\int_0^t g(s)\, ds$ for all $t\in T$, $x(t)\in K(t)$ and $\emptyset(\dot{x}(t))=\inf\{\emptyset(z): z\in F(x(t))\cap T_K(x(t))\}$ a.e.

In our first theorem we will establish the existence of such solutions for a large class of infinite dimensional differential inclusions. But first we will need two auxiliary lemmata.

Lemma α: *If Y, Z are metric spaces, $R: Y\to P_k(Z)$ is l.s.c. and $\emptyset: Z\to \mathrm{R}$ is continuous, then $y\to\lambda(y)=\inf\{\emptyset(z): z\in R(y)\}$ is u.s.c..*

Proof: We need to show that for every $\mu\in \mathrm{R}$, the upper level set $U(\mu)=\{y\in Y: \lambda(y)\geq\mu\}$ is closed. To this end let $y_n\to y$, $y_n\in U(\mu)$. Since by hypothesis $R(\cdot)$ is compact valued and $\emptyset(\cdot)$ is continuous, we can find $z\in R(y)$ s.t. $\emptyset(z)=\lambda(y)$. Also because $R(\cdot)$ is l.s.c. we have $R(y)\subseteq \underline{\lim}\, R(y_n)$ and so we can find $z_n\in R(y_n)$ s.t. $z_n\to z$. But note that $\mu\leq\lambda(y_n)\leq\emptyset(z_n)\Rightarrow\mu\leq\emptyset(z)=\lim\emptyset(z_n)\Rightarrow\mu\leq\lambda(y)\Rightarrow y\in U(\mu)\Rightarrow\lambda(\cdot)$ is indeed u.s.c. Q.E.D.

Lemma β: *If Y, Z are Hausdorff topological spaces, $F: Y\to 2^Z\setminus\{\emptyset\}$ is l.s.c., $G: Y\to 2^Z/\{\emptyset\}$ has open graph and for all $y\in Y$, $F(y)\cap G(y)\neq\emptyset$, then $y\to L(y)=F(y)\cap G(y)$ is l.s.c.*

Proof: See Flytzanis-Papageorgiou [6, Lemma 2].

Now we are ready for the theorem establishing the existence of $\varnothing$-minimal viable solutions for (*). Our result extends theorem 4.1 of Falcone—Saint Pierre [7], since our state space is infinite dimensional, the growth hypothesis on the orientor field $F(\cdot)$ is more general and $\varnothing(\cdot)$ need not be inf-compact as in [7]. Note that this last fact is very important, because it allows $\varnothing(\cdot)$ to be the norm of an infinite dimensional Banach space and so our existence theorem incorporates the results on the existence of slow solutions (see Aubin—Cellina [2]).

Theorem 3.1: *If* $K \in P_{fc}(X)$ *with* $\operatorname{int} K \neq \varnothing$, $\varnothing: X \to \mathrm{R}$ *continuous, convex and* $F: K \to P_{fc}(X)$ *is a multifunction s.t.*

(1) $F(\cdot)$ *is h-continuous,*

(2) $|F(X)| \leq c(1+\|x\|)$ $c>0$,

(3) $\alpha(F(B)) \leq k\ \alpha(B)$ *for all* $B \subseteq K$ *nonempty bounded,* $k>0$,

(4) $F(x) \cap \operatorname{int} T_K(x) \neq \varnothing$ *for all* $x \in K$,

then (*) *admits a* $\varnothing$*-minimal viable solution* $x(\cdot)$.

Proof: Let $G: K \to 2^X$ be defined by:

$$G(x)=\{y \in Y:\ \varnothing(y) \leq \inf\ (\varnothing(z):\ z \in R(x))=\lambda(x)\},$$

where $R(x)=F(x) \cap T_K(x)$. Since $F(\cdot)$ is h-continuous (hence l.s.c., too) and $x \to \operatorname{int} T_K(x)$ has an open graph (see Aubin—Ekeland [3, Proposition 7, p. 169]), from Lemma β we deduce that $x \to F(x) \cap \operatorname{int} T_K(x)$ is l.s.c.. Hence $x \to \overline{F(x) \cap \operatorname{int} T_K(x)} = F(x) \cap T_K(x) = R(x)$ is l.s.c. (see E. Klein — A. Thompson [10, Proposition 7.3.3, p. 85]). Also since $R(\cdot)$ is compact valued ($F(\cdot)$ being compact valued because of hypothesis (3)), there exists $\hat{z} \in R(x)$ (depending on x) s.t. $\varnothing(\hat{z})=\lambda(x) \Rightarrow G(x) \neq \varnothing$ and in fact since $\varnothing(\cdot)$ is also convex, it is easy to see that $G(x) \in P_{fc}(X)$.

