Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

A DIAGONAL PRINCIPLE FOR GENERALIZED SEQUENCES AND SOME OF ITS APPLICATIONS

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A diagonal principle is proved in the paper. This principle concerns choosing subsequences with suitable properties from generalized sequences. Some applications of the principle are exposed showing that it is essentially more general than the Tychonoff Theorem (for example, the existence of convergent generalized subsequences is proved in some cases when the convergence is not a topological one or compactness is not necessarily present).

Let $\{a_\alpha\}$ be a generalized sequence, where α ranges over A, and A is a directed system.[1] We shall say that $\{a_{\alpha_\beta}\}$ is a *subsequence* of $\{a_\alpha\}$ if α_β is a monotonically increasing function whose domain is some directed system and whose range is some cofinal subset of A.[2] If D is a non-empty subset of A such that $\alpha' \in D$ whenever $\alpha \in D, \alpha' \in A, \alpha \leqq \alpha'$, then the generalized sequence $\{a_\alpha\}_{\alpha \in D}$ is called a *tail* of the generalized sequence $\{a_\alpha\}$.[3]

A class Σ of generalized sequences is called *stable* if each subsequence of any sequence belonging to Σ also belongs to Σ.[4] Here are some examples of stable classes: the class of all convergent generalized sequences in a given topological space, the class of all generalized Cauchy sequences in a given metric space, the class of all bounded generalized sequences in such a space,[5] the class of all sequnces whose members belong to a fixed set, the class of all monotonic generalized sequences of real numbers.

To be given a *system* $\{\Sigma_t\}_{t \in M}$ *of stable classes*, it means that a set M is given, and a stable class Σ_t corresponds to each t in M.[6]

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* This paper is prepared for publication by Dimiter Skordev on the basis of the annual scientific reports [1-3] written by the late Professor Tagamlitzki in Bulgarian (the paper begins with the text of the last of these reports). Since the reports are not in a form intended for publication, and they are not completely consentient in terminology and notation, some minor changes in their text have been done and some footnotes have been appended while preparing the present paper (the synopsis at the beginning of the paper is also written by D. Skordev). Let us note that the only known publication of Tagamlitzki on the subject is the abstract [4].

[1] *A directed system* is, by the definition adopted by Tagamlitzki, a non-empty set supplied with a reflexive and transitive binary relation $\leqq$ on it such that, for every two elements α_1 and α_2 of the set, there is an element α of it satisfying the conditions $\alpha_1 \leqq \alpha$, $\alpha_2 \leqq \alpha$ (Tagamlitzki calls such a relation an *order* or, more precisely, a *partial order* on the given set; directed systems are sometimes called by him directed to the right).

[2] A subset A' of A is called *cofinal with* A if for each α in A there is an element α' of A' such that $\alpha \leqq \alpha'$. In the original text, functions whose ranges are cofinal subsets of A are called boundlessly increasing.

[3] This definition and the term "tail" are not to be found in the original text. Instead of saying "a tail of $\{a_\alpha\}$", Tagamlitzki said "$\{a_\alpha\}$ considered from some place on".

[4] It is convenient to interpret all set-theoretic considerations in this paper in the sense of Neumann-Bernays-Gödel axiomatization of Set Theory. It is obvious that a non-empty stable class of generalized sequences is always a proper class (i. e. a class which is not a set).

[5] In the original text, these examples are given without specifying the sort of the spaces which contain the members of the considered sequences.

[6] In Neumann-Bernays-Gödel axiomatization of Set Theory, such a system $\{\Sigma_t\}_{t \in M}$ of classes can be represented by a class S consisting of ordered pairs with first members in M and satisfying the condition that, for any fixed t in M, the second members of all pairs in S with first member t form a

SERDICA Bulgaricae mathematicae publicationes. Vol. 16, 1990, p. 201-209.

Diagonal Principle. Let $\{\Sigma_t\}_{t \in M}$ be a system of stable classes, and $\{a_\alpha\}$ be a generalized sequence such that, for each t in M and every subsequence $\{a_{\alpha_\beta}\}$ of $\{a_\alpha\}$, a sub-subsequence $\{a_{\alpha_{\beta_\gamma}}\}$ (depending on t in general) can be chosen which belongs to Σ_t.[7] Then there is a subsequence $\{a_{\alpha_\delta}\}$ of $\{a_\alpha\}$ (not depending on t) such that, for each t in M, some tail of $\{a_{\alpha_\delta}\}$ belongs to Σ_t (this tail may, in general, depend on t).

