Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

RANDOM CAP PROCESS AND GENERALIZED WICKSELL PROBLEM ON THE SURFACE OF A SPHERE

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The properties of the random process of the closed hemispherical caps on the surface of a two-dimensional Euclidean sphere are investigatied with using the theory of marked point processes. A first order moment measure of the marked point process of the parameters corresponding to the cap process is found. This measure permits to calculate the first order moment measure of the cap process for the spherical sets of the special forms. The properties of the random chord process on the big circle of the sphere induced by the cap process are investigated. The first order moment measure of the chord process is obtained. The well-known Wicksell problem is generalized for the surface of a two-dimensional Euclidean sphere.

1. Introduction. In this paper we study the properties of a random process $\mathscr{A}$ of the closed hemispherical caps on the surface of a two-dimensional Euclidean sphere of the unit radius. We use the methods of the theory of random marked point processes (MPP).

In Section 2 the random cap process $\mathscr{A}$ on the surface S^2 of a sphere is identified with the simple unordered MPP $(\mathscr{E}^*_{\mathscr{A}}, \mathscr{X}^*_{\mathscr{A}}, P^*_{\mathscr{A}})$ in the phase space $S^2\times\mathscr{K}$ where S^2 is the space of the position points (centres) and $K=[0, A]$ is the space of the marks (diameters) of the spherical caps. A simple unordered MPP of the parameters $\mathscr{D}=(\mathscr{E}^*_{\mathscr{D}}, \mathscr{X}^*_{\mathscr{D}}, P^*_{\mathscr{D}})$ corresponds to the cap process $\mathscr{A}=(\mathscr{E}^*_{\mathscr{A}}, \mathscr{X}^*_{\mathscr{A}}, P^*_{\mathscr{A}})$. The properties of the processes $\mathscr{A}$ and $\mathscr{D}$ are postulated. The first order moment measure of the parameter random MPP $\mathscr{D}=(\mathscr{E}^*_{\mathscr{D}}, \mathscr{X}^*_{\mathscr{D}}, P^*_{\mathscr{D}})$ is found.

In Section 3 the first order moment measure for the random cap process $\mathscr{A}$ is calculated for the spherical sets of special forms.

In Section 4 a generalized Wicksell problem for the cap process $\mathscr{A}$ on the surface S^2 of a sphere is solved.

The paper has be presented at the International Symposium "Stochastic and Integral Geometry" (Yerevan, 1985).

2. Random cap process on the surface of a sphere. Let $S^2=\{x=(x_1, x_2, x_3$ $:\{x_1^2, x_2^2, x_3^2=1\}$ be unit sphere with a centre at the origin of coordinates 0 in the three dimensional Euclidean space R^3 (fig. 1). We assume that the distance between non-antipodal points x, y of the sphere S^2 is the length of the (shorter great circle) arc joining them. Then the surface of a sphere will be a metric space. The position of any point $x\in S^2$ is determined by two spherical coordinates φ, θ (φ is a longitude and θ is a latitude), i. e. $x=x(\varphi, \theta)$. The two-dimensional Euclidean sphere may be represented by a function Φ on the rectangle

$$\Delta_{\varphi,\theta}=\{(\varphi, \theta): 0\leq\varphi<2\pi, -\frac{\pi}{2}<\theta<\frac{\pi}{2}\}\cup\{(0, \frac{\pi}{2}), (0, \frac{\pi}{2})\}$$

of the Euclidean plane R^2 as follows: $\Phi[x(\varphi, \theta)]=(\varphi, \theta)$. In this connection it is considered that the North pole $T_1=x(0, 0, 1)$ and the South pole

$T_2 = x(0,\ 0,\ -1)$ correspond to the spherical coordinate $(0, \frac{\pi}{2})$ and $(0, -\frac{\pi}{2})$, respectively.

Consider the random process $\mathscr{A} = (\mathscr{E}^*_{\mathscr{A}}, \mathscr{X}^*_{\mathscr{A}}, P^*_{\mathscr{A}})$ of closed hemispherical caps on the surface S^2 of a sphere whose realization $E^*_{\mathscr{A}}$ is an unordered set consisting of N closed mutually disjoint hemispherical caps [7] (see fig. 1), i. e.

