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LOCAL PROPERTIES OF ORTHOGONALLY REDUCIBLE LINEAR OPERATORS

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Local properties of linear operators T possessing the orthogonal complement of their fixed point set as an invariant subspace are investigated. A classification of elements with respect to such operators, local parameters and global types of such operators are introduced. The concepts of underrelaxation and overrelaxation are developed for the special case of operator relaxation playing an important role in the convergence theory of linear iterative methods.

1. Introduction. In the papers [2], [3] linear iterative methods of the form:

$$x_{n+1}=(I-D'_nA)x_n+D'_nb$$

are investigated assuming that the operators $T'_n=I-D'_nA$, or $S'_n=I-AD'_n$ represent orthoprojectors. Later additionally r e l a x a t i o n p a r a m e t e r s λ_n are introduced to influence the speed of convergence and the accuracy of results. In this case the operators D'_n are replaced by $D_n=\lambda_nD'_n$. This leads to the modified operators

$$T_n=I-D_nA=(1-\lambda_n)I+\lambda_nT'_n, \tag{1.1}$$

$$S_n=I-AD_n=(1-\lambda_n)I+\lambda_nS'_n. \tag{1.2}$$

It can be shown that certain convergence assertions hold for the modified methods

$$x_{n+1}=(I-D_nA)x_n+D_nb$$

if λ_n fulfils $|1-\lambda_n|<1$ (cf. [2], [3] with [6]). For real λ_n this means $0<\lambda<2$. It is convenient to speak of u n d e r r e l a x a t i o n for $0<\lambda_n<1$ and of o v e r r e l a x a t i o n for $1<\lambda_n<2$. This idea of relaxation is generalized in [4]. There the operators D_n are chosen in such a way that the corresponding operators T_n or S_n represent the so-called (abstract) relaxations of the orthoprojectors T'_n or S'_n. The operators T_n and S_n respectively in (1.1), (1.2) turn out to be special cases of such relaxations provided $|1-\lambda_n|<1$ holds.

In [7], [8] the relaxations are defined equivalently as in [4], but without any reference to an orthoprojector. Furthermore the concept of relaxations is developed. Important global properties are stated.

In the present paper we study local properties of relaxations and their relation to global properties. An essential aim is to make sense to the concepts of underrelaxation and overrelaxation in the general case too. Thereby it proves useful to define some basic concepts for a larger class of operators called orthogonally reducible.

2. Orthogonally reducible operators. Let H be a Hilbert space with the (real or complex) scalar field K. We consider operators T from $L(H)$ denoting the space of all linear continuous operators on H into itself. $\mathrm{N}(T)$ and $\mathrm{R}(T)$ denote the null spaces and ranges of T, respectively. The closure of a set $M\subseteq H$ is written as cl M.

At first we describe the kind of operators we are interested in.

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Definition 2.1. *T is said to be* orthogonally reducible *iff* $N(I-T)^{\perp}$ *is an invariant subspace of T. The number*

$$\nu = \nu(T) = \| T \mid N(I-T)^{\perp} \|$$

is said to be the reduced norm of T.

Obviously an orthogonally reducible operator T is completely reduced by the pair $(N(I-T), N(I-T)^{\perp})$ of orthogonal subspaces (for the notion see [9, p. 268]). It can be connected with the orthoprojector T' determined by $R(T') = N(I-T)$. It is easy to see that T and T' satisfy the relation

$$T' = TT' = T'T. \tag{2.1}$$

But then also

$$T' = T^*T' = T'T^*. \tag{2.2}$$

This implies

$$N(I-T) = R(T') \subseteq N(I-T^*), \tag{2.3}$$

$$N(I-T)^{\perp} = N(T') \supseteq N(I-T^*)^{\perp}. \tag{2.4}$$

Thereby the orthogonality equations

$$N(I-T)^{\perp} = \text{cl}R(I-T^*), \quad N(I-T^*)^{\perp} = \text{cl}R(I-T) \tag{2.5}$$

hold (see [9, p. 250]).

Under certain conditions the property of orthogonal reducibility is transferred from T to T^*. At first we need two notions known from the literature.

Definition 2.2. *T is said to be* decomposition regular *iff the direct sum representation*

$$H = N(T) \oplus \text{cl}\, R(T)$$

holds (see [5, p. 151]).

Definition 2.3. *T is said to be* asymptotically bounded *iff the sequence* $(\|T^n\|)$ *is bounded* (see [1, p. 566]).