Proof. Let

(1) $$t_1, t_2, \ldots$$

be a well-ordering of M, and let I be the set of ordinal numbers occurring as indices in (1).[8] Let A be the directed system where the index α of $\{a_\alpha\}$ ranges. For each z in I, we shall define a directed system B_z, and a function ρ_{xy} from B_y into B_x will be defined for each x, y in I with $x \leq y$, so that the following conditions will be satisfied:

1. Each directed system B_z has at least two elements,[9] and there is a unique first element O_z in it (denoted by O, for short).[10]

2. For each x, y in I with $x \leq y$, the equality $\rho_{xy}(O) = O$ holds.

3. Each function ρ_{xy} is monotonically increasing, and its range is a cofinal subset of B_x.

4. Whenever $x, y, z \in I$, $x \leq y \leq z$, $\beta \in B_z$ and $\rho_{yz}(\beta) \neq O$, then $\rho_{xy}(\rho_{yz}(\beta)) = \rho_{xz}(\beta)$ In addition, $\rho_{zz}(\beta) = \beta$ for all z in I and all $\beta \in B_z$.[11]

5. For each z in I, the image of $B_z \setminus \{O\}$ under ρ_{1z} is contained in A,[12] and the generalized sequence $\{a_{\rho_{1z}(\beta)}\}$, where β ranges over $B_z \setminus \{O\}$, belongs to Σ_{t_z}.

We shall define B_z and ρ_{xy} by induction. For that purpose, we set $B_1 = \{O\} \cup A$, where O is some object not belonging to A. Then, for all β_1, β_2 in B_1, we adopt that $\beta_1 \leq \beta_2$ in B_1 iff either $\beta_1, \beta_2 \in A$ and $\beta_1 \leq \beta_2$ in A, or $\beta_1 = O$. Of course, we set $\rho_{11}(\beta) = \beta$ for all β in B_1.

Now suppose that $p \in I$, $p > 1$, and B_z, ρ_{xy} are defined for all x, y, z in I with $z < p$ and $x \leq y < p$.

Let C_p be the set of all pairs $\sigma = (y, \beta)$, where $y \in I$, $y < p$ and $\beta \in B_y$. A binary relation $\leq$ on C_p is introduced by the convention that $(y_1, \beta_1) \leq (y_2, \beta_2)$ iff $y_1 \leq y_2$ and $\beta_1 \leq \rho_{y_1 y_2}(\beta_2)$. We shall show that C_p considered with the relation $\leq$ is a directed system. The reflexivity of the relation $\leq$ follows from the second clause of Condition 4.[13] The transitivity of the relation is seen as follows. Let

$$(y_1, \beta_1) \leq (y_2, \beta_2) \leq (y_3, \beta_3),$$

i. e. $y_1 \leq y_2$, $\beta_1 \leq \rho_{y_1 y_2}(\beta_2)$, $y_2 \leq y_3$, $\beta_2 \leq \rho_{y_2 y_3}(\beta_3)$. Then $y_1 \leq y_3$ and $\beta_1 \leq \rho_{y_1 y_2}(\rho_{y_2 y_3}(\beta_3))$. If $\rho_{y_2 y_3}(\beta_3) \neq O$, then $\rho_{y_1 y_2}(\rho_{y_2 y_3}(\beta_3)) = \rho_{y_1 y_3}(\beta_3)$, i. e. $\beta_1 \leq \rho_{y_1 y_3}(\beta_3)$. If $\rho_{y_2 y_3}(\beta_3) = O$ then $\beta_2 = O$

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stable class Σ_t of generalized sequences. The words "a set M is given" are added at the preparation of the paper.

[7] This possibility of choosing has to be interpreted in the following "functional" manner: there is a class F of ordered triples such that, for each t in M and every subsequence $\{a_{\alpha_\beta}\}$ of $\{a_\alpha\}$, there is exactly one sub-subsequence $\{a_{\alpha_{\beta_\gamma}}\}$ satisfying the condition $(t, \{a_{\alpha_\beta}\}, \{a_{\alpha_{\beta_\gamma}}\}) \in F$, and this sub-subsequence belongs to Σ_t.

[8] In the original text, no notation of this set of ordinal numbers is introduced.

[9] The condition that B_z has at least two elements is not explicitly formulated in the original text.

[10] The word "unique" is added at the preparation of the paper.

[11] The condition $\rho_{zz}(\beta) = \beta$ is not explicitly formulated in the original text.

[12] This condition is also not explicitly formulated in the original text.

