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NON-CRITICAL BELLMAN-HARRIS BRANCHING PROCESSES WITH STATE-DEPENDENT IMMIGRATION

MARUSYA NIKIFOROVA SLAVTCHOVA, NICKOLAY MIHAILOV YANEV

Non-critical Bellman-Harris branching processes with state-dependent immigration were investigated. The asymptotic behaviour of the first two factorial moments is obtained and limit theorems are also proved.

1. Introduction. Branching processes with immigration were first introduced and studied by Sevastyanov [14] in a continuous-time Markov case. Yanev [15] obtained similar results in a class of age-dependent branching processes with immigration.

A model with state-dependent immigration components was first investigated by Foster [4] and Pakes [10, 11]. They considered a modification of the Galton-Watson process allowing immigration whenever the number of particles is zero. The continuous-time analogue of this process was studied by Yamazato [12].

On the other hand, Mitov and Yanev [6] developed the Foster-Pakes processes with decreasing state-dependent immigration and Vatutin, Mitov and Yanev [7] generalized these results for the continuous-time Markov case with non-homogeneous state-dependent immigration.

Bellman-Harris branching processes with state-dependent immigration were introduced by Mitov and Yanev [8]. Their asymptotic results generalized those obtained by Foster [4] and Yamazato [12] in the critical Markov case. Mitov and Yanev [9] consider critical Bellman-Harris branching processes with a special type of state-dependent immigration.

It should be noticed that the Bellman-Harris branching processes with state-dependent immigration might be interpreted as mathematical models which describe cell proliferation (e. g. E. coli) in broth media where the moments of broth medium addition could be actually considered as moments of state-dependent immigration.

2. Model and equations. Now we shall briefly recall the definition of Bellman-Harris branching processes with state-dependent immigration which was given by Mitov and Yanev [8].

Let us have on the probability space $(\Omega, \mathfrak{U}, \mathsf{P})$ three independent sets of random variables where:

1) $X=\{X_i\}_{i\geq 1}$ is a set of independent identically distributed (i.i.d.) random variables with distribution function (d.f.) $K(t)=\mathsf{P}\{X_i\leq t\}$, $K(0)=0$;

2) $Y=\{Y_i\}_{i\geq 1}$ is a set of positive, integer-valued i.i.d. random variables with a p.g.f. $f(s)=\mathsf{E}s^{Y_i}=\sum_{k=1}^{\infty} f_k s^k$, $|s|\leq 1$:

and

3) $Z=\{Z_{ij}(t),\ t\geq 0,\ i,\ j\geq 1,\ Z_{ij}(0)=1\}$ is a set of i.i.d. Bellman-Harris branching processes defined by a particle-life distribution function $G(t)$, $G(0)=0$, and an offspring p.g.f. $h(s)=\sum_{k=0}^{\infty} p_k s^k$, $|s|\leq 1$.

Then

(2.1) $$Z_i(t)=\sum_{j=1}^{Y_i} Z_{ij}(t),\ t\geqq 0,\ i\geqq 1,$$

are i.i.d. Bellman-Harris processes starting with a random number $Y_i>0$ of ancestors.

Let T_i be the life-period of $Z_i(t)$, i.e.

(2.2) $$T_i=\inf\{t:\ Z_i(t)=0\},\ i=1,\ 2,\ \dots .$$

Observe that $U_i=X_i+T_i$, $i\geqq 1$ are i.i.d. random variables which form the renewal process $S_0=0$, $S_n=\sum_{i=1}^{n} U_i$, $n\geqq 1$, and

(2.3) $$N(t)=\max\{n\geqq 0:\ S_n\leqq t\},\ t\geqq 0.$$

Then Bellman-Harris branching processes with state-dependent immigration can be defined as follows:

(2.4) $$Z(0)=0,\ Z(t)=\begin{cases} Z_{N(t)+1}(t-S_{N(t)}-X_{N(t)+1}), & \text{if } S_{N(t)}+X_{N(t)+1}\leqq t,\\ 0, & \text{if } S_{N(t)}+X_{N(t)+1}>t.\end{cases}$$

Comment. The Foster-Pakes model follows from (2.4) with

$$G(t)=\begin{cases}0, & t\leqq 1,\\ 1, & t>1,\end{cases}\quad \text{and}\quad K(t)=\begin{cases}0, & t\leqq 0,\\ 1, & t>0.\end{cases}$$

Also we obtain the Yamazato process if we suppose in (2.4)

$$G(t)=\begin{cases}0, & t\leqq 0,\\ 1-e^{-\lambda t}, & t>0,\end{cases}\quad \text{and}\quad K(t)=\begin{cases}0, & t\leqq 0,\\ 1, & t>0.\end{cases}$$

Further, we shall use the following notations:

$$F(t,\ s)=\mathsf{E}s^{Z_{ij}(t)},\qquad F(0,\ s)=s,$$

$$\mathfrak{F}(t,\ s)=\mathsf{E}s^{Z_i(t)},\qquad \mathfrak{F}(0,\ s)=f(s),$$

$$\Phi(t,\ s)=\mathsf{E}s^{Z(t)},\qquad \Phi(0,\ s)=1,$$

$$R(t,\ s)=1-\Phi(t,\ s),\ Q(t,\ s)=1-\mathfrak{F}(t,\ s).$$

From (2.1) and (2.2) it follows that $V(t)=\mathsf{P}\{T_i\leqq t\}=\mathsf{P}\{Z_i(t)=0\}=\mathfrak{F}(t,\ 0)$ $=f(F(t,0))$, and $V(0)=0$.

