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CANONICAL CONNECTION AND THE CANONICAL CONFORMAL GROUP ON A RIEMANNIAN ALMOST-PRODUCT MANIFOLD

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On an almost Hermitian manifold (M, J, g) there exists a unique linear connection D with torsion tensor T such that $D_X J = D_X g = 0$ and $T(x, JY) = T(JX, Y)$ for all vector fields X, Y on M. This is the Hermitian connection on the manifold [1, 2]. Another proof of this result has been given in [3]. The group of the conformal transformations of the metric g generates the conformal group of transformations of D.

On an almost complex manifold with B-metric (M, J, g) there exists a unique canonical connection D such that $D_X J = D_X g = 0$, or equivalently $D_X g = 0$, $D_X \tilde{g} = 0$ for any vector field X on M [4]. Here $\tilde{g}$ is the associated metric to the metric g. Considering the general group of the conformal transformations of the B-metric g, in [4] there is also obtained the canonical conformal group of transformations of the canonical connection D and its remarkable subgroups.

In this paper we treat analogous problems on Riemannian almost-product manifolds.

1. Preliminaries. Let (M, g) be a differentiable manifold with metric g.

Definition 1. *An almost-product structure on M is a $(1,1)$-tensor field v of constant rank for which $v^2 = v$.*

If we denote $h = I - v$ and $\mathscr{P} = v - h$, then $h^2 = h$, so that h is also an almost-product structure on M and $\mathscr{P}^2 = I$. It is easy to be proved that

$$vh = hv = 0, \quad v = \frac{1}{2}(I + \mathscr{P}), \quad h = \frac{1}{2}(I - \mathscr{P}). \tag{1}$$

Definition 2. *A Riemannian almost-product structure on M is an almost-product structure on M such that*

$$g(X, Y) = g(vX, vY) + g(hX, hY), \tag{2}$$

where X, Y are arbitrary vector fields on M.

From (1) and (2) it follows immediately that

$$g(vX, Y) = g(X, vY), \quad g(hX, Y) = g(X, hY), \quad g(vX, hY) = 0, \tag{3}$$

and

$$g(\mathscr{P}X, \mathscr{P}Y) = g(X, Y) \tag{4}$$

for all differentiable vector fields X, Y on M.

In this case we shall say that $\mathscr{P}$ and g are compatible, and that $\mathscr{P}$ is the tensor field defining a Riemannian almost-product structure on a Riemannian manifold M with metric tensor field g.

Because of (4) and (3) we can also define a symmetric 2-covariant tensor field $\tilde{g}$ on M by

$$\tilde{g}(X, Y) = g(\mathcal{P}X, Y). \tag{5}$$

In what follows $(M, \mathcal{P}, g)$ will be a 2n-dimensional Riemannian almost-product manifold, i. e. $\mathcal{P}$ will be the tensor field defining the Riemannian almost-product structure on M, and g will be the metric tensor field on M such that $\mathcal{P}^2 = I$, $g(\mathcal{P}X, Y) = g(X, \mathcal{P}Y)$ for all vector fields X, Y on M. The associated metric tensor field $\tilde{g}$ on the manifold is given by (5). The metric $\tilde{g}$ is necessarily of signature (n, n).

Further X, Y, Z will stand for arbitrary differentiable vector fields on M.

The Levi—Civita connections of g and $\tilde{g}$ will be denoted by ∇ and $\tilde{\nabla}$, respectively.

The difference $\tilde{\nabla}_X Y - \nabla_X Y$ is a tensor field of type (1, 2) on M and it will be denoted by

$$\Phi(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y. \tag{6}$$

This is the fundamental tensor field of the manifold. The corresponding tensor field of type (0,3) will be denoted by the same letter: $\Phi(X, Y, Z) = g(\Phi(X, Y), Z)$. The fundamental tensor field has the following symmetries:

$$\begin{aligned} &\Phi(X, Y) = \Phi(Y, X) \\ &\Phi(X, Y, Z) + \Phi(X, \mathcal{P}Y, \mathcal{P}Z) + \Phi(Z, X, Y) + \Phi(\mathcal{P}Z, X, \mathcal{P}Y) = 0, \\ &\Phi(X, Y, Z) + \Phi(\mathcal{P}X, \mathcal{P}Y, Z) + \Phi(\mathcal{P}X, Y, \mathcal{P}Z) + \Phi(X, \mathcal{P}Y, \mathcal{P}Z) = 0. \end{aligned} \tag{7}$$

2. Partial decomposition of the space of torsion tensors. The canonical linear connection in our considerations will have a torsion. Thus, we have to study the properties of the torsion tensors.