We claim that $G(\cdot)$ has a closed graph. To this end let $(x_n, y_n) \in \operatorname{Gr} G$ $(x_n, y_n) \xrightarrow{s} (x, y)$ in $K \times X$. We have $\varnothing(y_n) \leq \lambda(x_n)$ for all $n \geq 1$. Since $R(\cdot)$ is l.s.c., from Lemma α above we have that $\lambda(\cdot)$ is u.s.c.. So passing to the limit, we get:

$$\lim \varnothing(y_n)=\varnothing(y) \leq \overline{\lim}\ \lambda(x_n) \leq \lambda(x) \Rightarrow (x, y) \in \operatorname{Gr} G \Rightarrow \operatorname{Gr} G \text{ is closed in } K \times X.$$

Invoking Theorem 1, p. 41 of Aubin—Cellina [2], we get that $x \to L(x)=F(x) \cap G(x)$ is u.s.c. Also because of hypothesis (4) $L(x) \cap T_K(x) \neq \varnothing$ for all $x \in K$. Furthermore, we have $|L(x)|=\sup\{\|z\|:\ z \in L(x)\} \leq |F(x)|=\sup\{\|z'\|:\ z' \in F(x)\} \leq c(1+\|x\|)$ (hypothesis (2)), while for $B \subseteq K$ nonempty bounded since the Kuratowski measure of noncompactness is monotone, we have $\alpha(L(B)) \leq \alpha(F(B)) \leq k\ \alpha(B)$. So if we consider the following viability problem

$$(*)' \qquad \left\{ \begin{array}{l} \dot{x}(t) \in L(x(t)) \text{ a.e.} \\ x(0)=x_0 \in K \\ x(t) \in K,\ t \in T=[0,\ b] \end{array} \right\},$$

we see that all hypotheses of theorem 1 of K. Deimling [4] are satisfied and so according to that theorem, there exists solution $x(\cdot)$ for (*)'. It is easy to see that $x(\cdot)$ is the desired $\varnothing$-minimal viable trajectory for (*). Q.E.D.

We can also have an integral selection criterion.

So as before, let $\varnothing: X \to \mathrm{R}$ be a continuous, convex function and set $I_\varnothing(v) = \int_0^a \varnothing(v(t))\,dt$, for all $v(\cdot) \in L^1(X)$ if the integral exists, permitting $\pm\infty$. We say that a trajectory $x(\cdot)$ of (*) is "$I_\varnothing$-minimal viable" if and only if $I_\varnothing(\dot{x})=\inf\{I_\varnothing(v):\ v \in S^1_{R(x(\cdot))}\}$, where $R(x)=F(x) \cap T_K(x)$ and $S^1_{R(x(\cdot))}=\{g \in L^1(X):\ g(t) \in R(x(t)) \text{ a.e.}\}$.

Our existence result concerning $I_\varnothing$-minimal viable trajectories of (*), reads as follows:

Theorem 3.2: *If X is a separable Banach space, $K \in P_{fc}(X)$ with* int $K \neq \emptyset$, $\emptyset: X \to \mathbf{R}$ *is continuous, convex, for all $z(\cdot)$ viable trajectories of (*) and all $v \in S^1_{R(z(\cdot))}$, $I_\emptyset(v)$ is defined and finite for at least one such v and $F: K \to P_{fc}(X)$ is a multifunction satisfying hypotheses (1)→(4) of Theorem 3.1, then (*) admits a $I_\emptyset$-minimal viable trajectory.*

Proof: From Theorem 3.1 we know that there exists $\emptyset$-minimal viable trajectory $x(\cdot)$ of (*). So $\emptyset(\dot{x}(t)) = \inf\{\emptyset(z): z \in R(x(t))\}$ a.e. Note that $R(\cdot)$ being l.s.c. is measurable. So we can apply Theorem 2.2 of F. Hiai, H. Umegaki [9] and get that:

$$\inf\{I_\emptyset(v): v \in S^1_{R(x(\cdot))}\} = \int_0^b \inf\{\emptyset(z): z \in R(x(t))\}\, dt = \int_0^b \emptyset(\dot{x}(t))\, dt = I_\emptyset(\dot{x})$$

$\Rightarrow x(\cdot)$ is $I_\emptyset$-minimal viable solution of (*). Q.E.D.

If the underlying state space X is finite dimensional, then we can improve Theorem 3.1 by replacing hypothesis (4), with a standard Nagumo type hypothesis. So we have the following existence result.