Now we can present the following results:

Theorem 2.4. *Let $I-T$ be decomposition regular. Then T is orthogonally reducible iff T^* is orthogonally reducible. Besides the carriers of T and T^* coincide if one of these equivalent conditions is fulfilled.*

Proof. The assumption leads to

$$H = N(I-T) \oplus \text{cl}\, R(I-T).$$

Let T be orthogonally reducible. Then (2.4) and (2.5) imply

$$N(I-T)^{\perp} = N(I-T^*)^{\perp}, \quad N(I-T) = N(I-T^*).$$

Furthermore T^* has the invariant subspace

$$\text{cl}\, R(I-T^*) = N(I-T)^{\perp} = N(I-T^*)^{\perp}$$

in view of (2.5). Thus T^* is orthogonally reducible and has the same carriers as T.

Now $I-T$ is decomposition regular iff $I-T^*$ is decomposition regular (see [5, p. 157]). Therefore it suffices to show one direction since the symmetry relation $(T^*)^* = T$ is satisfied.

Corollary 2.5. *Let T be asymptotically bounded. Then the assertions of Theorem 2.4 hold.*

Proof. According to [10, p. 214] the operator $I-T$ is decomposition regular if T is asymptotically bounded.

There are many examples for orthogonally reducible operators. Trivially all operators with the fixed point set $\mathrm{N}(I-T)=\{0\}$ are orthogonally reducible.

Normal operators T (i. e. $TT^*=T^*T$) have the invariant subspaces $\mathrm{cl}\,\mathrm{R}(I-T^*)$. In view of (2.5) they are orthogonally reducible too.

Nonexpansive operators T (i. e. $\|T\|\leq 1$) are asymptotically bounded and orthogonally reducible (see [7]). In connection with iterative methods we are interested in the following classes of operators.

Definition 2.6. *T is said to be a* scalar operator *iff there exist a projector P and a scalar $\lambda\in K\setminus\{0\}$ such that*

$$T=(1-\lambda)I+\lambda P.$$

P and λ are said to be the base *and the* parameter *of T respectively.*

Definition 2.7. a) *T is said to be a* relaxation *iff $\|Tx\|<\|x\|$ holds for all $x\notin \mathrm{N}(I-T)$ (see [7], [8]).*

b) *T is called a strong* relaxation *iff the relations*

$$T\mathrm{N}(I-T)^{\perp}\subseteq \mathrm{N}(I-T)^{\perp}, \quad \|T\mid \mathrm{N}(I-T)^{\perp}\|<1$$

are satisfied (see [8]).

c) *T is said to be a* scalar relaxation *iff T is both a scalar operator and a relaxation.*

The next statement can easily be seen. The proof is omitted here.

Lemma 2.8. *For a scalar operator T the following statements are equivalent:*

a) *T is orthogonally reducible.*

b) *T is normal.*

c) *The base P is normal*

d) *The base P is selfadjoint (an orthoprojector).*

Orthogonally reducible scalar operators $T=T_{\lambda,p}$ have the reduced norm $\nu(T)=|1-\lambda|$ provided $P\neq I$. A relaxation T is orthogonally reducible since it is nonexpansive. Its reduced norm $\nu(T)$ lies between 0 and 1. It is a measure for the strength of the relaxation T (with respect to the connected orthoprojector T'). An orthoprojector T is a relaxation with $\nu(T)=0$. Therefore $\nu(T)$ is said to be the relaxation degree in [8].

A strong relaxation T is a relaxation with $\nu(T)<1$ (see [8]) or also an orthogonally reducible operator with $\nu(T)<1$ (see Definition 2.1 and Definition 2.7). A scalar operator $T\neq I$ is a (scalar) relaxation iff $P=P^*$ and $|1-\lambda|<1$ (see [7]). Under these conditions T is even a strong relaxation (see [8]).

A scalar relaxation $T=T_\lambda\neq I$ is studied here to prepare the concepts of the following sections. Let $T'=P$ denote the connected orthoprojector. Then the elements x, $T'x$ and $T_\lambda x$ are linearly dependent. They belong to an one-dimensional affine subspace N_x which is orthogonal to $\mathrm{R}(T')$.

For real Hilbert space H there is a natural order relation between the elements of N_x. Depending on whether $\lambda<1$ or $\lambda>1$ the order sequence is x, $T_\lambda x$, $T'x$ or x, $T'x$, $T_\lambda x$. In the first case we speak of (scalar) underrelaxation or relaxation of u-type, in the second case of (scalar) overrelaxation or relaxation of o-type (see also section 1). The reduced norm $\nu(T_\lambda)$ of T_λ grows with the distance between $T_\lambda x$ and $T'x$.