Denote $L(t)=\mathsf{P}\{X_i+T_i\leqq t\}=\int_0^t V(t-u)\,dK(u)$ and suppose that $L(t)$ is non-lattice with $L(0)=0$.

It is known (see Mitov and Yanev [8]) that the p.g.f. $\Phi(t,\ s)$, $|s|\leqq 1$, satisfies the renewal equation

(2.5) $$\Phi(t,\ s)=\int_0^t \Phi(t-u,\ s)\,dL(u)+1-L(t)-K(t)+\int_0^t \mathfrak{F}(t-u,\ s)\,dK(u),$$

where $\mathfrak{F}(t,\ s)=f(F(t,\ s))$ and $F(t,\ s)$, $|s|\leqq 1$, is the unique solution (in the class of p.g.f.) of the equation

(2.6) $$F(t,\ s)=\int_0^t h(F(t-u,\ s))\,dG(u)+s(1-G(t)),\ F(0,\ s)=s.$$

Observe that (2.5) can be given in the following equivalent form:

(2.7) $$R(t, s)=\int_0^t R(t-u, s)\,dL(u)+D(t, s),$$

where

(2.8) $$D(t, s)=\int_0^t Q(t-u, s)\,dK(u).$$

Hence

(2.9) $$R(t, s)=\int_0^t D(t-u, s)\,dH(u),$$

where

(2.10) $$H(t)=\sum_{n=0}^{\infty} L^{*n}(t)=\mathsf{E}N(t)+1.$$

Theorem 2.1. *The p.g.f.* $\Phi(t.\ \tau;\ s_1,\ s_2)=\mathsf{E}\{s_1^{Z(t)}s_2^{Z(t+\tau)}\mid Z(0)=0\}$, $|s_k|\leq 1$, $k=1$, 2, *satisfies the equation*

(2.11) $$\Phi(t,\ \tau;\ s_1,\ s_2)=1-K(t+\tau)+\int_0^{t+\tau}[\mathfrak{F}(t+\tau-u,\ s_2)-\mathfrak{F}(t+\tau-u,\ 0)]\,dK(u)$$
$$+\int_0^{t+\tau}\Phi(t+\tau-u,\ s_2)\,dL(u)+\int_0^t \mathfrak{F}(t-u,\ \tau;\ s_1,\ s_2)\,dK(u)$$
$$+\int_0^t \mathfrak{F}(t-u,\ s_1)\Big(\int_{t-u}^{t+\tau-u}\Phi(t+\tau-u-x,\ s_2)\,dV(x)\Big)\,dK(u)$$
$$+\int_0^t \Phi(t-u,\ \tau;\ s_1,\ s_2)\,dL(u),$$

with initial conditions $\Phi(0,\ \tau;\ s_1,\ s_2)=\mathsf{E}s_2^{Z(\tau)}=\Phi(\tau,\ s_2)$,
where

$$\mathfrak{F}(t,\ \tau;\ s_1,\ s_2)=\mathsf{E}\{s_1^{Z_1(t)}s_2^{Z_1(t+\tau)}\mid Z_1(0)=Y_1\},$$
$$\mathfrak{F}(0,\ \tau;\ s_1,\ s_2)=\mathfrak{F}(\tau,\ s_2)\ \textit{and}\ \mathfrak{F}(0,\ 0;\ s_1,\ s_2)=f(s_2).$$

Proof. From Definition (2.3) we obtain

(i) If $0\leq t<t+\tau<X_1$ then $Z(t)=Z(t+\tau)=0$ and $\mathsf{E}\{s_1^{Z(t)}s_2^{Z(t+\tau)}\mid 0\leq t<t+\tau<X_1\}=1$;

(ii) If $0<t<X_1\leq t+\tau<X_1+T_1$ then $Z(t)=0$,

$$Z(t+\tau)=Z_1(t+\tau-X_1)=\sum_{j=1}^{Y_1} Z_{1j}(t+\tau-X_1)>0,\ \text{and}$$

$$\mathsf{E}\{s_1^{Z(t)}s_2^{Z(t+\tau)}\mid 0<t<X_1\leq t+\tau<X_1+T_1\}=\mathfrak{F}(t+\tau-X_1,\ s_2)-\mathfrak{F}(t+\tau-X_1,\ 0);$$

(iii) If $0<t<X_1<X_1+T_1\leq t+\tau$ then $Z(t)=0$, $Z(t+\tau)$ has the same distribution as $Z(t+\tau-X_1-T_1)$, and

$$\mathsf{E}\{s_1^{Z(t)}s_2^{Z(t+\tau)}\mid 0<t<X_1<X_1+T_1\leq t+\tau\}=\Phi(t+\tau-X_1-T_1;\ s_2);$$

(iv) If $X_1\leq t<t+\tau<X_1+T_1$, then $Z(t)=Z_1(t-X_1)$, $Z(t+\tau)=Z_1(t+\tau-X_1)$ and $\mathsf{E}\{s_1^{Z(t)}s_2^{Z(t+\tau)}\mid X_1\leq t<t+\tau<X_1+T_1\}=\mathfrak{F}(t-X_1,\ t+\tau-X_1;\ s_1,\ s_2)$;

(v) If $X_1\leq t<X_1+T_1\leq t+\tau$, then $Z(t)=Z_1(t-X_1)$, and $Z(t+\tau)=\tilde{Z}(t+\tau-X_1-T_1)$ where the process $\tilde{Z}(t)$ is stochastically equivalent to $Z(t)$ and independent of $Z_1(t)$. Hence

$$\mathsf{E}\{s_1^{Z(t)}s_2^{Z(t+\tau)}\mid X_1\leq t<X_1+T_1\leq t+\tau\}=\mathsf{E}\{s_{1_1}^{Z(t-X_1)}\,{}_{S_2}^{\tilde{Z}(t+\tau-X_1-T_1)}\};$$

(vi) If $X_1+T_1\leq t<t+\tau$ then $\mathsf{E}\{s_1^{Z(t)}s_2^{Z(t+\tau)}\mid X_1+T_1\leq t<t+\tau\}=\Phi(t-X_1-T_1,t+\tau-X_1-T_1;s_1,s_2)$.