[13] In the original text, there is no explicit mentioning of the verification of the reflexivity.

hence (by Condition 2) $\rho_{y_1y_2}(\beta_2)=O$, and therefore $\beta_1=O$, i. e. $\beta_1\leq\rho_{y_1y_3}(\beta_3)$ again. So

$$(y_1, \beta_1)\leq(y_3, \beta_3)$$

in both cases. Thus the transitivity of the relation $\leq$ on C_p is established. Consider now arbitrary elements (y_1, β_1) and (y_2, β_2) of C_p. We shall construct an element (y, β) of C_p such that

$$(y_1, \beta_1)\leq(y, \beta), (y_2, \beta_2)\leq(y, \beta).$$

Let, for example, $y_1\leq y_2$. By Condition 3, the range of $\rho_{y_1y_2}$ is a cofinal subset of B_{y_1}, and therefore we can choose an element β_3 of B_{y_2} such that $\beta_1\leq\rho_{y_1y_2}(\beta_3)$. Now choose β from B_{y_2} so that $\beta_2\leq\beta$ and $\beta_3\leq\beta$. Then, by the monotonicity of $\rho_{y_1y_2}$, we get $\beta_1\leq\rho_{y_1y_2}(\beta)$. Hence

$$(y_1, \beta_1)\leq(y_2, \beta).$$

On the other hand, we have also

$$(y_2, \beta_2)\leq(y_2, \beta)$$

since $\beta_2\leq\beta$ and $\rho_{y_2y_2}(\beta)=\beta$.

Now set

$$\psi(\sigma)=\rho_{1y}(\beta),$$

where $\sigma=(y, \beta)\in C_p$. Then ψ is a function defined on C_p, and the range of ψ is contained in B_1. The function ψ is monotonically increasing. Indeed, let

$$\sigma_1=(y_1, \beta_1)\leq\sigma_2=(y_2, \beta_2)$$

be elements of C_p. Then $y_1\leq y_2$ and $\beta_1\leq\rho_{y_1y_2}(\beta_2)$. Therefore, if $\rho_{y_1y_2}(\beta_2)\neq O$ then

$$\rho_{1y_1}(\beta_1)\leq\rho_{1y_1}(\rho_{y_1y_2}(\beta_2))=\rho_{1y_2}(\beta_2),$$

i. e. $\psi(\sigma_1)\leq\psi(\sigma_2)$, On the other hand, if $\rho_{y_1y_2}(\beta_2)=O$ then $\beta_1=O$, and, consequently $\rho_{1y_1}(\beta_1)=O$; hence $\psi(\sigma_1)=O$, and $\psi(\sigma_1)\leq\psi(\sigma_2)$ again.

Denote by D the set of the elements $\sigma=(y, \beta)$ of C_p such that $\beta\neq O$. We shall show that the image of D under ψ is equal to A. Indeed, the values of $\psi(\sigma)$ for $\sigma\in D$ belong to A due to the first part of Condition 5. On the other hand, let $\alpha_1\in A$. Then $\sigma_1=(1, \alpha_1)$ is a pair from D, and

$$\psi(\sigma_1)=\rho_{11}(\alpha_1)=\alpha_1.$$ [14]

So $\{a_{\psi(\sigma)}\}_{\sigma\in D}$ is a subsequence of $\{a_\alpha\}$.[15] Making use of the assumption of the theorem, we choose a sub-subsequence $\{a_{\psi(\sigma_\gamma)}\}$ which belongs to Σ_{t_p}. Let Γ be the directed system where γ ranges; then $\sigma_\gamma\in D$ for all γ in Γ. Set $B_p=\{O\}\cup\Gamma$, where O is some object not belonging to Γ, and adopt that, for all β_1, β_2 in B_p, $\beta_1\leq\beta_2$ in B_p iff either $\beta_1, \beta_2\in\Gamma$ and $\beta_1\leq\beta_2$ in Γ, or $\beta_1=O$. Let $\sigma_\gamma=(y_\gamma, \beta_\gamma)$ for each γ in Γ. For all x in I with $x<p$, we set

$$\rho_{xp}(\gamma)=\rho_{xy_\gamma}(\beta_\gamma)$$

if $\gamma\in\Gamma$ and $x\leq y_\gamma$. Otherwise, i. e. if $\gamma=O$ or $x>y_\gamma$, we set

$$\rho_{xp}(\gamma)=O.$$

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[14] In the original text, the set D is not considered, and, instead of the last statement, it is proved that the range of ψ is a cofinal subset of B_1 (in fact, coinciding with B_1).

[15] In the original text, $\{a_{\psi(\sigma)}\}$ is considered instead of $\{a_{\psi(\sigma)}\}_{\sigma\in D}$. We regard this as somewhat vague since $\psi(\sigma)\notin A$ for some values of σ in C_p, and therefore we impose the restriction $\sigma\in D$.

Finally, we set

$$\rho_{pp}(\gamma)=\gamma$$

for all γ in B_p.