3. A classification of elements with respect to orthogonally reducible operators. Let $T\in L(H)\setminus\{I\}$ be an orthogonally reducible operator. By T' we denote the corresponding orthoprojector determined by $\mathsf{R}(T')=\mathsf{N}(I-T)$. Then (2.1) is satisfied. Furthermore $T\neq I$ means $H\setminus\mathsf{R}(T')\neq\emptyset$.

Now we consider a fixed element x in $H\setminus\mathsf{R}(T')$ and the sets

$$H(x)=H(T,\ x)=\{v\in H\colon\ \mathrm{Re}\,(v,\ x-T'x)=0\},\tag{3.1}$$

$$H_\pm(x)=H_\pm(T,\ x)=\{v\in H\colon\ \mathrm{Re}\,(v,\ x-T'x)\gtrless 0\},\tag{3.2}$$

where $x-T'x\neq 0$ in view of the assumption about x. The classification of the elements x with respect to T is as follows.

Definition 3.1. *We call* $x\in H\setminus\mathsf{R}(T')$ *an u-element, o-element, n-element of* T *iff* $Tx\in H_+(x)$, $Tx\in H_-(x)$, $Tx\in H(x)$, *respectively*.

If H is a real Hilbert space, there is a simple geometric interpretation. Thereby the real parts of scalars z can be replaced by z itself. Namely, then $H(x)$ is the hyperplane through 0 with the position element $x-T'x$, or in other words, the maximal subspace of H orthogonal to span $\{x-T'x\}$ (for the notions see [9, p. 137]). Furthermore $H_\pm(x)$ is the halfspace consisting of all elements v which form an acute/obtuse angle with $x-T'x$. Depending on whether x is an u-element, o-element or n-element of T the element Tx lies before, behind or on the hyperplane $H(T,\ x)$. If T is a relaxation, Tx means a local underrelaxation/overrelaxation in x (compared with $T'x$) for u-elements/o-elements x.

The definitions have some simple consequences.

Theorem 3.2. *The following statements are equivalent*:

a) x *is an u-element of* T,

b) $\mathrm{Re}\,(Tx,\ x-T'x)>0$,

c) $\mathrm{Re}\,(Tx-T'x,\ x)>0$,

d) $\mathrm{Re}\,(T(I-T')x,\ (I-T')x)>0$,

e) $\mathrm{Re}\,(T\alpha x,\ \alpha x-T'\alpha x)>0$ *for any* $\alpha\in K\setminus\{0\}$.

Proof. From Definition 3.1 and (3.2) the equivalence of a) and b) is evident. In view of

$$(I-T')^2=(I-T')^*=I-T'$$

and (2.1) we get

$$(Tx,\ x-T'x)=(Tx,\ (I-T')x)=((I-T')Tx,\ x)=(Tx-T'x,\ x),$$

$$(Tx,\ x-T'x)=(Tx,\ (I-T')^2x)=((I-T')Tx,\ (I-T')x)=(T(I-T')x,\ (I-T')x)$$

and therefore the equivalence of b), c) and d). The relations

$$(T\alpha x,\ \alpha x-T'\alpha x)=(\alpha Tx,\ \alpha(x-T'x))=|\alpha|^2(Tx,\ x-T'x)$$

show the equivalence of b) and e).

Corollary 3.3. *The following statements are equivalent*:

a) x *is an u-element of* T,

b) αx *is an u-element of* T *for any* $\alpha\neq 0$,

c) $(I-T')x$ *is an u-element of* T,

d) y *is an u-element of* T *for all* $y\in$ *span* $\{x\}+\mathsf{R}(T')$.

Proof. Theorem 3.2 implies the equivalence of a), b) and c) if we take

$$(T(I-T')x,\ (I-T')x)=(T(I-T')x,\ (I-T')x-T'(I-T')x)$$

into account. Statement d) arises by combination of a), b) and c). The inversion is clear.

Corresponding statements can be formulated for o-elements and n-elements.

In the sequel we give the notions based on all three classes of elements but restrict us essentially to the results based on the class of u-elements. Other cases can be treated analogously.

Definition 3.4. *T is said to be of* u-type (o-type, n-type) *iff all $x \notin R(T')$ are u-elements (o-elements, n-elements) of T. A (strong) relaxation T of u-type (o-type) is also called (strong)* underrelaxation (overrelaxation).

Besides we consider the sets

$$U(T)=\{x \in H \setminus R(T') : x \text{ is an } u\text{-element of } T\}, \tag{3.3}$$

$$K(T)=\{x \in N(T') : \|x\|=1\}. \tag{3.4}$$

In view of Corollary 3.3 we have

$$U(T)=U(T \mid N(T'))+R(T') \text{ for } U(T)\neq\emptyset,$$

$$U(T)\neq\emptyset \text{ iff } U(T)\cap K(T)\neq\emptyset.$$

An operator T of u-type is characterized by one of the equivalent relations

$$U(T)=H\setminus R(T'),\ U(T \mid N(T'))=N(T')\setminus\{0\}.$$
$$U(T)\cap K(T)=K(T).$$

Thus T is of u-type iff $T \mid N(T')$ is.