Therefore

$$\begin{aligned}\Phi(t,\tau;s_1,s_2)=1-K(t+\tau)&+\int_0^{t+\tau}[\mathfrak{F}(t+\tau-u,s_2)-\mathfrak{F}(t+\tau-u,0)]\,dK(u)\\&+\int_0^{t+\tau}\Phi(t+\tau-u,s_2)\,dL(u)+\int_0^t\mathfrak{F}(t-u,\tau;s_1,s_2)\,dK(u)\\&+\int_0^t\mathfrak{F}(t-u,s_1)\Big(\int_{t-u}^{t+\tau-u}\Phi(t+\tau-u-x,s_2)\,dV(x)\Big)\,dK(u)\\&+\int_0^t\Phi(t-u,\tau;s_1,s_2)\,dL(u),\end{aligned}$$

which proves (2.11).

From now on it will be assumed:

1°. $0<A=h'(1)<\infty$, $m=\mathsf{E}Y_i=f'(1)<\infty$,

2°. $G(t)$ and $K(t)$ are non-lattice,

3°. $0<B=h''(1)<\infty$, $n=f''(1)<\infty$,

4°. $r=\int_0^\infty x\,dG(x)<\infty$, $a=\mathsf{E}X_i=\int_0^\infty t\,dK(t)<\infty$.

Denote the moments

$$M(t)=\frac{\partial}{\partial s}\Phi(t,s)\big|_{s=1}=\mathsf{E}Z(t),\quad M_2(t)=\frac{\partial^2}{\partial s^2}\Phi(t,s)\big|_{s=1}=\mathsf{E}Z(t)[Z(t)-1],$$

$$A(t)=\frac{\partial}{\partial s}F(t,s)\big|_{s=1}=\mathsf{E}Z_{ij}(t),\quad B(t)=\frac{\partial^2}{\partial s^2}F(t,s)\big|_{s=1}=\mathsf{E}Z_{ij}(t)[Z_{ij}(t)-1].$$

Under the conditions 1°-3°, from (2.4) and (2.5) by differentiating and setting $s=1$ it follows that

$$M(t)=\int_0^t M(t-u)\,dL(u)+m\int_0^t A(t-u)\,dK(u),\tag{2.12}$$

$$M_2(t)=\int_0^t M_2(t-u)\,dL(u)+m\int_0^t B(t-u)\,dK(u)+n\int_0^t A^2(t-u)\,dK(u),\tag{2.13}$$

$$A(t)=A\int_0^t A(t-u)\,dG(u)+1-G(t),\tag{2.14}$$

$$B(t)=A\int_0^t B(t-u)\,dG(u)+B\int_0^t A^2(t-u)\,dG(u).\tag{2.15}$$

3. Moments. In this section we investigate the asymptotic behavior of the first two factorial moments in the subcritical ($A<1$) and supercritical ($A>1$) cases and of the moment $\mathsf{E}\{Z(t)Z(t+\tau)\}$ in the supercritical case. In the critical case detailed asymptotic results are obtained by Mitov and Yanev [8].

For the extinction probability in the supercritical case we have (see Athreya and Ney [2]) $q=\lim_{t\to\infty}\mathsf{P}\{Z_{ij}(t)=0\}<1$. Therefore $V(t)\to f(q)$, $t\to\infty$, $V(0)=\mathsf{P}\{Z_i(0)=0\}=f(0)$

and $L_0=L(\infty)=f(q)$.

From (2.8) with $s=0$ it follows that $\lim_{t\to\infty} D(t,0)=1-f(q)$. Then from equation (2.7) with $s=0$ we obtain (see Feller [3], Section XI. 6) $\lim_{t\to\infty} R(t,0)=\lim_{t\to\infty} D(t,0)/(1-L_0)=1$.

On the other hand, in the subcritical case the processes $Z_{ij}(t)$ degenerate, i.e. $F(t,0)\uparrow 1$, $t\to\infty$ (see Athreya and Ney [2]). Hence $V(0)=L(0)=0$ and $V(\infty)=L(\infty)=1$.

Define the Malthusian parameter α as the root of equation

$$A\int_0^\infty e^{-\alpha u}dG(u)=1. \tag{3.1}$$

Due to the monotonicity of the left side of (3.1), such a root always exists (in this case $\alpha\geqq 0$). If $A<1$, then α may not exist (if it does, $\alpha<0$). Further we assume that equation (3.1) has a unique solution α.

Theorem 3.1. *Under assumptions* 1^0, 2^0 *and* 4^0 *if* $A<1$, *then*

$$\lim_{t\to\infty} M(t)=mr/(1-A)\,\nu_0, \tag{3.2}$$

where $\nu_0=\int_0^\infty t\,dL(t)<\infty$.