Now we shall show that Conditions 1, 2, 3, 4, 5 are satisfied. Concerning Conditions 1 and 2, this is obvious. In the case of Condition 3, we have to show that, for any fixed x in I with $x\leqq p$, the function ρ_{xp} is monotonically increasing and its range is cofinal with B_x. If $x=p$, then this is obvious. Let $x<p$ and $\beta_1\leqq\beta_2$ $(\beta_1, \beta_2\in B_p)$.[16] If $\beta_1=O$ then $\rho_{xp}(\beta_1)=O$ and therefore $\rho_{x_p}(\beta_1)\leqq\rho_{x_p}(\beta_2)$. Consider now the case of $\beta_1\neq O$. Then $\beta_2\neq 0$ too, and we may write: $\beta_1=\gamma_1\in\Gamma$, $\beta_2=\gamma_2\in\Gamma$. If $x>y_{\gamma_1}$, then $\rho_{xp}(\gamma_1)=0$ and therefore $\rho_{xp}(\beta_1)\leqq p_{xp}(\beta_2)$ again. Now consider the case of $x\leqq y_{\gamma_1}$. But means $\gamma_1\leqq\gamma_2$, and this implies $\sigma_{\gamma_1}\leqq\sigma_{\gamma_2}$, i. e. $(y_{\gamma_1}, \beta_{\gamma_1})\leqq(y_{\gamma_2}, \beta_{\gamma_2})$. Consequently, $y_{\gamma_1}\leqq y_{\gamma_2}$ $\beta_1\leqq\beta_2$ and therefore $x\leqq y_{\gamma_2}$. Hence

$$\rho_{xp}(\beta_1)=\rho_{xy_{\gamma_1}}(\beta_{\gamma}),$$

$$\rho_{xp}(\beta_2)=\rho_{xy_{\gamma_2}}(\beta_{\gamma_2}).$$

On the other hand, $\sigma_{\gamma_1}\leqq\sigma_{\gamma_2}$ entails also

$$\beta_{\gamma_1}\leq\rho_{y_{\gamma_1}y_{\gamma_2}}(\beta_{\gamma_2}). \tag{2}$$

But $\beta_{\gamma_1}\neq O$ since $\sigma_{\gamma_1}=(y_{\gamma_1}, \beta_{\gamma_1})$ belongs to D. So (2) implies

$$\rho_{y_{\gamma_1}y_{\gamma_2}}(\beta_{\gamma_2})\neq O,$$

and we have

$$\rho_{xp}(\beta_1)=\rho_{xy_{\gamma_1}}(\beta_{\gamma_1})\leqq\rho_{xy_{\gamma_1}}(\rho_{y_{\gamma_1}y_{\gamma_2}}(\beta_{\gamma_2}))=\rho_{xy_{\gamma_2}}(\beta_{\gamma_2})=\rho_{xp}(\beta_2).$$

Thus the monotonic increasing of ρ_{xp} is demonstrated.

Now we shall show that the range of ρ_{xp} is cofinal with B_x. Indeed, let $\beta_0\in B_x$. If $\beta_0\neq O$ then (x, β_0) belongs to D, and therefore $(x, \beta_0)\leqq\sigma_\gamma$ for some γ in Γ. Since $\sigma_\gamma=(y_\gamma, \beta_\gamma)$, we have the inequality $\beta_0\leq\rho_{xy_\gamma}(\beta_\gamma)$, i. e. $\beta_0\leq\rho_{xp}(\gamma)$. On the other hand if $\beta_0=O$ then $\beta_0\leqq\rho_{xp}(O)$. The cofinality of the range of ρ_{xp} is thus proved.

We now go to the verification of Condition 4. We have to show that, whenever $x, y\in I$, $x\leqq y\leqq p$, $\gamma\in B_p$ and $\rho_{yp}(\gamma)\neq O$, then

$$\rho_{xy}(\rho_{yp}(\gamma))=\rho_{xp}(\gamma) \tag{3}$$

the second clause of Condition 4 is obviously satisfied).

In case $y=p$, the equality (3) is obvious, due to the convention $\rho_{pp}(\gamma)=\gamma$. Consider now the case of $y<p$. From $\rho_{yp}(\gamma)\neq O$ we conclude that $y\leqq y_\gamma$ and therefore $x\leqq y_\gamma$. So we get

$$\rho_{yp}(\gamma)=\rho_{yy_\gamma}(\beta_\gamma),\ \rho_{xp}(\gamma)=\rho_{xy_\gamma}(\beta_\gamma),$$

and (3) takes the form

$$\rho_{xy}(\rho_{yy_\gamma}(\beta_\gamma))=\rho_{xy_\gamma}(\beta_\gamma).$$

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[16] Strictly speaking, the present use of notations as β_1 and β_2 is not completely legitimate after the notation β_γ was introduced for all γ in Γ. The trouble is that, for example, 1 or 2 could happen to belong to Γ, and then ambiguity could arise. We, however, regard this notational problem as non-essential, and therefore we preserved the notations from the original text.