Let $(V, \mathcal{P}, g) = T_pM$, $p \in M$ and $\mathcal{T}$ be the vector space of all tensors T of type (0, 3) over V with the property $T(x, y, z) = -T(y, x, z)$, $x, y, z \in V$. The metric g induces on $\mathcal{T}$ an inner product $\langle\, .\,,\, .\,\rangle$ in the following way:

$$\langle T', T'' \rangle = g^{iq} g^{jr} g^{ks} T'(l_i, l_j, l_k) T''(l_q, l_r, l_s), \quad T', T'' \in \mathcal{T},$$

where $\{l_i\}$ $(i = 1, \dots, 2n)$ is a basis of V.

The natural representation of the group $G = \{a \in O(2n) / a = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$, A, B are $n \times n$-matrices$\}$ in V induces a representation λ of G in $\mathcal{T}$:

$$(\lambda a)T(x, y, z) = T(a^{-1}x, a^{-1}y, a^{-1}z), \quad a \in G, \ T \in \mathcal{T}, \ x, y, z \in V,$$

so that $\langle (\lambda a)T', (\lambda a)T'' \rangle = \langle T', T'' \rangle$, $a \in G$, $T', T'' \in \mathcal{T}$.

Let L be the linear operator $L: \mathcal{T} \to \mathcal{T}$ defined by

$$L(T)(x, y, z) = T(\mathcal{P}x, \mathcal{P}y, z), \quad T \in \mathcal{T}, \ x, y, z \in V.$$

The following proposition follows by simple computations.

Lemma 1. *The operator L is an involutive isometry of $\mathcal{T}$ and commutes with the action of G, i. e. $L^2 = id$,*

$$\langle L(T'), L(T'') \rangle = \langle T', T'' \rangle, \quad L((\lambda a)T) = (\lambda a)(L(T)),$$

where $T, T', T'' \in \mathcal{T}$, $a \in G$.

This lemma implies that L has two eigenvalues (± 1) and the corresponding eigenspaces

$$\mathcal{T}^- = \{T \in \mathcal{T} / L(T) = -T\}, \quad \mathcal{T}^+ = \{T \in \mathcal{T} / L(T) = T\}$$

are invariant orthogonal subspaces of $\mathcal{T}$.

In order to decompose $\mathcal{T}^-$ we consider the linear operator $L_1: \mathcal{T}^- \to \mathcal{T}^-$ defined by

$$L_1(T)(x, y, z) = \frac{1}{2}\{T(y, z, x) + T(z, x, y) - T(\mathcal{P}y, z, \mathcal{P}x) - T(z, \mathcal{P}x, \mathcal{P}y)\},$$

where $T \in \mathcal{T}^-$, $x, y, z \in V$. We have

Lemma 2. *The operator L_1 is an involutive isometry and computes with the action of G.*

This lemma implies that the eigenspaces

$$\mathcal{T}_1 = \{T \in \mathcal{T}^- \mid L_1(T) = -T\},\ \mathcal{T}_2 = \{T \in \mathcal{T}^- \mid L_1(T) = T\}$$

are invariant and orthogonal subspaces of $\mathcal{T}^-$.

To decompose $\mathcal{T}^+$ we define the linear operator $L_2: \mathcal{T}^+ \to \mathcal{T}^+$ in the following way

$$L_2(T)(x, y, z) = T(\mathcal{P}x, y, \mathcal{P}z), \quad T \in \mathcal{T}^+, \quad x, y, z \in V.$$

We have

Lemma 3. *The operator L_2 is an involutive isometry and commutes with the action of G.*

This lemma implies that the eigenspaces

$$\mathcal{T}_3 = \{T \in \mathcal{T}^+ \mid L_2(T) = -T\}, \quad \mathcal{T}_4 = \{T \in \mathcal{T}^+ \mid L_2(T) = T\}$$

are invariant and orthogonal subspaces of $\mathcal{T}^+$.