Proof. Denote $\varphi(\lambda)=\int_0^\infty e^{-\lambda t}M(t)\,dt$ and $\psi(\lambda)=\int_0^\infty e^{-\lambda t}A(t)\,dt$. Equation (2.12) yields

$$\varphi(\lambda)=m\psi(\lambda)\int_0^\infty e^{-\lambda t}dK(t)/(1-\int_0^\infty e^{-\lambda t}dL(t)). \tag{3.3}$$

Similarly from equation (2.14) we obtain

$$\psi(\lambda)=\int_0^\infty e^{-\lambda t}(1-G(t))\,dt/(1-A\int_0^\infty e^{-\lambda t}dG(t)).$$

Hence

$$\lim_{\lambda\to 0}\psi(\lambda)=r/(1-A). \tag{3.4}$$

Since $\bar\varphi(\lambda)=\int_0^\infty e^{-\lambda t}dM(t)=\lambda\varphi(\lambda)$, then

$$\bar\varphi(\lambda)=\frac{m\psi(\lambda)\int_0^\infty e^{-\lambda t}dK(t)}{(1/\lambda)[1-\int_0^\infty e^{-\lambda t}dL(t)]}. \tag{3.5}$$

On the other hand, $(1/\lambda)[1-\int_0^\infty e^{-\lambda t}dL(t)]=\int_0^\infty e^{-\lambda t}[1-L(t)]\,dt\to\nu_0$, $\lambda\to 0$.

Now, from (3.5) it follows that

$$\lim_{\lambda\to 0}\bar\varphi(\lambda)=\lim_{t\to\infty} M(t)=\frac{mr}{\nu_0(1-A)},$$

which implies (3.2).

It is well known (see Harris [5], Chapter VI, Theorem 17.1) that for supercritical Bellman-Harris branching processes

(3.6)
$$\lim_{t\to\infty} A(t)\,e^{-\alpha t}=A_0,$$

where α is the Malthusian parameter of (3.1) and

(3.7)
$$A_0=\frac{\int_0^\infty e^{-\alpha u}[1-G(u)]\,du}{A\int_0^\infty ue^{-\alpha u}dG(u)}.$$

Denote $\widetilde{L}(t)=L(t)/L(\infty)$, so that $\widetilde{L}(+\infty)=1$.

Theorem 3.2. *Under assumptions* 1^0 *and* 2^0 *and* $A>1$ *it follows*

(3.8)
$$\lim_{t\to\infty} M(t)\,e^{-\alpha t}=\overline{A},$$

where α *is the Malthusian parameter and*

(3.9)
$$\overline{A}=\frac{m\int_0^\infty e^{-\alpha u}(1-G(u))\,du\int_0^\infty e^{-\alpha u}dK(u)}{A\int_0^\infty ue^{-\alpha u}dG(u)[1-L_0\int_0^\infty e^{-\alpha u}d\widetilde{L}(u)]}.$$

Proof. Let $\overline{a}=\int_0^\infty e^{-\alpha u}d\widetilde{L}(u)$ and $\overline{L}(t)=\int_0^t e^{-\alpha u}d\widetilde{L}(u)/\overline{a}$. Applying the substitutions $M(t)=\overline{a}e^{\alpha t}\overline{M}(t)$ and $A(t)=e^{\alpha t}\overline{A}(t)$ to (2.12), we obtain

(3.10)
$$\overline{M}(t)=c_0\int_0^t\overline{M}(t-u)\,d\overline{L}(u)+(m/\overline{a})\int_0^t e^{-\alpha u}\overline{A}(t-u)\,dK(u),$$

where $c_0=L_0\overline{a}<1$.

On the other hand, from (3.6) and (3.7) we have $\int_0^t e^{-\alpha u}\overline{A}(t-u)\,dK(u)$ $<A\int_0^t e^{-\alpha u}dK(u)<\infty$, then applying Lemma 4 (see Harris [5], Chapter VI) we obtain

$$\lim_{t\to\infty}\overline{M}(t)=\frac{A_0 m\int_0^\infty e^{-\alpha u}dK(u)}{\overline{a}(1-a\overline{L}_0)},$$

which proves the theorem.

Denote

$$\mu_0=\int_0^\infty e^{-2\alpha u}\,d\widetilde{L}(u)<1,$$

(3.11)
$$B_0=\frac{BA_0\int_0^\infty e^{-2\alpha u}dG(u)}{1-A\int_0^\infty e^{-2\alpha u}dG(u)},$$

$$\widetilde{L}_1(t)=(1/\mu_0)\int_0^t e^{-2\alpha u}d\widetilde{L}(u),$$

where α is the Malthusian parameter.

Theorem 3.3. *Let* 1^0-3^0 *hold and* $A>1$. *Then*

$$\lim_{t\to\infty} M_2(t)e^{-2\alpha t}=\bar{N}, \tag{3.12}$$

where

$$\bar{N}=\frac{(mB_0+nA_0^2)\int\limits_0^\infty e^{-2\alpha u}dK(u)}{1-L_0\mu_0}, \tag{3.13}$$

A_0 *is defined by* (3.7), μ_0 *and* B_0 *are introduced by* (3.11).