An operator T with the eigenspace $N(T')$, or in other words, a scalar operator T with the base T' is of n-type iff $U(T)\neq\emptyset$. A selfadjoint operator T is of u-type iff $T \mid N(T')$ is positive (i. e. $(Tx, x)>0$ for all $x \in N(T')\setminus\{0\}$). An orthoprojector is an operator of n-type.

4. Local parameters of orthogonally reducible operators. Let $T \in L(H)\setminus\{I\}$ be an orthogonally reducible operator with the connected orthoprojector T'. Let x be a fixed element in $H\setminus R(T')$.

Definition 4.1. *The scalar*

$$\lambda_x=\lambda_x(T)=(x-Tx,\ x-T'x)/\|x-T'x\|^2$$

is said to be the local parameter *of T with respect to x.*

An alternative definition would be

$$\lambda_x=(x-T'x,\ x-Tx)/\|x-T'x\|^2.$$

The decision for one of these definitions is arbitrary. We will see that the real part of λ_x has a special meaning. Therefore it is not essential which of the two possibilities we prefer.

If T and T^* are both orthogonally reducible and possess the same carrier, then λ_x is a local parameter of T iff $\bar{\lambda}_x$ is a local parameter of T^*. Section 2 contains conditions ensuring these assumptions.

Theorem 4.2. *The local parameters λ_x of T have the following properties:*

a) $\lambda_x=(x-Tx,\ x)/(x-T'x,\ x)=1-(Tx,\ x-T'x)/\|x-T'x\|^2$,

b) $(Tx,\ x)=(T_{\lambda_x}x,\ x)$, *where* $T_\lambda=(1-\lambda)I+\lambda T'$,

c) $\lambda_x=\lambda_{(I-T')x}=\lambda_{\alpha x}$ *for all* $\alpha \in K\setminus\{0\}$,

d) $\lambda_x=\lambda_y$ for all $y \in \operatorname{span}\{x\}+R(T')$.

Proof. 1. Observing

$$(I-T')^2=(I-T')^*=I-T'$$

and (2.1) we obtain

$$(x-Tx,\ x-T'x)=((I-T)x,\ (I-T')x)=((I-T')(I-T)x,\ x)=((I-T)x,\ x)=(x-Tx,\ x)$$

$$\|x-T'x\|^2=(x-T'x,\ x-T'x)=((I-T')x,\ (I-T')x)=((I-T')^2x,\ x)=((I-T')x,\ x),$$

$$\|x-T'x\|^2=(x-T'x,\ x)=(x,\ x-T'x),$$

$$(x-Tx,\ x-T'x)=(x,\ x-T'x)-(Tx,\ x-T'x)=\|x-T'x\|^2-(Tx,\ x-T'x).$$

Thus the statements a) are valid.

2. By virtue of a) we have

$$(T_{\lambda_x}x,\ x)=((1-\lambda_x)x+\lambda_x T'x,\ x)=(x,\ x)-\lambda_x(x-T'x,\ x)=(x,\ x)-(x-Tx,\ x)=(Tx,\ x).$$

3. From (2.1) and a) we find

$$(x-Tx,\ x-T'x)=((I-T')x-T(I-T')x,\ (I-T')x),$$

$$\|x-T'x\|^2=(x-T'x,\ x-T'x)=((I-T')x-T'(I-T')x,\ (I-T')x)$$

and therefore $\lambda_x=\lambda_{(I-T)x}$. For $\alpha\neq 0$ the equations

$$(\alpha x-A\alpha x,\ \alpha x)=(\alpha(x-Ax),\ \alpha x)=|\alpha|^2(x-Ax,\ x)$$

are fulfilled for $A=T$ or $A=T'$. From a) we conclude now that $\lambda_x=\lambda_{\alpha x}$.

4. Assertion d) is an immediate consequence of c).

T acts locally in the stated way as the normal scalar operator T_{λ_x}.

There is a close connection between the local parameters and the classification of elements given in section 3.

C o r o l l a r y 4.3. *Iff* $\operatorname{Re}\lambda_x<1$, *then* $x\in U(T)$.