From the definitions of the spaces $\mathcal{T}_k$ ($k = 1, 2, 3, 4$) we obtain

Lemma 4. *Let $T \in \mathcal{T}$. Then*

1) $T \in \mathcal{T}_1$ *iff* $T(\mathcal{P}x, \mathcal{P}y, z) = -T(x, y, z)$, $T(x, y, z) + T(y, z, x) + T(z, x, y) = 0$;

2) $T \in \mathcal{T}_2$ *iff* $T(\mathcal{P}x, \mathcal{P}y, z) = -T(x, y, z)$, $T(\mathcal{P}x, y, z) + T(\mathcal{P}y, z, x) + T(\mathcal{P}z, x, y) = 0$;

3) $T \in \mathcal{T}_3$ *iff* $T(\mathcal{P}x, \mathcal{P}y, z) = T(x, y, z) = -T(\mathcal{P}x, y, \mathcal{P}z)$, or equivalently $T(\mathcal{P}x, y, z) = T(x, \mathcal{P}y, z) = -T(x, y, \mathcal{P}z)$;

4) $T \in \mathcal{T}_4$ *iff* $T(\mathcal{P}x, \mathcal{P}y, z) = T(x, y, z) = T(\mathcal{P}x, y, \mathcal{P}z)$, *or equivalently* $T(\mathcal{P}x, y, z) = T(x, \mathcal{P}y, z) = T(x, y, \mathcal{P}z)$.

Using Lemmas 1, 2 and 3 we obtain the following decomposition of $\mathcal{T}$:

Theorem 1. $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3 \oplus \mathcal{T}_4$, *where $\mathcal{T}_i$ ($i = 1, 2, 3, 4$) are invariant orthogonal subspaces of $\mathcal{T}$.*

The projection operators of $\mathcal{T}$ in $\mathcal{T}_i$ ($i = 1, 2, 3, 4$) are given in the following theorem.

Theorem 2. *Let $T \in \mathcal{T}$ and denote by p_i ($i = 1, 2, 3, 4$) the projection operators of $\mathcal{T}$ in $\mathcal{T}_i$. Then*

$$p_1(x, y, z) = \frac{1}{8}\{2T(x, y, z) - T(y, z, x) - T(z, x, y) + T(\mathcal{P}y, z, \mathcal{P}x) + T(z, \mathcal{P}x, \mathcal{P}y)$$
$$- 2T(\mathcal{P}x, \mathcal{P}y, z) + T(\mathcal{P}y, \mathcal{P}z, x) + T(\mathcal{P}z, \mathcal{P}x, y) - T(y, \mathcal{P}z, \mathcal{P}x) - T(\mathcal{P}z, x, \mathcal{P}y)\}$$

$$p_2(x, y, z) = \frac{1}{8}\{2T(x, y, z) + T(y, z, x) + T(z, x, y) - T(\mathcal{P}y, z, \mathcal{P}x) - T(z, \mathcal{P}x, \mathcal{P}y)$$
$$- 2T(\mathcal{P}x, \mathcal{P}y, z) - T(\mathcal{P}y, \mathcal{P}z, x) - T(\mathcal{P}z, \mathcal{P}x, y) + T(y, \mathcal{P}z, \mathcal{P}x) + T(\mathcal{P}z, x, \mathcal{P}y)\},$$

$$p_3(x, y, z) = \frac{1}{4}\{T(x, y, z) + T(\mathcal{P}x, \mathcal{P}y, z) - T(\mathcal{P}x, y, \mathcal{P}z) - T(x, \mathcal{P}y, \mathcal{P}z)\},$$

$$p_4(x, y, z) = \frac{1}{4}\{T(x, y, z) + T(\mathcal{P}x, \mathcal{P}y, z) + T(\mathcal{P}x, y, \mathcal{P}z) + T(x, \mathcal{P}v, \mathcal{P}z)\}.$$

P r o o f. We will compute for example v_1. The other projections can be obtained in a similar way.