The last equality is true by the inductive assumption, since

$$\rho_{yy_\gamma}(\beta_\gamma)=\rho_{yp}(\gamma)\neq O.$$

Thus Condition 4 is verified.

At last, it remains to verify Condition 5, i. e. to show that $\rho_{1p}(\gamma)\in A$ for all $\gamma\in\Gamma$, and

$$\{a_{\rho_{1p}}(\gamma)\}_{\gamma\in\Gamma}\in\Sigma_{t_p}.$$

For all $\gamma\in\Gamma$, we have

$$\rho_{1p}(\gamma)=\rho_{1y_\gamma}(\beta_\gamma),$$

since the condition $1\leqq y_\gamma$ is satisfied. Therefore

$$\rho_{1p}(\gamma)=\psi(\sigma_\gamma)\in A$$

for all γ in Γ, and

$$\{a_{\rho_{1p}}(\gamma)\}_{\gamma\in\Gamma}=\{a_{\psi(\sigma\gamma)}\}\in\Sigma_{t_p}.$$

So the inductive construction of the sets B_z and of the functions ρ_{xy} is completed.

After this preliminary work, we are ready to prove the Diagonal Principle. For that purpose, let C denote the set of all pairs $\sigma=(y,\beta)$, where $y\in I$ and $\beta\in B_y$. We introduce an order in C in the same way as we did in C_p. Then C becomes a directed system. We set

$$\psi_x(\sigma)=\rho_{xy}(\beta).$$

This definition is correct when $y\geqq x$, and, for any fixed x, the inequality $y\geqq x$ is surely satisfied if the value of σ is sufficiently great. We shall show that, for any fixed x, the function ψ_x is monotonically increasing and its range is equal to B_x.[17] Indeed, let $\sigma_1\leqq\sigma_2$, where $\sigma_1=(y_1,\beta_1)$, $\sigma_2=(y_2,\beta_2)$. Then $\beta_1\leqq\rho_{y_1y_2}(\beta_2)$. If $\rho_{y_1y_2}(\beta)_2\neq O$ then

$$\psi_x(\sigma_1)=\rho_{xy_1}(\beta_1)\leqq\rho_{xy_1}(\rho_{y_1y_2}(\beta_2))=\rho_{xy_2}(\beta_2)=\psi_x(\sigma_2).$$

On the other hand, if $\rho_{y_1y_2}(\beta_2)=O$ then $\beta_1=O$ and $\psi_x(\sigma_1)=O$, i. e. $\psi_x(\sigma_1)\leqq\psi_x(\sigma_2)$ again

For showing that the range of ψ_x is equal to B_x, take $\beta_0\in B_x$ and set $\sigma_0=(x,\beta_0)$ Then

$$\psi_x(\sigma_0)=\rho_{xx}(\beta_0)=\beta_0.$$

Let us fix some x in I. Then, for all sufficiently great values of $\sigma=(y,\beta)$, we have $\psi_x(\sigma)\neq O$, i. e. $\rho_{xy}(\beta)\neq O$, and, consequently,

$$\psi_1(\sigma)=\rho_{1y}(\beta)=\rho_{1x}(\rho_{xy}(\beta))=\rho_{1x}(\psi_x(\sigma)).$$

In the sequel, let D_x be the set of those σ in C, for which $\psi_x(\sigma)$ is defined and $\psi_x(\sigma)\neq O$.[18]

Now consider the subsequence $\{a_{\psi_1(\sigma)}\}_{\sigma\in D_1}$. We shall show that, for any fixed x in I, some tail of this subsequence belongs to Σ_{t_x}.

Indeed, we know that

$$\{a_{\rho_{1x}(\beta)}\}_{\beta\in B_x\setminus\{O\}}\in\Sigma_{t_x}.$$

By the stability of Σ_{t_x}, we may assert that

$$\{a_{\rho_{1x}(\psi_x(\sigma))}\}_{\sigma\in D_x}\in\Sigma_{t_x}.$$

[17] Instead of the equality between the range of ψ_x and the set B_x, only the cofinality of the range of ψ_x with B_x is mentioned in the original text.

[18] No denotation of this set is introduced in the original text.

But $\rho_{1x}(\psi_x(\sigma))=\psi_1(\sigma)$ for all σ in D_x, and therefore

$$\{a_{\psi_1(\sigma)}\}_{\sigma \in D_x} \in \Sigma_{t_x}.$$

The proof of the Diagonal Principle is completed.[19]

In the applications, the following equivalent form of the Diagonal Principle turns out to be useful.