P r o o f. By Corollary 3.3 and Theorem 4.2 we can restrict us to elements x in $K(T)$ (see (3.4)). Then Theorem 4.2a) implies $\lambda_x=1-(Tx,\ x)$. According to Theorem 3.2 the relation $x\in U(T)$ is equivalent to $\operatorname{Re}(Tx,\ x)>0$. But the last inequality means $\operatorname{Re}\lambda_x=1-\operatorname{Re}(Tx,\ x)<1$.

Thus T is an operator of u-type iff $\operatorname{Re}\lambda_x<1$ for all $x\in K(T)$. It is useful to introduce one more local parameter of T.

D e f i n i t i o n 4.4. *The real number*

$$\nu_x=\nu_x(T)=\|Tx-T'x\|/\|x-T'x\|$$

is said to be the r e d u c e d l o c a l p a r a m e t e r o f T *with respect to* x.

We will see that ν_x can be related to both the local parameter λ_x and the reduced norm ν (see Definition 2.3). The next statement contains some properties of ν_x.

T h e o r e m 4.5. *It holds*

a) $\nu_x=\nu_{(I-T')x}=\nu_{\alpha x}$ *for all* $\alpha\in K\setminus\{0\}$,

b) $\nu_x=\nu_y$ *for all* $y\in\operatorname{span}\{x\}+R(T')$,

c) $|1-\lambda_x|\leq\nu_x,\ 1-\nu_x\leq\operatorname{Re}\lambda_x\leq 1+\nu_x$,

d) $|1-\lambda_x|=\nu_x$ iff $x-T'x$ *is an eigenelement of* T,

e) $\nu=\sup\{\nu_x:\ x\in H\setminus R(T')=\sup\{\nu_x:\ x\in K(T)\}$.

P r o o f. From $T'^2=T$ and (2.1) the assertion a) is obvious. Assertion b) is an immediate consequence of a). In view of Theorem 4.2 a) it suffices to show assertion c) for $x\in K(T)$. Paying attention to Theorem 4.2a) and using Schwarz's inequality we get

$$|1-\lambda_x|=|(Tx,\ x)|\leq\|Tx\|=\nu_x.$$

Thus it is also $1-\nu_x \leqq \operatorname{Re} \lambda_x \leqq 1+\nu_x$. As it is well known $|(Tx, x)|=\| Tx \|$ is satisfied iff Tx and x are linearly dependent. But in this case x is an eigenelement of T. Now d) follows also for $x \notin \mathrm{N}(T')$ if x is replaced by $(I-T')x$. Taking a) and the norm definition into account assertion e) is evident.

Corollary 4.6. *Let $T \neq I$ be a relaxation. Then*

a) $0 \leqq \nu_x < 1$,

b) $|1-\lambda_x| < 1$, $0 < \operatorname{Re} \lambda_x < 2$.

T is an underrelaxation iff $0 < \operatorname{Re} \lambda_x < 1$ for all $x \in K(T)$.

Proof. Without loss of generality, we can assume $x \in K(T)$. Then $\nu_x = \| Tx \| < \| x \| = 1$ for a relaxation T. Assertion b) results from Theorem 4.5c). The last assertion is true by virtue of b) and Corollary 4.3.

To characterize T we need the parameter spectrum

(4.1) $$\Lambda(T) = \operatorname{cl} \{\lambda_x : x \in H \setminus \mathrm{R}(T')\},$$

its real projection

(4.2) $$\Lambda_R(T) = \operatorname{cl} \{\operatorname{Re} \lambda : \lambda \in \Lambda(T)\}$$

and its radius

(4.3) $$r_\Lambda = r_\Lambda(T) = \sup \{ |1-\lambda| : \lambda \in \Lambda(T)\}.$$

Besides the restricted operator

(4.4) $$\tilde{T} = T \mid \mathrm{N}(T') = T \mid \mathrm{N}(I-T)^\perp,$$

its spectrum $\sigma(\tilde{T})$, its spectral radius $r_\sigma = r_\sigma(\tilde{T})$ and the set

$$V(\tilde{T}) = \operatorname{cl} \{(\tilde{T}x, x) : \| x \| = 1\} = \operatorname{cl} \{(Tx, x) : x \in K(T)\}$$

are used.

Theorem 4.7. *$\Lambda(T)$ and $\Lambda_R(T)$ are convex sets satisfying the relations*

a) $\Lambda(T) = \operatorname{cl} \{\lambda_x : x \in K(T)\} = 1 - V(\tilde{T})$,

b) $1-\sigma(\tilde{T}) \subseteq \Lambda(T) \subseteq \{\lambda : |1-\lambda| \leqq r_\Lambda\}$,

c) $\Lambda_R(T) \subseteq \{\mu : 1-r_\Lambda \leqq \mu \leqq 1+r_\Lambda\}$.