From Lemma 1 it follows that the tensor $\frac{1}{2}(T-L(T))$ is the projection of T into $\mathcal{T}^{-}=\mathcal{T}_1\oplus\mathcal{T}_2$. Using Lemma 2 we find

$$p_1=\frac{1}{4}\{T-L(T)-L_1(T)+L_1(L(T))\},$$

which gives the required expression of p_1.

We need the following characteristic of the class $\mathcal{T}_1\oplus\mathcal{T}_3$.

L e m m a 5. *Let* $T\in\mathcal{T}$. *Then* $T\in\mathcal{T}_1\oplus\mathcal{T}_3$ *iff*

$$T(x, y, z)+T(y, z, x)+T(\mathcal{P}x, y, \mathcal{P}z)+T(y, \mathcal{P}z, \mathcal{P}x)=0. \tag{8}$$

P r o o f. Let $T\in\mathcal{T}_1\oplus\mathcal{T}_3$, i. e. $T=p_1+p_3$. It can be directly seen that p_1 and p_3 satisfy (8).

Conversely, let T satisfy (8). Since $T=T'+T''$, where T' and T'' are the components of T in $\mathcal{T}_1\oplus\mathcal{T}_2$ and $\mathcal{T}_3\oplus\mathcal{T}_4$ respectively, the substitutions of T' and T'' into (8) imply $T'=p_1$ and $T''=p_3$, i. e. $T=p_1+p_3$.

3. Natural connection on a Riemannian almost-product manifold.

D e f i n i t i o n 3. *A linear connection D on the Riemannian almost-product manifold* $(M, \mathcal{P}, g)$ *is said to be natural if* $D\mathcal{P}=0$ *and* $Dg=0$.

Since for an arbitrary linear connection

$$(D_X\tilde{g})(Y, Z)=(D_Xg)(\mathcal{P}Y, Z)+g((D_X\mathcal{P})Y, Z),$$

it follows immediately that the linear connection D is natural iff $Dg=D\tilde{g}=0$.

In this section we shall study the set of the natural connections on a Riemannian almost-product manifold $(M, \mathcal{P}, g)$.

Let T be the torsion tensor of a natural connection D. We denote by the same letter the corresponding tensor of type $(0, 3)$; $T(X, Y, Z)=g(T(X, Y), Z)$. The corresponding tensors p_i $(=1, 2, 3, 4)$ associated with T are defined in Section 2.

T h e o r e m 3. *The linear connection D with torsion tensor T is natural iff*

$$v_1(X, Y, Z)=\frac{1}{4}\{-\Phi(X, Y, Z)+\Phi(Y, Z, X)-\Phi(X, \mathcal{P}Y, \mathcal{P}Z) -\Phi(Y, \mathcal{P}Z, \mathcal{P}X)+2\Phi(Z, \mathcal{P}X, \mathcal{P}Y)\}, \tag{9}$$

$$p_3(X, Y, Z)=-\frac{1}{2}\{\Phi(Z, X, Y)+\Phi(Z, \mathcal{P}X, \mathcal{P}Y)\}. \tag{10}$$

P r o o f. We denote

$$D_XY-\nabla_XY=S(X, Y). \tag{11}$$

Since $\nabla g=0$, the condition $Dg=0$ is equivalent to

$$g(S(X, Y), Z)+g(S(X, Z), Y)=0.$$

Hence

$$g(S(X, Y), Z)=\frac{1}{2}\{T(X, Y, Z)-T(Y, Z, X)+T(Z, X, Y)\}, \tag{12}$$

which is the Theorem of Hayden [5].