Theorem 3.3: *If* dim $X < \infty$, $K \in P_{fc}(X)$ *with* int $K \neq \emptyset$, $\emptyset: X \to \mathbf{R}$ *is continuous strictly convex,* inf-*compact and $F: K \to P_{fc}(X)$ is a multifunction s.t.*

(1) *$F(\cdot)$ is h-continuous*

(2) $|F(x)| \leq c(1 + \|x\|)$ $c > 0$,

(3) *$F(x) \cap T_K(x) \neq \emptyset$ for all $x \in K$,*

then (*) admits a $\emptyset$-minimal viable trajectory $x(\cdot)$

Proof: Let $F_n(x) = F(x) + \frac{1}{n} B_1$, where B_1 is the closed unit ball in X. Clearly, $F_n(\cdot)$ is h-continuous, $|F_n(x)| \leq \left(c + \frac{1}{n}\right) + c\|x\|$ and $F_n(x) \cap \text{int}\, T_K(x) \neq \emptyset$ for all $x \in K$. Consider the following approximating viability problems:

$$(*)_n \qquad \left\{ \begin{array}{l} \dot{x}_n(t) \in F_n(x_n(t)) \text{ a.e.} \\ x_n(0) = x_0 \in K \\ x_n(t) \in K,\ t \in T = [0, b] \end{array} \right\}$$

From Theorem 3.1 (note that hypothesis (3) of the theorem is automatically satisfied with $k = 0$, because of the finite dimensionality of X), we know that for every $n \geq 1$, $(*)_n$ admits a $\emptyset$-minimal viable solution $x_n(\cdot)$. Then for all $n \geq 1$, we have:

$$\|\dot{x}_n(t)\| \leq (c+1) + c\|x_n(t)\| \text{ a.e.} \Rightarrow \|x_n(t)\| \leq \|x_0\| + (c+1)\, b + \int_0^t c\|x_n(s)\|\, ds.$$

So from Gronwall's inequality we get that for all $n \geq 1$ and all $t \in T$

$$\|x_n(t)\| \leq (\|x_0\| + (c+1)b) \exp(cb) = M.$$

Thus $\|\dot{x}_n(t)\| \leq (c+1) + cM = \bar{M}$ a.e. Therefore $\{\dot{x}_n\}(\cdot)\}_{n\geq 1}$ is uniformly integrable in $L^1(X)$ and so $\{x_n(\cdot)\}_{n\geq 1}$ is equicontinuous in $C(T, X)$. It is also bounded. So from the Arzela—Ascoli theorem, we deduce that $\{x_n\}_{n\geq 1}$ is compact in $C(T, X)$. Hence by passing to a subsequence if necessary, we may assume that $x_n \to x$ in $C(T, X)$.

Note that $F_n(x) \xrightarrow{K} F(x)$ (convergence in the sense of Kuratowski, see K. Kuratowski [11, p. 339]). Because int $T_K(x) \neq \emptyset$, from Lemma 1.4 of U. Mosco [12]' we have that $F_n(x) \cap T_K(x) = R_n(x) \xrightarrow{K} F(x) \cap T_K(x) = R(x)$ for all $x \in K$.

Now we claim that the minimization problem min $\{\emptyset(z): z \in R(x)\}$ is Tihonov well-posed, i.e. it admits a unique solution $z \in R(x)$ and every minimizing sequence converges to it. That a solution exists, follows from the continuity of $\emptyset(\cdot)$ and the compactness of $R(x)$. That it is unique, is a consequence of the strict convexity of $\varnothing(\cdot)$. Finally, let $\{z_n\}_{n\geq 1}$ be a minimizing sequence, i.e. $\varnothing(z_n)\downarrow\lambda(x)$, where $\lambda(x)$ is the value of the problem. Without any loss of generality we may assume that for all $n\geq 1$, $\varnothing(z_n)\leq\lambda(x)+1$. Since $\varnothing(\cdot)$ is inf-compact, $\{z_n\}_{n\geq 1}$ is relatively compact and so we may assume that $z_n\to\hat{z}$. Then $\varnothing(z_n)\to\varnothing(\hat{z})=\lambda(x)$, i.e. $\hat{z}$ is the unique solution of the problem. Therefore min $\{\varnothing(z): z\in R(x)\}$ is Tichonov well-posed. Without any loss of generality assume $\varnothing(0)=0$.