Furthermore

d) $r_\sigma \leqq r_\Lambda \leqq \nu \leqq \| T \|$ *is valid.*

If T is normal, then $\Lambda(T)$ is the closed convex hull $\operatorname{cl} \operatorname{co}(1-\sigma(\tilde{T}))$ of $1-\sigma(\tilde{T})$.

If additionally H is a complex Hilbert space or if T is selfadjoint, the equations

$$r_\sigma = r_\Lambda = \nu$$

are fulfilled.

Proof. 1. It is

$$\{\lambda_x : x \in H \setminus \mathrm{R}(T')\} = \{\lambda_x : x \in K(T)\}$$

by Theorem 4.2. The same theorem shows

$$\{\lambda_x : x \in K(T)\} = \{1-(Tx, x) : x \in K(T)\}.$$

Therefore a) is valid.

It is known that $V(\tilde{T})$ is a convex set (see [9, p. 330]). Hence $\Lambda(T)=1-V(\tilde{T})$ and $\Lambda_R(T)$ are convex.

2. It can be shown that $\sigma(\tilde{T})\subseteq V(\tilde{T})$ (see [9, p. 330]). Thus $\Lambda(T)=1-V(\tilde{T}) \supseteq 1-\sigma(\tilde{T})$. The second inclusion in b) is clear if (4.3) is observed. Statement c) is a simple consequence of b).

3. By virtue of Theorem 4.5

$$\begin{aligned} r_\sigma(\tilde{T}) &= \sup\{|\lambda|: \lambda\in\sigma(\tilde{T})\}\leqq \sup\{|\lambda|: \lambda\in V(\tilde{T})\} \\ &= \sup\{|1-\lambda|: \lambda\in 1-V(\tilde{T})\} \\ &= \sup\{|1-\lambda|: \lambda\in\Lambda(T)\} = r_\Lambda(T) \end{aligned}$$

and

$$r_\Lambda(T)=\sup\{|1-\lambda_x|: x\in K(T)\}\leqq \sup\{\nu_x: x\in K(T)\}=\nu(T)=\|\tilde{T}\|\leqq\|T\|$$

hold.

4. If T is normal, then $\tilde{T}$ is normal. Hence $V(\tilde{T})$ is the closed convex hull of $\sigma(\tilde{T})$ (see [9, p. 330]). Besides in this case we have $r_\sigma(\tilde{T})=\|\tilde{T}\|=\nu(T)$ provided H is complex or T is selfadjoint (see [9, p. 331]).

It is easy to see that $\sigma(T)$ and $\sigma(\tilde{T})$ fulfil the equation $\sigma(T)=\sigma(\tilde{T})\cup\{1\}$ if $N(I-T)\neq\{0\}$.

Using $\Lambda_R(T)$, Corollary 4.3 gives immediately

Theorem 4.8. *If $\Lambda_R(T)\subseteq(-\infty, 1)$, then T is an operator of u-type. If T is an operator of u-type, then*

$$\Lambda_R(T)\subseteq(-\infty, 1].$$

In the case $\Lambda_R(T)\subseteq(-\infty, 1]$ the operator T can be of u-type but can possess also n-elements.

Theorem 4.9. *Let H be a complex Hilbert space and T be a normal operator. With the notations*

$$\Lambda_0=\{\lambda\in\mathbf{C}: |\lambda|<1,\ \operatorname{Re}\lambda>0\},$$
$$\Lambda_1=\{\lambda\in\mathbf{C}: |\lambda|<1,\ \operatorname{Re}\lambda\geqq 0\}$$

the following statements hold:

If $\sigma(\tilde{T})\subseteq\Lambda_0$, then T is a strong relaxation of u-type, that is a strong underrelaxation. Conversely, if T is a strong underrelaxation, then $\sigma(\tilde{T})\subseteq\Lambda_1$.

Proof. According to Theorem 4.7, the general assumptions ensure the equations

$$\Lambda(T)=\operatorname{cl}\operatorname{co}(1-\sigma(\tilde{T})),\ \nu(T)=r_\sigma(\tilde{T}).$$

Then $\sigma(\tilde{T})\subseteq\Lambda_0$ implies $\Lambda(T)\subseteq\Lambda_0$ and therefore $\Lambda_R(T)\subseteq(0, 1)$. Thus T is an operator of u-type by Theorem 4.8. On the other hand, $\sigma(\tilde{T})\subseteq\Lambda_0$ means also $\nu(T)=r_\sigma(\tilde{T})<1$, since $\sigma(\tilde{T})$ is closed. Hence T is a strong relaxation (see section 2).