Further we denote

$$D_XY-\tilde{\nabla}_XY=\tilde{S}(X, Y). \tag{13}$$

Analogously, because of $\tilde{\nabla}\tilde{g}=0$, the condition $D\tilde{g}=0$ is equivalent to

$$g(\tilde{S}(X, Y), \mathcal{P}Z) + g(\tilde{S}(X, Z), \mathcal{P}Y)=0.$$

Then

$$(14) \qquad g(\tilde{S}(X, Y), Z)=\frac{1}{2}\{T(X, Y, Z)-T(Y, \mathcal{P}Z, \mathcal{P}X)+T(\mathcal{P}Z, X, \mathcal{P}Y)\}.$$

Using (11), (13) and (6), we get

$$\Phi(X, Y, Z)=g(S(X, Y, Z)-g(\tilde{S}(X, Y), Z).$$

From the last equality and (12), (14) it follows

$$(15) \quad \Phi(X, Y, Z)=\frac{1}{2}\{-T(Y, Z, X)+T(Z, X, Y)+T(Y, \mathcal{P}Z, \mathcal{P}X)-T(\mathcal{P}Z, X, \mathcal{P}Y)\}.$$

From Theorem 1 and (15), using Lemma 4, we get

$$\Phi(X, Y, Z)=\frac{1}{2}\{-p_1(Y, Z, X)+p_1(Z, X, Y)+p_1(Y, \mathcal{P}Z, \mathcal{P}X)-p_1(\mathcal{P}Z, X, \mathcal{P}Y)$$
$$-2p_3(Y, Z, X)+2p_3(Z, X, Y)\}.$$

From the last formula we easily find (9) and (10).

For the inverse, let the projections p_1 and p_3 of T be (9) and (10). Substituting this projections into $T=p_1+p_2+p_3+p_4$ we get (15), which implies D is natural.

Theorem 4. *Let $Q'(X, Y, Z)$ and $Q''(X, Y, Z)$ be tensor fields on $(M, \mathcal{P}, g)$ having the properties*:

$$\text{(i)} \qquad Q'(X, Y, Z)=-Q'(Y, X, Z)=-Q'(\mathcal{P}X, \mathcal{P}Y, Z),$$
$$Q'(\mathcal{P}X, Y, Z)+Q'(\mathcal{P}Y, Z, X)+Q'(\mathcal{P}Z, X, Y)=0,$$

$$\text{(ii)} \quad Q''(X, Y, Z)=-Q''(Y, X, Z),\ Q''(\mathcal{P}X, Y, Z)=Q''(X, \mathcal{P}Y, Z)=Q''(X, Y, \mathcal{P}Z).$$

Then there exists a unique natural connection D such that the components p_2 and p_4 of its torsion tensor T are the given tensors Q' and Q'', respectively.

Proof. *Existence.* Taking into account Theorem 3, we construct the connection D

$$g(D_X, Y, Z)=g(\nabla_X Y, Z) + \frac{1}{4}\{\Phi(X, Y, Z) - 2\Phi(Z, X, Y) - \Phi(X, \mathcal{P}Y, \mathcal{P}Z)\}$$
$$+\frac{1}{2}\{Q'(X, Y, Z)-Q'(Y, Z, X)+Q'(Z, X, Y)\}$$
$$+\frac{1}{2}\{Q''(X, Y, Z)-Q''(Y, Z, X)+Q''(Z, X, Y)\}.$$

It is easy to be checked that D is a natural connection with torsion tensor

$$T(X, Y, Z)=\frac{1}{4}\{\Phi(Y, Z, X)-\Phi(Z, X, Y)-\Phi(Y, \mathcal{P}Z, \mathcal{P}X)+\Phi(\mathcal{P}Z, X, \mathcal{P}Y)\}$$
$$+Q'(X, Y, Z)+Q''(X, Y, Z).$$

From Theorem 2 it follows that $p_2=Q'$ and $p_4=Q''$.

Uniqueness. Let D' be another natural connection with torsion tensor T' satisfying the conditions of the theorem. If p_i' $(i=1, 2, 3, 4)$ are the components of T', Theorem 3 implies $p'_1=p_1$, $p'_3=p_3$. Since $p'_2=Q'=p_2$, $p'_4=Q''=p_4$, then $T'=T$. Hence $D'\equiv D$.

Theorem 4 gives a one-to-one mapping of the set of the natural connections onto the set of pairs (Q', Q'') having the properties (i) and (ii).