Diagonal Principle for Generalized Sequences of Functions. Let $\{f_\alpha\}$ be a generalized sequence of functions having one and the same domain M. Let $\{\Sigma_t\}_{t \in M}$ be a system of stable classes, and let, for each fixed t in M and each subsequence $\{f_{\alpha_\beta}(t)\}$ of the generalized sequence $\{f_\alpha(t)\}$, a sub-subsequence $\{f_{\alpha_{\beta_\gamma}}'(t)\}$ can be chosen which belongs to Σ_t.[20] Then there is a subsequence $\{f_{\alpha_\delta}\}$ of $\{f_\alpha\}$ (not depending on t) such that, for each t in M, some tail of the generalized sequence $\{f_{\alpha_\delta}(t)\}$ belongs to Σ_t.[21]

Application 1. Let a set Ω be given, and let L be a system of subsets, which will be called *open*. The system L is not obliged to be a topology. Let $\{x_\alpha\}$ be a generalized sequence of elements of Ω. Since L is, in general, no topology, it is not generally true that the existence of an accumulation point of $\{x_\alpha\}$ implies the existence of a convergent generalized subsequence of $\{x_\alpha\}$.[22] However, the following theorem is valid.

If each subsequence of $\{x_\alpha\}$ has some accumulation point, then a convergent generalized subsequence of $\{x_\alpha\}$ exists.

P r o o f. Let V belong to L. We set $f_\alpha(V)=1$ if $x_\alpha \in V$, and $f_\alpha(V)=0$ if $x_\alpha \notin V$. Let Σ be the class of all stationary generalized sequences.[23] For each fixed V, the sequence of functional values $\{f_\alpha(V)\}$ has the property that each its subsequence has some sub-subsequence belonging to Σ (this is guaranteed by the fact that there are only finitely many distinct values of members of $\{f_\alpha(V)\}$). By the Diagonal Principle for Generalized Sequences of Functions, a subsequence $\{f_{\alpha_\delta}\}$ can be chosen such that, for each V, some tail of the sequence of functional values $\{f_{\alpha_\delta}(V)\}$ belongs to Σ. Consider now the subsequence $\{x_{\alpha_\delta}\}$ of $\{x_\alpha\}$. By the assumption of the theorem, this subsequence has some accumulation point c. We shall show that c is a limit of $\{x_{\alpha_\delta}\}$ Indeed, let $c \in V_0 \in L$. Then there are arbitrarily large indices δ such that $x_{\alpha_\delta} \in V_0$ and, consequently, $f_{\alpha_\delta}(V_0)=1$. Since some tail of $\{f_{\alpha_\delta}(V_0)\}$ is stationary, we conclude that, for all sufficiently large indices δ, we have $f_{\alpha_\delta}(V_0)=1$, i. e. $x_{\alpha_\delta} \in V_0$. The proof is completed.

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[19] This is the end of the text of [3]. After a sentence inserted at the preparation of the paper, a part of the text of [2] follows.

[20] This possibility of choosing has to be interpreted in the the spirit of the seventh footnote (concerning the formulation of the Diagonal Principle).

[21] The mutual equivalence of both forms of the Diagonal Principle can be established as follows. Assuming the first form of this principle, we can obtain the second one by considering the new system $\{\Sigma_t'\}_{t \in M}$ of stable classes defined by the convention that Σ_t' consists of all generalized sequences $\{f_\alpha\}$ of functions defined on M such that $\{f_\alpha(t)\} \in \Sigma_t$. Assuming the second form of the Diagonal Principle, we can obtain the first one by considering the generalized sequence $\{f_\alpha\}$ of constant functions on M defined by the condition that $f_\alpha(t)=a_\alpha$ for all t in M. The Diagonal Principle for Generalized Sequences of Functions is formulated without proof in [1], and a direct proof of it is exposed in [2] (this direct proof will be omitted in the present paper).

[22] The definitions of accumulation point and convergency are the usual ones. An element c of Ω is called an ***accumulation point of*** $\{x_\alpha\}$ iff, for each V from L, there are arbitrarily large α such that $x_\alpha \in V$. The element c is called a ***limit of*** $\{x_\alpha\}$ iff, for each V from L, $x \in V$ for all sufficiently large α. The generalized sequence $\{x_\alpha\}$ is called ***convergent*** iff there is at least one limit of this sequence.

[23] I. e. the class of the generalized sequences having all their members equal each other.

Application 2. A subset B of a given topological space will be called *precompact*[24] if, from each generalized sequence of elemens of B, some convergent subsequence can be chosen (not necessarily convergent to some element of B). Under this definition, the following theorem is valid.

Let $\{f_\alpha\}$ be a generalized sequence of functions, having one and the same domain M, and let, for every fixed t in M, the values of the functions $f_\alpha(t)$ belong to some precompact set (which may depend on t). Then there is a subsequence $\{f_{\alpha_\delta}\}$ of $\{f_\alpha\}$ such that, for each t, the sequence of functional values $\{f_{\alpha_\delta}(t)\}$ is convergent.