Let T be a strong underrelaxation. Then it follows $\Lambda_R(T)\subseteq(-\infty, 1]$ by Theorem 4.8 and $r_\sigma(\tilde{T})=\nu(T)<1$. Therefore $\sigma(\tilde{T})\subseteq\Lambda_1$ is fulfilled.

Theorem 4.10. *Let T be a selfadjoint operator. If $\sigma(\tilde{T})\subseteq(0, 1)$, then T is a strong underrelaxation. Conversely, if T is a strong underrelaxation, then $\sigma(\tilde{T})\subseteq[0,1)$*

Proof. The assertions can be shown quite similar to those of Theorem 4.9. In this case $\sigma(\tilde{T})$ and $\Lambda(T)$ are sets of real numbers.

We see that there is a gap of equivalence in the Theorems 4.8, 4.9 and 4.10 This gap cannot be eliminated (see section 5).

5. Examples. The theoretical considerations of sections 3 and 4 are illustrated by the following examples. The results can easily be shown. That's why they are mentioned here without proof.

a) Let T be a scalar operator with the selfadjoint base $T' \neq I$ and the parameter $\lambda \neq 0$ (see Definition 2.6). Then T has the form

$$T=(1-\lambda)I+\lambda T'=T'+(1-\lambda)(I-T').$$

T is orthogonally reducible by Lemma 2.8. We are interested in the following derived operators:

$$\tilde{T} = T \mid \mathsf{N}(T') = (1-\lambda)\tilde{I}, \quad \tilde{I}=I \mid \mathsf{N}(T'),$$

$$T^* = (1-\bar{\lambda})I + \bar{\lambda}T' = T' + (1-\bar{\lambda})(I-T'),$$

$$TT^* = T^*T = (1-\alpha)I + \alpha T' = T' + (1-\alpha)(I-T'),$$

$$\alpha = 1-|1-\lambda|^2.$$

T is a normal and for real λ also a selfadjoint operator. We get the local parameters

$$\lambda_x=\lambda, \quad \nu_x=|1-\lambda|, \quad x \in K(T)$$

and the global sets

$$\sigma(T)=\{1,\ 1-\lambda\}, \quad \sigma(\tilde{T})=\{1-\lambda\},$$

$$\sigma(TT^*)=\sigma(T^*T)=\{1,\ |1-\lambda|^2\},$$

$$\Lambda(T)=\{\lambda\}, \quad \Lambda_R(T)=\{\mathrm{Re}\,\lambda\},$$

Furthermore the quantitative characteristics are

$$r_\sigma(\tilde{T})=r_\Lambda(T)=\nu(T)=|1-\lambda|,$$

$$r_\sigma(T)=\|T\|=\max\{1,\ |1-\lambda|\}.$$

Evidently an arbitrary $x \in K(T)$ is an u-element of T iff $\mathrm{Re}\,\lambda<1$. Moreover, T is an operator of u-type iff $\mathrm{Re}\,\lambda<1$. Finally T is a strong underrelaxation iff $\mathrm{Re}\,\lambda<1$ and $|1-\lambda|<1$.

b) Let H be a separable Hilbert space with the complete orthonormal set $\{e_i\colon i \in I\}$, where I is a finite or a countably infinite index set. Then every element x has the representation

$$x=\sum_{i \in I} x_i e_i, \quad x_i=(x,\ e_i),$$

Let T be the diagonal operator defined by

$$Tx=\sum_{i \in I} \lambda_i x_i e_i$$

with the bounded sequence $(\lambda_i\colon i \in I)$ and the nonempty index set $I'=\{i \in I\colon \lambda_i \neq 1\}$. Hence $T \neq I$. Obviously T is orthogonally reducible. We are interested in the following derived operators:

$$T'x=\sum_{i \in I \setminus I'} x_i e_i,$$

$$\tilde{T}x=\sum_{i \in I'} \lambda_i x_i e_i, \quad x=\sum_{i \in I'} x_i e_i \in \mathsf{N}(T'),$$

$$T^*Tx=TT^*x=\sum_{i \in I} |\lambda_i|^2 x_i e_i.$$

T is a normal and for real λ_i $(i \in I)$ a selfadjoint operator. We obtain the local parameters

$$\lambda_x = 1 - \sum_{i \in I'} \lambda_i |x_i|^2, \quad x \in K(T),$$

$$\nu_x^2 = \sum_{i \in I'} |\lambda_i|^2 |x_i|^2, \quad x \in K(T)$$

and the global sets

$$\sigma(T) = \mathrm{cl}\{\lambda_i : i \in I\}, \quad \sigma(\widetilde{T}) = \mathrm{cl}\{\lambda_i : i \in I'\},$$

$$\sigma(TT^*) = \sigma(T^*T) = \mathrm{cl}\{|\lambda_i|^2 : i \in I\},$$

$$\Lambda(T) = 1 - \mathrm{cl\,co}\{\lambda_i : i \in I\} = \mathrm{cl}\{1 - \sum_{i \in I'} \lambda_i |x_i|^2, \sum_{i \in I'} |x_i|^2 = 1\}.$$