Definition 4. *A natural connection D with torsion tensor T is said to be canonical if*

$$T(X, Y, Z)+T(Y, Z, X)+T(\mathscr{P}X, Y, \mathscr{P}Z)+T(Y, \mathscr{P}Z, \mathscr{P}X)=0. \tag{16}$$

Taking into account Lemma 5, we find that (16) is equivalent to $T\in\mathscr{T}_1\oplus\mathscr{T}_3$, i. e. $p_2=p_4=0$. Applying Theorem 4, we check

Theorem 5. *There exists a unique canonical connection on a Riemannian almost-product manifold.*

From Theorem 4 and (7), (11), (12), (13), (14) it follows that the cannonical connection D on a Riemannian almost-product manifold is given by one of the following equalities:

$$g(D_X Y, Z)=g(\nabla_X Y, Z)+\frac{1}{4}\{\Phi(X, Y, Z)-2\Phi(Z, X, Y)-\Phi(X, \mathscr{P}Y, \mathscr{P}Z)\},$$

$$g(D_X Y, Z)=g(\tilde{\nabla}_X Y, Z)-\frac{1}{4}\{3\Phi(X, Y, Z)+2\Phi(Z, X, Y)+\Phi(X, \mathscr{P}Y, \mathscr{P}Z)\}, \tag{17}$$

$$g(D_X Y, Z)=\frac{1}{2}g(\tilde{\nabla}_X Y+\nabla_X Y, Z)-\frac{1}{4}\{\Phi(Z, X, Y)-\Phi(\mathscr{P}Z, X, \mathscr{P}Y)\}.$$

The torsion tensor T of D is given by

$$T(X, Y, Z)=\frac{1}{4}\{\Phi(Y, Z, X)-\Phi(Z, X, Y)-\Phi(Y, \mathscr{P}Z, \mathscr{P}X)+\Phi(\mathscr{P}Z, X, \mathscr{P}Y)\}. \tag{18}$$

4. Canonical conformal group. In this section we consider the group of transformations of the canonical connections generated by the general conformal transfor mations of the metric.

Let $(M, \mathscr{P}, g)$ be a Riemannian almost-product manifold. The general conformal transformations of the metric g are defined by

$$\bar{g}=u(\operatorname{ch}v\,.\,g+\operatorname{sh}v\,.\,\tilde{g}),\quad \tilde{\bar{g}}=u(\operatorname{sh}v\,.\,g+\operatorname{ch}v\,.\,\tilde{g}), \tag{19}$$

where u, v are differentiable functions on M. By $v=0$ (19) is the usual conformal change of the metric g. The manifold $(M, \mathscr{P}, \bar{g})$ is also a Riemannian almost-product manifold.

We take in consideration only local conformal transformations.

Let $(M, \mathscr{P}, g)$ and $(M, \mathscr{P}, \bar{g})$ be conformally related as in (19). The Levi—Civita connections of $\bar{g}$ and $\tilde{\bar{g}}$ are denoted by $\bar{\nabla}$ and $\tilde{\bar{\nabla}}$ respectively. Applying the formula

$$2g(\nabla_X Y, Z)=Xg(Y, Z)+Yg(X, Z)-Zg(X, Y)+g([X, Y], Z)-g([Y, Z], X)+g([Z, X], Y)$$

to $\bar{\nabla}$ and $\tilde{\bar{\nabla}}$ we find

$$\begin{aligned}2\bar{g}(\bar{\nabla}_X Y, Z) &= 2u\operatorname{ch}v\,.\,g(\nabla_X Y, Z)+2u\operatorname{sh}v\,.\,\tilde{g}(\tilde{\nabla}_X Y, Z)\\ &+X(u\operatorname{ch}v)g(Y, Z)+Y(u\operatorname{ch}v)g(X, Z)-Z(u\operatorname{ch}v)g(X, Y)\\ &+X(u\operatorname{sh}v)\tilde{g}(Y, Z)+Y(u\operatorname{sh}v)\tilde{g}(X, Z)-Z(u\operatorname{sh}v)\tilde{g}(X, Y),\end{aligned} \tag{20}$$

and

(21) $$2\tilde{\tilde{g}}(\tilde{\tilde{\nabla}}_X Y,\ Z)=2u\,\mathrm{sh}\,v\,.\,g(\nabla_X Y,\ Z)+2u\,\mathrm{ch}\,v\,.\,\tilde{g}(\tilde{\nabla}_X Y,\ Z)$$
$$+X(u\,\mathrm{sh}\,v)g(Y,\ Z)+Y(u\,\mathrm{sh}\,v)g(X,\ Z)-Z(u\,\mathrm{sh}\,v)\,g(X,\ Y)$$
$$+X(u\,\mathrm{ch}\,v)\tilde{g}(Y,Z)+Y(u\,\mathrm{ch}\,v)\,\tilde{g}(X,Z)-Z(u\,\mathrm{ch}\,v)\,\tilde{g}(X,\ Y).$$