Set $\lambda_n(x)=\min\{\varnothing(z): z\in R_n(x)\}$ and $\lambda(x)=\min\{\varnothing(z): z\in R(x)\}$. Since $R_n(x)\xrightarrow{K}R(x)$ and the limit problem is Tihonov well-posed, we can apply Theorem 3 of T. Zolezzi [15] and get that $\lambda_n(x)\uparrow\lambda(x)$. Now note that for every $n\geq 1$, we have:

$$\lambda_n(x_n(t))\leq\lambda(x_n(t)).$$

Recall (see Lemma α) that $\lambda(\cdot)$ is u.s.c., so we get that:

$$\overline{\lim}\ \lambda_n(x_n(t))\leq\ \overline{\lim}\ \lambda(x_n(t))\leq\lambda(x(t)).$$

Also from the Dunford—Pettis compactness criterion and by passing to a subsequence if necessary, we may assume that $\dot{x}_n\xrightarrow{w}\dot{x}$ in $L^1(X)$. Then for all $A\subseteq T$ Lebesgue measurable we have $\chi_A\dot{x}_n\xrightarrow{w}\chi_A\dot{x}$ in $L^1(X)$. Recalling that $I_\varnothing(v)$ is weakly l.s.c. we get:

$$\int_0^b\varnothing(\chi_A(t)\,\dot{x}(t))\,dt\leq\underline{\lim}\int_0^b\varnothing(\chi_A(t)\,\dot{x}_n(t))\,dt\leq\overline{\lim}\int_A\varnothing(\dot{x}_n(t))\,dt$$

$$=\overline{\lim}\int_A\lambda_n(x_n(t))\,dt\leq\int_A\overline{\lim}\ \lambda_n(x_n(t))\,dt\ \text{(Fatou's lemma)}\leq\int_A\lambda(x(t))\,dt$$

$$\Rightarrow\int_A\varnothing(\dot{x}(t))\,dt\leq\int_A\lambda(x(t))\,dt\Rightarrow\varnothing(\dot{x}(t))\leq\lambda(x(t))\ \text{a.e.}$$

On the other hand, note that for every $n\geq 1$

$$\dot{x}_n(t)\in F_n(x_n(t))\ \text{a.e.}$$

and recall that $\dot{x}_n\xrightarrow{w}\dot{x}$ in $L^1(X)$ and $\|\dot{x}_n(t)\|\leq M$ a.e. for all $n\geq 1$. So from Theorem 3.1 of [13], we get that:

$$\dot{x}(t)\in\overline{\text{conv}}\ \overline{\lim}\ \{\dot{x}_n(t)\}_{n\geq 1}\subseteq\overline{\text{conv}}\ \overline{\lim}\ F_n(x_n(t))\ \text{a.e.}$$

But we claim that $F_n(x_n(t))\xrightarrow{h}F(x(t))$ as $n\to\infty$ To this end note that for every $n\geq 1$, $x\to u_n(x)=h(F_n(x),F(x))$ is continuous and $u_n(x)\downarrow 0$. So from Dini's theorem we have $u_n(x)\to 0$ uniformly on compacta. Then note that:

$$h(F_n(x_n(t)),F(x(t))\leq h(F_n(x_n(t)),F(x_n(t)))+h(F(x_n(t)),F(x(t)))$$
$$=u_n(x_n(t))+h(F(x_n(t)),F(x(t))).$$

We see that $u_n(x_n(t))\to 0$, while from hypothesis (1) we have $h(F(x_n(t)),F(x(t))\to 0$ as $n\to\infty\Rightarrow F_n(x_n(t))\xrightarrow{h}F(x(t))$. Therefore

$$\dot{x}(t)\in\overline{\text{conv}}\ \overline{\lim}\ F_n(x_n(t))=\overline{\text{conv}}\ F(x(t))=F(x(t))\ \text{a.e.}$$

Furthermore, $x(t)\in K$ for all $t\in T$. Then for $\lambda>0$ we have:

$$\frac{d_K(x(t)+\lambda\dot{x}(t))}{\lambda}=\frac{d_K(x(t+\lambda)-\lambda\epsilon(\lambda))}{\lambda}\leq\frac{\lambda\epsilon(\lambda)}{\lambda}=\epsilon(\lambda)\ \text{a.e.,}$$

where $\epsilon(\lambda)\to 0$. So we have:

$$\lim_{\lambda \downarrow 0} \frac{d_K(x(t)+\lambda \dot{x}(t))}{\lambda} = 0 \text{ a.e.}$$

$$\Rightarrow \dot{x}(t) \in T_K(x(t)) \text{ a.e.}$$

$$\Rightarrow \dot{x}(t) \in F(x(t)) \cap T_K(x(t)) = R(x(t)).$$

Because of the uniqueness of the solution of min $\{\varnothing(z): z \in R(x(t))\}$ $\lambda(x(t))$ and since as we saw above, $\varnothing(\dot{x}(t)) \leq \lambda(x(t))$ a.e., we conclude that $x(\cdot)$ is the desired $\varnothing$-minimal viable solution of (*). Q.E.D.

Remark: If in Theorem 3.3 X is strictly convex and $\varnothing(z) = \|z\|$, then the result applies and we get slow viable solutions for (*). By the way, note that there is a minor inaccuracy in the work of Falcone – Saint Pierre [7]. The state space X has to be strictly convex, or otherwise the metric projection need not be single valued.

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