This theorem is an obvious corollary from the Diagonal Principle for Generalized Sequences of Functions.[25] Note that Tychonoff Theorem is not applicable in the case, since the closure of a precompact set is not necessarily compact.

Application 3.[26] Let $\{f_\alpha\}$ be a generalized sequence of real-valued functions with a common domain M. Then a subsequence can be chosen from $\{f_\alpha\}$ such that for each t in M some tail of the corresponding sequence of functional values is monotonic.

Indeed, for applying the Diagonal Principle for Generalized Sequences of Functions, it is sufficient to show that a monotonic subsequence can be chosen from each generalized sequence of real numbers. This can be shown as follows. Let c be an accumulation point of the sequence $\{x_\alpha\}$ whose members are real numbers (the cases $c=\pm\infty$ are not excluded). If there are arbitrarily large indices α with $x_\alpha=c$, then a stationary subsequence can be chosen, and such a subsequence is clearly monotonic. Otherwise, there are arbitrarily large indices α with $x_\alpha<c$ or there are arbitrarily large indices α with $x_\alpha>c$. Consider the first case. For all α with $x_\alpha<c$, we set $\beta=(\alpha, x_\alpha)$. Let $(\alpha_1, x_{\alpha_1})\leq(\alpha_2, x_{\alpha_2})$ means that $\alpha_1\leq\alpha_2$ and $x_{\alpha_1}\leq x_{\alpha_2}$. The set of the pairs $\beta=(\alpha, x_\alpha)$ becomes thus a directed system. For each $\beta=(\alpha, x_\alpha)$, set $\alpha_\beta=\alpha$. So we get a subsequence $\{x_{\alpha_\beta}\}$ which is monotonically increasing. The second case can be considered in a similar way.

Again Tychonoff Theorem is not applicable, since monotonicity is not a topological convergence.

Application 4. Tychonoff Theorem follows from the Diagonal Principle for Generalized Sequences of Functions in an obvious way. [27]

Application 5. Let S be a metric space, and let $\rho(x, y)$ be the metric in S. A generalized sequence $\{x_\alpha\}$ of elements of S is called *absolutely fundamental* if there is an upper bound of the set of all finite sums of the form

$$\rho(x_{\alpha_1}, x_{\alpha_2})+\rho(x_{\alpha_2}, x_{\alpha_3})+\cdots+\rho(x_{\alpha_{n-1}}, x_{\alpha_n}),$$

where $\alpha_1\leq\alpha_2\leq\cdots\leq\alpha_n$.[28]

We shall prove that each generalized Cauchy sequence $\{x_\alpha\}$ of elements of S has some absolutely fundamental subsequence.[29] For the proof, we take a convergent series $\sum_{\nu=1}^{\infty}\lambda_\nu$ with positive members. Consider all pairs $\beta=(\alpha, \nu)$, where ν is a natural number

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[24] In the original text, another term is used whose literal translation is "completely bounded"

[25] The "functional" interpretation of choosing subsequences, which is meant in the formulations of the Diagonal Principle, creates no problems again. This can be seen by using reduction of generalized sequences to open filters and applying the Axiom of Choice.

[26] As in the case of the previous two applications, the exposition of this application is taken from [2]. However, an earlier version of its exposition can be found in [1].

[27] Now we interrupt the exposition of text from [2] and we go to the exposition of an application taken from [1] (with a modification connected with a difference between the notions of directed set in [1] and in the present paper, where the definition from [2] is adopted).

[28] Obviously, if the generalized sequence $\{x_\alpha\}$ is absolutely fundamental, then $x_{\alpha_1}=x_{\alpha_2}$ for every two indices α_1, α_2 satisfying both inequalities $\alpha_1\leq\alpha_2$, $\alpha_2\leq\alpha_1$.

[29] A generalized sequence $\{x_\alpha\}$ of elements of S is called a *Cauchy sequence* iff it satisfies the Cauchy condition: for each positive real number ε, the inequality $\rho(x_\alpha, x_\mu)<\varepsilon$ holds for all sufficiently large indices α, μ.

and the index α is chosen in such a way that

$$\rho(x_\alpha, x_\mu) < \lambda_\nu$$

for all $\mu \geqq \alpha$. Let $(\alpha_1, \nu_1) \leqq (\alpha_2, \nu_2)$ means that either $\alpha_1 \leqq \alpha_2$, $\nu_1 < \nu_2$, or $\alpha_1 = \alpha_2$, $\nu_1 = \nu_2$.[30] Set $\alpha_\beta = \alpha$. Then $\{x_{\alpha_\beta}\}$ is an absolutely fundamental subsequence of $\{x_\alpha\}$. Indeed, let $\beta_1 \leqq \beta_2 \leqq \cdots \leqq \beta_n$, and let $\beta_i = (\alpha_i, \nu_i)$. Then