From (6) and (20) we have

(22) $$\bar{\nabla}_X Y=\nabla_X Y-\mathrm{sh}^2 v\,.\,\Phi(X,\ Y)+\frac{1}{2}\,\mathrm{sh}\,2v\,.\,\mathscr{P}\Phi(X,\ Y)+\frac{1}{2}\{\frac{du(X)}{u}Y+\frac{du(Y)}{u}X$$
$$+dv(X)\mathscr{P}Y+dv(Y)\mathscr{P}X\}+\frac{1}{2}\,g(X,\ Y)\{-\mathrm{ch}^2 v\frac{\mathrm{grad}\,u}{u}+\frac{1}{2}\,\mathrm{sh}\,2v\frac{\mathscr{P}\,\mathrm{grad}\,u}{u}$$
$$-\frac{1}{2}\,\mathrm{sh}\,2v\,.\,\mathrm{grad}\,v+\mathrm{sh}^2 v\,.\,\mathscr{P}\,\mathrm{grad}\,v\}+\frac{1}{2}\tilde{g}(X,\ Y)\{-\frac{1}{2}\,\mathrm{sh}\,2v\frac{\mathrm{grad}\,u}{u}$$
$$+\mathrm{sh}^2 v\frac{\mathscr{P}\,\mathrm{grad}\,u}{u}-\mathrm{ch}^2 v\,.\,\mathrm{grad}\,v+\frac{1}{2}\,\mathrm{sh}\,2v\,.\,\mathscr{P}\,\mathrm{grad}\,v\}.$$

Analogously, from (6) and (21) we check

(23) $$\tilde{\tilde{\nabla}}_X Y=\tilde{\nabla}_X Y+\mathrm{sh}^2 v\,.\,\Phi(X,\ Y)-\frac{1}{2}\,\mathrm{sh}\,2v\,.\,\mathscr{P}\Phi(X,\ Y)+\frac{1}{2}\{\frac{du(X)}{u}Y+\frac{du(Y)}{u}X$$
$$+dv(X)\mathscr{P}Y+dv(Y)\mathscr{P}X\}+\frac{1}{2}g(X,\ Y)\{\mathrm{sh}^2 v\frac{\mathrm{grad}\,u}{u}-\frac{1}{2}\,\mathrm{sh}\,2v\frac{\mathscr{P}\,\mathrm{grad}\,u}{u}$$
$$+\frac{1}{2}\,\mathrm{sh}\,2v\,.\,\mathrm{grad}\,v-\mathrm{ch}^2 v\,.\,\mathscr{P}\,\mathrm{grad}\,v\}+\frac{1}{2}\tilde{g}(X,\ Y)\{\frac{1}{2}\,\mathrm{sh}\,2v\frac{\mathrm{grad}\,u}{u}$$
$$-\mathrm{ch}^2 v\frac{\mathscr{P}\,\mathrm{grad}\,u}{u}+\mathrm{sh}^2 v\,.\,\mathrm{grad}\,v-\frac{1}{2}\,\mathrm{sh}\,2v\,.\,\mathscr{P}\,\mathrm{grad}\,v\}.$$

Subtracting (22) from (23) and taking into account the defining condition (6) of Φ, we obtain