Proof. Applying the substitutions $M_2(t)=\mu_0 e^{2\alpha t}\bar{M}_2(t)$, $B(t)=e^{2\alpha t}\bar{B}(t)$ and $A(t)=e^{\alpha t}\bar{A}(t)$ to equation (2.13), we obtain

$$\bar{M}_2(t)=c_1\int\limits_0^t \bar{M}_2(t-u)\,d\tilde{L}_1(u)+S(t), \tag{3.14}$$

where $c_1=L_0\mu_0<1$, $\tilde{L}_1(t)$ is defined in (3.11) and

$$S(t)=(m/\mu_0)\int\limits_0^t e^{-2\alpha u}\bar{B}(t-u)\,dK(u)+(n/\mu_0)\int\limits_0^t e^{-2\alpha u}\bar{A}^2(t-u)\,dK(u).$$

It is known (see Harris [5], Chapter VI) that for supercritical Bellman-Harris branching processes

$$\lim_{t\to\infty} B(t)\,e^{-2\alpha t}=B_0, \tag{3.15}$$

where B_0 is defined by (3.11).

The relations (3.6), (3.7) and (3.15) show that as $t\to\infty$

$$\int\limits_0^t e^{-2\alpha u}\bar{B}(t-u)\,dK(u)\to B_0\int\limits_0^\infty e^{-2\alpha u}dK(u),$$

and

$$\int\limits_0^t e^{-2\alpha u}\bar{A}^2(t-u)\,dK(u)\to A_0^2\int\limits_0^\infty e^{-2\alpha u}dK(u).$$

Therefore

$$\lim_{t\to\infty} S(t)=\frac{(mB_0+nA_0^2)\int\limits_0^\infty e^{-2\alpha u}dK(u)}{\mu_0}=K_1<\infty.$$

Applying Lemma 4 (see Harris [5], Chapter VI) to equation (3.14), we obtain that $\lim\limits_{t\to\infty}\bar{M}_2(t)=K_1/(1-L_0\mu_0)$. Hence $M_2(t)\to\bar{N}e^{-2\alpha t}$, $t\to\infty$, where $\bar{N}$ is defined by (3.13).

It is known that for the supercritical Bellman-Harris processes (see Harris [5], Chapter VI) we have

$$B(t,\ \tau)=\mathsf{E}\{Z_{ij}(t)\,Z_{ij}(t+\tau)\}=e^{\alpha(2t+\tau)}B_0[1+o(1)]. \tag{3.16}$$

uniformly for $\tau\geq 0$ where B_0 is defined by (3.11).

Denote $N(t,\ \tau)=\mathsf{E}\{Z(t)\,Z(t+\tau)\}$, $\tau>0$.

Theorem 3.4. *Under the conditions of Theorem 3.3*

$$N(t,\ \tau)=e^{\alpha(2t+\tau)}\bar{N}[1+o(1)], \tag{3.17}$$

uniformly for $\tau\geqq 0$, *where* $\bar{N}$ *is defined by* (3.13).

Proof. From (2.11) by differentiating and setting $s_1=s_2=1$ we obtain

$$N(t,\tau)=L_0\int_0^t N(t-u,\tau)\,d\tilde{L}(u)+m\int_0^t B(t-u,\tau)\,dK(u)$$
$$+n\int_0^t A(t-u)A(t+\tau-u)\,dK(u)$$
$$+m\int_0^t A(t-u)\Big(\int_{t-u}^{t+\tau-u} M(t+\tau-u-x)\,dV(x)\Big)\,dK(u). \tag{3.18}$$

Now applying the substitutions

$$N(t,\tau)=\mu_0\bar{N}(t,\tau)e^{\alpha(2t+\tau)},\quad B(t,\tau)=\bar{B}(t,\tau)e^{\alpha(2t+\tau)},$$
$$A(t)=e^{\alpha t}\bar{A}(t)\ \text{and}\ M(t)=e^{\alpha t}\bar{M}(t)$$

to equation (3.18), we have

$$\bar{N}(t,\tau)=C_1\int_0^t \bar{N}(t-u,\tau)\,d\tilde{L}_1(u)+S(t,\tau), \tag{3.19}$$

where

$$S(t,\tau)=(m/\mu_0)\int_0^t \bar{B}(t-u,\tau)e^{-2\alpha u}dK(u)$$
$$+(n/\mu_0)\int_0^t \bar{A}(t-u)\bar{A}(t+\tau-u)e^{-2\alpha u}dK(u)$$
$$+(m/\mu_0)\int_0^t \bar{A}(t-u)e^{-2\alpha u}\Big(\int_{t-u}^{t+\tau-u}\bar{M}(t+\tau-u-x)e^{-\alpha x}dV(x)\Big)\,dK(u) \tag{3.20}$$

and $C_1=\mu_0L_0$.

Let

$$I(t,\tau)=\int_0^t \bar{A}(t-u)e^{-2\alpha u}\Big(\int_{t-u}^{t+\tau-u}\bar{M}(t+\tau-u-x)e^{-\alpha x}dV(x)\Big)\,dK(u).$$

Using the relations (3.5)-(3.9), we obtain

$$|I(t,\tau)|\leqq C\int_0^t e^{-2\alpha u}\Big(\int_{t-u}^{t+\tau-u} e^{-\alpha x}dV(x)\Big)\,dK(u)$$
$$\leqq C\int_0^t e^{-2\alpha u}[V(t+\tau-u)-V(t-u)]\,dK(u)$$
$$\leqq C\Big[\int_0^{t+\tau} e^{-2\alpha u}[V(t+\tau-u)\,dK(u)-\int_0^t e^{-2\alpha u}V(t-u)]\,dK(u)\Big],$$

where $C=A_0\bar{A}>0$. Since $\lim_{t\to\infty}\int_0^t e^{-2\alpha u}dK(u)<\infty$ and $\lim_{t\to\infty}V(t)=f(q)$, then $\lim_{t\to\infty}I(t,\tau)=0$, uniformly for $\tau\geqq 0$.