$$\rho(x_{\alpha_{\beta_i}}, x_{\alpha_{\beta_{i+1}}}) = \rho(x_{\alpha_i}, x_{\alpha_{i+1}}) < \lambda_{\nu_i}$$

since $\alpha_{i+1} \geqq \alpha_i$. Without loss of generality, we may assume that $\nu_1 < \nu_2 < \cdots < \nu_n$.[31] Then

$$\sum_{i=1}^{n} \rho(x_{\alpha_{\beta_i}}, x_{\alpha_{\beta_{i+1}}}) < \sum_{i=1}^{n} \lambda_{\nu_i} < \sum_{\nu=1}^{\infty} \lambda_\nu,$$

because all numbers $\nu_1, \nu_2, \ldots, \nu_n$ are distinct.

From the above result and the Diagonal Principle for Generalized Sequences of Functions, we get the following theorem.

Let $\{f_\alpha\}$ be a generalized sequence of functions with a common domain M, and let for every fixed t in M the sequence of functional values $\{f_\alpha(t)\}$ be a Cauchy sequence in some metric space (which may depend on t). Then a subsequence $\{f_{\alpha_\delta}\}$ can be chosen in such a way that for each fixed t some tail of the sequence $\{f_{\alpha_\delta}(t)\}$ is absolutely fundamental.[32]

The Diagonal Principle can be generalized for sequences of higher order in the following way. Let A be a set with a partial order (which is supposed to be transitive and reflexive, but not necessarily antisymmetric). This partially ordered set will be called a *directed system of first order* if each denumerable subset of A has an upper bound and each ordinary monotonically increasing sequence of elements of A has least upper bound. A *generalized sequence of first order* $\{x_\alpha\}$ is determined by a function x_α whose argument α ranges over a directed system of first order. The generalized sequences, considered until now, will be called *sequences of order zero.*

A *subsequence* $\{x_{\alpha_\beta}\}$ of a sequence of first order $\{x_\alpha\}$ is determined by a monotonically increasing function α_β from some directed system of first order B to the domain of x_α, such that the range of α_β is cofinal with the domain of x_α and the following continuity condition is satisfied: whenever $\beta_1, \beta_2, \beta_3, \ldots$ is a monotonically increasing ordinary sequence of elements of B and β_0 is the least upper bound of this sequence, then α_{β_0} is the least upper bound of the sequence $\alpha_{\beta_1}, \alpha_{\beta_2}, \alpha_{\beta_3}, \ldots$

The Diagonal Principle remains valid for sequences of first order. The proof can be carried out in the same way, but with the following supplement. The sets B_z, which we defined, are directed systems of order zero. We enlarge them to directed systems of first order by considering the monotonically increasing sequences of their elements.

As an example of a directed system of first order, the set of all denumerable subsets of a given set can be mentioned (the partial ordering is by inclusion). Another example is the set of the countable transfinite numbers.

Of course, these investigations can be generalized in an obvious way for sequences of arbitrary order.

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[30] In [1], where the order (denoted by $<$) in a directed set is not obliged to be reflexive, it is adopted that the inequality $(\alpha_1, \nu_1) < (\alpha_2, \nu_2)$ means $\alpha_1 < \alpha_2$, $\nu_1 < \nu_2$. In the present notations, this would look so: $(\alpha_1, \nu_1) \leq (\alpha_2, \nu_2)$ means $\alpha_1 \leq \alpha_2$, $\nu_1 < \nu_2$. Of course, the relation $\leq$, defined in this way, is not reflexive, and therefore we adopted a slightly more complicated definition.

[31] Since the inequalities $\nu_1 \leq \nu_2 \leq \ldots \leq \nu_n$ hold, and $\nu_i = \nu_{i+1}$ implies that $\alpha_i = \alpha_{i+1}$.

[32] From here on, the last page of [2] follows (without changes or remarks).

REFERENCES[33]

1. Я. Тагамлицки. Върху общия диагонален принцип, 3 стр. В: Отчет на сектор „Реален и функционален анализ" при ЕЦММ за изпълнението на научноизследователския план, 1977.
2. Я. Тагамлицки. Някои приложения на общия диагонален принцип, 5 стр. В: Отчет на сектор „Реален и функционален анализ" при ЕЦММ за изпълнението на научноизследователския план, 1980.
3. Я. Тагамлицки. Диагонален принцип за обобщени редици, 6 стр. В: Отчет на сектор „Реален и функционален анализ" при ЕЦММ за изпълнението на научноизследователския план, 1982.
4. Y. Tagamlitzki. The diagonal principle for generalized sequences. ICM—82, Warsaw, 1983, Short communications (Abstracts), Section 9: Real and functional analysis, Part I, p. 22.

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Received 6. 12. 1989

[33] Added at the preparation of the paper.