(23) $$\bar{\Phi}(X,\ Y)=\mathrm{ch}\,2v\,.\,\Phi(X,\ Y)-\mathrm{sh}\,2v\,.\,\mathscr{P}\Phi(X,\ Y)+\frac{1}{2}g(X,\ Y)\{\mathrm{ch}\,2v\frac{\mathrm{grad}\,u}{u}$$
$$-\mathrm{sh}\,2v\frac{\mathscr{P}\mathrm{grad}\,u}{u}+\mathrm{sh}\,2v\,.\,\mathrm{grad}\,v-\mathrm{ch}\,2v\,.\,\mathscr{P}\,\mathrm{grad}\,v\}+\frac{1}{2}\,\tilde{g}(X,\ Y)\{\mathrm{sh}\,2v\frac{\mathrm{grad}\,u}{u}$$
$$-\mathrm{ch}\,2v\frac{\mathscr{P}\mathrm{grad}\,u}{u}+\mathrm{ch}\,2v\,.\,\mathrm{grad}\,v-\mathrm{sh}\,2v\,.\,\mathscr{P}\,\mathrm{grad}\,v\}.$$

Adding (22) and (23), we get

(25) $$\frac{1}{2}(\tilde{\tilde{\nabla}}_X Y+\bar{\nabla}_X Y)=\frac{1}{2}(\tilde{\nabla}_X Y+\nabla_X Y)+\frac{1}{2}\{\frac{du(X)}{u}Y+\frac{du(Y)}{u}X+dv(X)\mathscr{P}Y+dv(Y)\mathscr{P}X\}$$
$$-\frac{1}{4}g(X,\ Y)\{\frac{\mathrm{grad}\,u}{u}+\mathscr{P}\,\mathrm{grad}\,v\}-\frac{1}{4}\tilde{g}(X,\ Y)\{\frac{\mathscr{P}\mathrm{grad}\,u}{u}+\mathrm{grad}\,v\}.$$

The third equality of (17) and (25) imply

Theorem 6. *The group of the general conformal transformations* (19) *generates the group of conformal transformations of the canonical connection*

$$(26)\quad \bar{D}_X Y = D_X Y + \frac{1}{2}\{\frac{du(X)}{u}Y + dv(X)\mathscr{P}Y\} + \frac{1}{4}\{\frac{du(Y)}{u} + dv(\mathscr{P}Y)\}X + \frac{1}{4}\{\frac{du(\mathscr{P}Y)}{u}$$

$$+ dv(Y)\}\mathscr{P}X - \frac{1}{4}g(X, Y)\{\frac{\operatorname{grad} u}{u} + \mathscr{P}\operatorname{grad} v\} - \frac{1}{4}\tilde{g}(X, Y)\{\frac{\mathscr{P}\operatorname{grad} u}{u} + \operatorname{grad} v\}.$$

As a consequence of (26) the torsion tensors $\bar{T}$ and T of $\bar{D}$ and D, respectively, are related as follows:

$$(27)\quad \bar{T}(X, Y) = T(X, Y) - \frac{1}{4}\{\frac{du(Y)}{u} - dv(\mathscr{P}Y)\}X + \frac{1}{4}\{\frac{du(X)}{u} - dv(\mathscr{P}X)\}Y$$

$$+ \frac{1}{4}\{\frac{du(\mathscr{P}Y)}{u} - dv(Y)\}\mathscr{P}X - \frac{1}{4}\{\frac{du(\mathscr{P}X)}{u} - dv(X)\}\mathscr{P}Y.$$

The group of the usual conformal transformations of the metric g is characterized by the conditions $u = e^{2f}$, $v = 0$ in (19) In this case the formulas (24), (26) and (27) are reduced to

$$\bar{\Phi}(X, Y) = \Phi(X, Y) + g(X, Y)\operatorname{grad} f - \tilde{g}(X, Y)\mathscr{P}\operatorname{grad} f,$$

$$\bar{D}_X Y = D_X Y + df(X)Y + \frac{1}{2}df(Y)X - \frac{1}{2}g(X, Y)\operatorname{grad} f$$

$$+ \frac{1}{2}df(\mathscr{P}Y)\mathscr{P}X - \frac{1}{2}g(\mathscr{P}X, Y)\mathscr{P}\operatorname{grad} f,$$

$$\bar{T}(X, Y) = T(X, Y) - \frac{1}{2}\{df(Y)X - df(X)Y - df(\mathscr{P}Y)\mathscr{P}X + df(\mathscr{P}X)\mathscr{P}Y\}.$$

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