Using relations (3.20), (3.6) and (3.7) and from Theorems 3.1, 3.2, and 3.3, we obtain

$$\lim_{t\to\infty}S(t,\tau)=\frac{(mB_0+nA_0^2)\int_0^\infty e^{-2\alpha u}dK(u)}{\mu_0}=K_1.$$

From (3.19) and Lemma 4 (see Harris [5], Chapter VI) we obtain (3.17).

4. Limit theorems.

Theorem 4.1. *Let conditions 1°, 2° and 4° hold. If the process $Z(t)$ is subcritical, then*

$$\lim_{t\to\infty} \mathsf{P}\{Z(t)=k\}=\Phi_k, \quad \sum_{k=0}^{\infty}\Phi_k=1,$$

where

$$\Phi(s)=1-\frac{Q(s)}{\nu_0}, \quad \nu_0=\int_0^\infty [1-L(t)]\,dt<\infty, \tag{4.1}$$

and

$$Q(s)=\int_0^\infty Q(t, s)\,dt, \quad |s|\leq 1. \tag{4.2}$$

Proof. From equation (2.5) we have

$$\Phi(t, s)=\int_0^t \Phi(t-u, s)\,dL(u)+J(t, s), \tag{4.3}$$

where

$$J(t, s)=1-L(t)-K(t)+\int_0^t Q(t-u, s)\,dK(u).$$

On the other hand,

$$J(t, s)=1-L(t)-\int_0^t Q(t-u, s)\,dK(u)=1-L(t)-D(t, s).$$

It is well known (see Athreya and Ney [2], Theorem 1, p. 162) that there exists a $C_1>0$ such that

$$|1-\mathfrak{F}(t, s)|\leq mA(t)\,|1-s|\leq C_1e^{\alpha t}, \tag{4.4}$$

where $\alpha<0$ is the Malthusian parameter.

Using (4.4), we obtain

$$\Big|\int_0^\infty D(t, s)\,dt\Big|\leq\Big|\int_0^\infty\Big[\int_0^t Q(t-u, s)\,dK(u)\Big]dt\Big|$$

$$=\Big|\int_0^\infty dK(u)\int_u^\infty Q(t-u, s)\,dt\Big|=\Big|\int_0^\infty Q(x, s)\,dx\Big|\leq C_1\int_0^\infty e^{\alpha x}dx<\infty.$$

Applying the basic renewal theorem (see Feller [3, Section XI. 1]) to the equation (4.3), we conclude that

$$\lim_{t\to\infty}\Phi(t, s)=\frac{\int_0^\infty J(t, s)\,dt}{\nu_0}=\frac{\nu_0-\int_0^\infty Q(t, s)\,dt}{\nu_0}=1-Q(s)/\nu_0=\Phi(s). \tag{4.5}$$

From (3.4), (4.4) and (4.5) it follows that

$$Q(s)\,|\leq\Big|\int_0^\infty J(t, s)\,dt\Big|\leq\Big|\int_0^\infty Q(t)\,dt\Big|\leq|1-s|\,m\int_0^\infty A(t)\,dt\leq\frac{|1-s|\,mr}{1-A}.$$

Obviously $\Phi(s)\to 1$ as $s\to 1$, which completes the theorem.

Corollary 4.1. *Under the conditions of Theorem 4.1*

$$\mathsf{E}Z(\infty)=\Phi(1)=-\frac{Q(1)}{\nu_0}=\frac{mr}{\nu_0(1-A)}.$$

P r o o f. From (4.1) and (4.2) by differentiating and using (3.4), we have

$$Q'(1)=-\int_0^\infty f'(1)A(x)\,dx=-m\int_0^\infty A(x)\,dx=-\frac{mr}{\nu_0(1-A)}.$$

Finally we obtain $\Phi'(1)=\frac{mr}{\nu_0(1-A)}$, which is equivalent to (3.2).

T h e o r e m 4.2. *Assume conditions* 1^0-3^0 *and* $A>1$. *If* $\alpha>0$ *is the Malthusian parameter, then the process* $W(t)=Z(t)e^{-\alpha t}$ *converges in mean square to a random variable* W *whose Laplace transform* $\varphi(\lambda)=\mathsf{E}e^{-\lambda W}$ *satisfies the equation:*

$$\varphi(\lambda)=\int_0^\infty \varphi(\lambda e^{-\alpha u})\,dL(u)+\int_0^\infty f(\psi(\lambda e^{-\alpha u}))\,dK(u)-f(q), \tag{4.6}$$

where $\psi(\lambda)$ *is the unique solution of the equation*

$$\psi(\lambda)=\int_0^\infty h(\psi(\lambda e^{-\alpha u}))\,dG(u) \tag{4.7}$$

in the class $C=\{\psi:\psi(u)=\int_0^\infty e^{-ut}dF(t),\ F(0+)<1,\ \int_0^\infty t\,dF(t)=1\}$.

P r o o f. It is not difficult to show that

$$\begin{aligned}\mathsf{E}\{W(t+\tau)-W(t)\}^2&=e^{-2\alpha t}M_2(t)+e^{-2\alpha t}M(t)+e^{-2\alpha(t+\tau)}M_2(t+\tau)\\&\quad+e^{-2\alpha(t+\tau)}M(t+\tau)-2e^{-\alpha(2t+\tau)}N(t,\tau).\end{aligned} \tag{4.8}$$

Now, using Theorems 3.2, 3.3, 3.4 for the right side of (4.8), we obtain that $\lim_{t\to\infty}\mathsf{E}\{W(t+\tau)-W(t)\}^2=0$, uniformly for $\tau>0$, which is equivalent to the mean square convergence to a random variable W.