Furthermore the quantitative characteristics are

$$r_\sigma(\widetilde{T}) = r_\Lambda(T) = \nu(T) = \sup\{\lambda_i : i \in I'\},$$

$$r_\sigma(T) = \|T\| = \sup\{|\lambda_i| : i \in I\}.$$

Evidently x is an u-element of T iff

$$\sum_{i \in I'} (\mathrm{Re}\,\lambda_i) |x_i|^2 > 0.$$

Thus T is an operator of u-type iff $\mathrm{Re}\,\lambda_i > 0$ for all $i \in I'$. Especially T is an underrelaxation iff $\mathrm{Re}\,\lambda_i > 0$ and $|\lambda_i| < 1$ for all $i \in I'$. Finally T is a strong underrelaxation iff $\mathrm{Re}\,\lambda_i > 0$ for all $i \in I'$ and

$$\sup\{|\lambda_i| : i \in I'\} < 1.$$

In the case of finite I the closure operation can be omitted and the supremum can be replaced by the maximum. If we choose the scalars λ_i such that

$$\inf\{|\lambda_i| : i \in I'\} = 0,$$

then it is $0 \in \sigma(\widetilde{T})$ and $1 \in \Lambda_R(T)$. Therefore the assertions in the Theorem 4.8, 4.9 and 4.10 cannot be improved.

c) Let H be an n-dimensional Hilbert space with the orthonormal basis $\{e_1, \ldots, e_n\}$. Let T be the operator defined by

$$Tx = \sum_{i=1}^{n-1} \lambda_i x_i e_{i+1}, \quad x_i = (x, e_i),$$

where at least one λ_i does not vanish, that is $T \neq 0$. In this case we have $N(I - T) = \{0\}$. Thus T is orthogonally reducible. Besides we get

$$T' = 0, \quad \widetilde{T} = T,$$

$$T^*x = \sum_{i=1}^{n-1} \bar{\lambda}_i x_{i+1} e_i,$$

$$T^*Tx = \sum_{i=1}^{n-1} |\lambda_i|^2 x_i e_i, \quad TT^*x = \sum_{i=1}^{n-1} |\lambda_i|^2 x_{i+1} e_{i+1}.$$

Hence T is not normal. The local parameters are of the form

$$\lambda_x = 1 - \sum_{i=1}^{n-1} \lambda_i x_i \bar{x}_{i+1}, \quad x \in K(T),$$

$$\nu_x^2 = \sum_{i=1}^{n-1} |\lambda_i|^2 |x_i|^2, \quad x \in K(T),$$

T and T^* are nilpotent operators. Here we find

$$\sigma(T) = \sigma(\tilde{T}) = \sigma(T^*) = \{0\},$$

$$\sigma(T^*T) = \sigma(TT^*) = \{|\lambda_i|^2 : i = 1, \dots, n-1\},$$

$$\Lambda(T) = \{1 - \sum_{i=1}^{n-1} \lambda_i x_i \bar{x}_{i+1}, \quad \sum_{i=1}^{n} |x_i|^2 = 1\}$$

and

$$r_\sigma(T) = r_\sigma(\tilde{T}) = 0,$$

$$r_\Lambda(T) = \max\{|\sum_{i=1}^{n-1} \lambda_i x_i \bar{x}_{i+1}|, \quad \sum_{i=1}^{n} |x_i|^2 = 1\},$$

$$\nu(T) = \|T\| = \|T^*\| = \max\{|\lambda_i| : i = 1, \dots, n-1\}.$$

T is a (strong) relaxation iff T is contractive ($\|T\| < 1$). Furhermore x is an u-elemen of T iff the relation

$$\mathrm{Re} \sum_{i=1}^{n-1} \lambda_i x_i \bar{x}_{i+1} > 0$$

is satisfied.

$\Lambda(T)$ contains 1 and further numbers. Besides $\Lambda(T)$ is symmetric to 1, that is, $1-\alpha \in \Lambda(T)$ implies $1+\alpha \in \Lambda(T)$.
If

$$x = \sum_{i=1}^{n} x_i e_i$$

is an u-element, then

$$y = \sum_{i=1}^{n} (-1)^i x_i e_i$$

is an o-element. Therefore T is always of mixed type, i. e. T is neither of u-type nor of o-type or n-type.

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