From here it follows that

$$\lim_{t\to\infty}\Phi(t,\exp(-\lambda e^{-\alpha t}))=\varphi(\lambda), \tag{4.9}$$

where $\varphi(\lambda)=\mathsf{E}e^{-\lambda W}$, $\lambda>0$.

It is known (see Athreya and Ney [2]) that under the conditions of the theorem we have

$$\lim_{t\to\infty}F(t,\exp(-\lambda e^{-\alpha t}))=\psi(\lambda) \tag{4.10}$$

where $\psi(\lambda)$ satisfies (4.7).

Setting $s=\exp\{-\lambda e^{-\alpha t}\}$ in equation (2.4), we obtain

$$\begin{aligned}\Phi(t,\exp(-\lambda e^{-\alpha t}))&=\int_0^t\Phi(t-u,\exp(-\lambda e^{-\alpha(t-u)}e^{-\alpha u}))\,dL(u)+1-K(t)-L(t)\\&\quad+\int_0^t f(F(t-u,\exp(-\lambda e^{-\alpha(t-u)}e^{-\alpha u})))\,dK(u).\end{aligned} \tag{4.11}$$

As $t\to\infty$ from (4.11), using (4.9) and (4.10), we obtain (4.6).

Equations (4.6) and (4.7) when $\lambda\to 0$ yield $\varphi(\lambda)\to 1$, i.e. the limit distribution $S(x)=\mathsf{P}\{W\leq x\}$ is non-degenerated.

C o r o l l a r y 4.2. *Under the conditions of Theorem* 4.2 $\mathsf{E}W=\bar{A}$, $\operatorname{Var}W=\bar{N}-\bar{A}^2$, *where* $\bar{A}$ *and* $\bar{N}$ *are constants defined by* (3.9) *and* (3.13), *respectively.*

P r o o f. Differentiating (4.6) and setting $\lambda=0$, we observe that $-\mathsf{E}W=\varphi'(0)$ $=\int_0^\infty \varphi'(0)e^{-\alpha u}dL(u)+\int_0^\infty f'(\psi(0)e^{-\alpha u})\psi'(0)\,dK(u)$.

Since $\psi(\lambda)=\mathsf{E}\exp(-\lambda)\lim\{Z_{ij}(t)e^{-\alpha t}\})$ and $\psi(0)=1$, then we have

$$\mathsf{E}W=\frac{mA_0\int_0^\infty e^{-\alpha u}dK(u)}{1-\int_0^\infty e^{-\alpha u}dL(u)}. \tag{4.12}$$

By differentiating once more (4.6) and setting $\lambda=0$, we obtain $\mathsf{E}W^2=\varphi''(0)$ $=\int_0^\infty \varphi''(0)e^{-2\alpha u}dL(u)+n\int_0^\infty[\psi'(0)]^2e^{-2\alpha u})\,dK(u)+m\int_0^\infty \psi''(0)e^{-2\alpha u}dK(u)$.

Using that

$\psi''(0)=\mathsf{E}\{\lim_{t\to\infty}[Z_{ij}(t)e^{-\alpha t}]\}^2=B_0$ (see Harris (1963)), we obtain

$$\mathsf{E}W^2=\mathsf{E}W^2\int_0^\infty e^{-2\alpha u}dL(u)+nA_0^2\int_0^\infty e^{-2\alpha u}dK(u)+mB_0\int_0^\infty e^{-2\alpha u}dK(u).$$

Finally, we have

$$\mathsf{E}W^2=\frac{(mB_0+nA_0^2)\int_0^\infty e^{-2\alpha u}dK(u)}{1-\int_0^\infty e^{-2\alpha u}dL(u)}, \tag{4.13}$$

where the constants A_0 and B_0 are given by (3.7) and (3.11). From (4.12) and (4.13) we obtain Var $W=\mathsf{E}W^2-(\mathsf{E}W)^2=\bar{N}-\bar{A}^2$, which completes the proof.

Since $\psi(\infty)=q<1$ (see Harris [5], Chapter VI, §20) from equation (4.6) as $\lambda\to\infty$, we obtain $\varphi(\infty)=\varphi(\infty)L_0$, which implies that $\varphi(\infty)=0$ because of $L_0>0$.

C o r o l l a r y 4.3. *(Uniqueness) Equation* (4.6) *has a unique solution in the class*

$$D=\{\varphi:\varphi(u)=\int_0^\infty e^{-ut}dF(t),\quad \varphi(0)=1\}. \tag{4.14}$$

P r o o f. Suppose φ_1 and φ_2 are solutions of (4.6) in the class D. Let $\theta(\lambda)=_\infty|\varphi_1(\lambda)$ $-\varphi_2(\lambda)|$ for $\lambda\geqq 0$. Using (4.6), we have

$$\theta(\lambda)\leqq\int_0^\infty \theta(\lambda e^{-\alpha u})\,dL(u). \tag{4.15}$$

We can rewrite (4.15) as $\theta(\lambda)\leqq\mathsf{E}\theta(\lambda e^{-\alpha U})$, where U is a random variable with d. f. $L(t)$. Iteration yields $\theta(\lambda)\leqq\mathsf{E}\theta(\lambda e^{-\alpha S_n})$, where $S_n=\sum_{i=1}^n U_i$ and $\{U_i\}$ are i.i.d. random variables with d.f. $L(t)$. Since φ_1 and φ_2 are in D, θ is bounded and $\lim_{x\to 0}\theta(x)=0$.

By the strong law of large numbers $S_n\to\infty$ a.s. Then the bounded convergence theorem yields $\theta(\lambda)\leqq\lim\mathsf{E}[\theta(\lambda e^{-\alpha S_n})]=0$, thus the uniqueness is proved.

C o r o l l a r y 4.4. *The distribution of* W *is absolutely continuous on* $[0,\infty)$.

P r o o f. Denote $\varphi(it)=\mathsf{E}e^{itW}$. Equation (4.6) yields

$$\varphi'(it)=L_0\int_0^\infty \varphi'(ite^{-\alpha u})d\tilde{L}(u) \tag{4.16}$$

$$+\int_0^\infty f'(\psi(ite^{-\alpha u}))\,\psi'(ite^{-\alpha u})\,e^{-\alpha u}dK(u),$$

where $\widetilde{L}(t)=L(t)/L_0$.

On the other hand, from equation (4.6) we have

$$\varphi(it)=L_0\mathsf{E}\varphi(it\xi)+\int_0^t f(\psi(ite^{-\alpha u}))\,dK(u)-f(q),$$

where $\xi=e^{-\alpha\widetilde{U}}$ and $\mathsf{P}\{\widetilde{U}\leq x\}=\widetilde{L}(x)$.

Now, from (4.16) we have $\varphi'(it)=L_0\mathsf{E}[\varphi'(it\xi)\xi]+R(t)$, where

$$R(t)=\int_0^\infty f'(\psi(ite^{-\alpha u}))\,\psi'(ite^{-\alpha u})\,e^{-\alpha u}\,dK(u).$$

Let $J(T)=\int_{-T}^{T}|\varphi'(it)|\,dt$. Using Theorem 2 (see Athreya and Ney [2], Chapter IV, § 11), it is not difficult to show that $\int_{-T}^{T}|R(t)|\,dt\leqq C$, where C is a positive constant.

Then $J(T)\leqq L_0\int_{-T}^{T}\mathsf{E}[\,|\varphi'(it\xi)\,\xi|\,]\,dt+\int_{-T}^{T}|R(t)|\,dt$

$$\leqq L_0\mathsf{E}\Big(\int_{-T\xi}^{T\xi}|\varphi'(iy)|\,dy\Big)+C=L_0\mathsf{E}J(T\xi)+C.$$

An iteration yields

$$J(T)\leqq L_0^n\mathsf{E}J\Big(T\prod_{i=1}^{n}\xi_i\Big)+C\,(L_0^{n-1}+\ \dots\ +L_0+1).$$

Since $\prod_{i=1}^{n}\xi_i=\exp\{-\alpha\sum_{i=1}^{n}U_i\}$, then by the strong law of the large numbers we have $\prod_{i=1}^{n}\xi_i\to 0$ a.s. as $n\to\infty$.

Hence $J(T)\leqq C/(1-L_0)$. As $T\to\infty$ we obtain $\int_{-\infty}^{+\infty}|\varphi'(it)|\,dt<\infty$ and by Lemma 3 (see Athreya [1]) it follows that W has an absolutely continuous distribution on $[0,\infty)$.

Denote $\widetilde{W}(t)=Z_{ij}(t)\,e^{-\alpha t}$ and $\lim_{t\to\infty}\widetilde{W}(t)=\widetilde{W}$ a.s.

Theorem 4.3. *Assume conditions* 1^0-3^0 *and* $A>1$. *If the Harris condition*

$$\int_0^\infty \mathsf{E}[\widetilde{W}(t)-\widetilde{W}]^2dt<\infty \tag{4.17}$$

holds, then $\lim_{t\to\infty}Z(t)\,e^{-\alpha t}=W$ a.s.

Proof. In the case $p_0=0$ it follows that $Z(t)$ is a classical Bellman-Harris process and it is known (see Harris [5], Chapter VI) that (4.17) is sufficient condition for a.s. convergence.

For the Bellman-Harris process $Z_i(t)$ let $\zeta_i^{(1)}(t)$ be the number of particles which are born up to time t, and let $\zeta_i^{(2)}(t)$ be the number of particles dead up to time t. Denote $S_k(t)=\sum_{i=1}^{N(t)}\zeta_i^{(k)}(t)$, $k=1,\,2$, where $N(t)$ is defined in (2.3) and the process $N(t)$ is independent of $\{\zeta_i^{(k)}(t)\}$, $k=1,\,2$.

When $p_0>0$, we have the representation $Z(t)=S_1(t)-S_2(t)$.

Under the conditions of the theorem $N(t)\to\nu$ a.s. as $t\to\infty$ and $E\nu=1/(1-L_0)<\infty$ (see Feller [3], Section XI. 6). On the other hand, we have $\eta_i^{(k)}=\lim_{t\to\infty}[\zeta_i^{(k)}e^{-\alpha t}]$, $k=1, 2$, a.s. (see Harris [5], Chapter VI).

Therefore as $t\to\infty$

$$\frac{S_k(t)}{e^{\alpha t}}=\sum_{i=1}^{N(t)}\zeta_i^{(k)}(t)\,e^{-\alpha t}\to\sum_{i=1}^{\nu}\eta_i^{(k)}\ \text{a.s.}$$

Hence $\lim_{t\to\infty}W(t)=W$ a.s., which completes the proof.

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Institute of Mathematics
Bulgarian Academy of Sciences
P. O. Box 373
1090 Sofia, Bulgaria

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