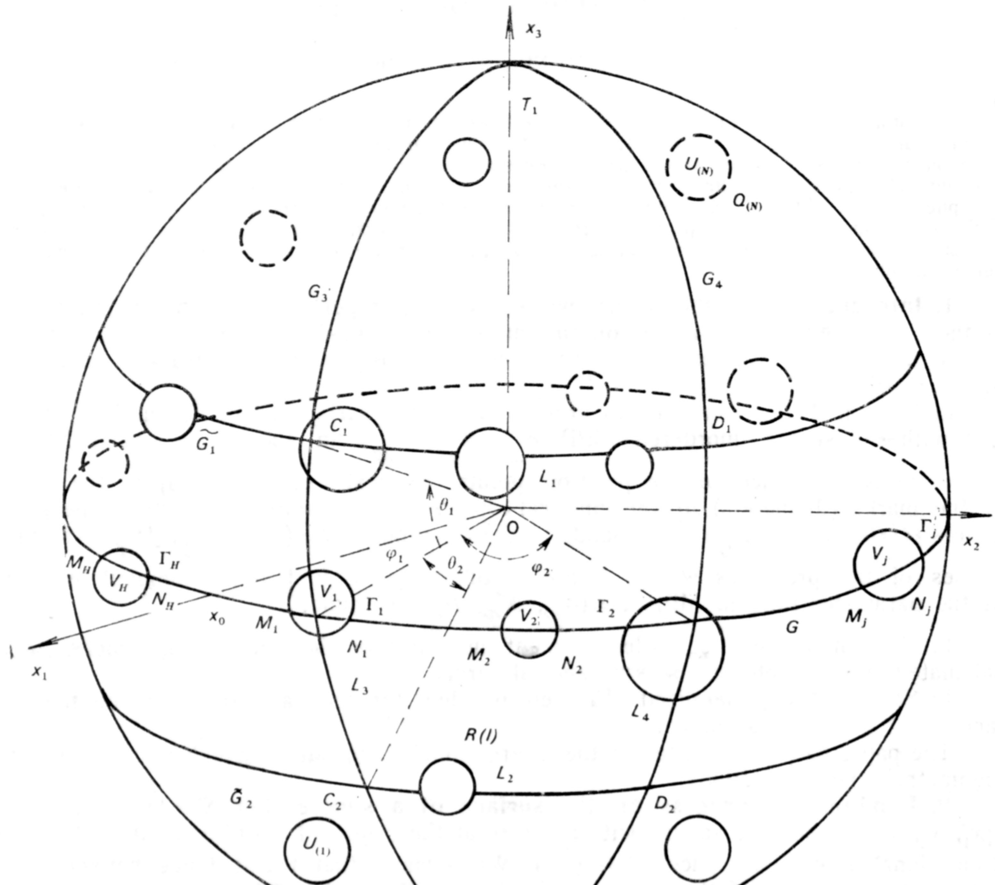

Fig. 1. The intersection of the trajectory $E^*_{\mathscr{A}}\{Q_{(1)}, \ldots, Q_{(N)}\}$ of the random cap process $\mathscr{A}$ with the region $R(l)$ bounded by the contour $\mathscr{K} = C_1D_1D_2C_1C_2$ and the circumference G of the big circle

$$E^*_{\mathscr{A}} = \{Q_{(1)}(U_{(1)},\ a_{(1)}),\ \ldots,\ Q_{(i)}(U_{(i)},\ a_{(i)}),\ \ldots,\ Q_{(N)}(U_{(N)},\ a_{(N)})\}.$$

Moreover, these caps are randomly located on S^2 and, in addition, the diameters $\{a_{(1)}, \ldots, a_{(i)}, \ldots, a_{(N)}\}$ of the caps belong to the interval $K=[0, A]$ where $A<\pi$. Then N is a nonnegative integer-valued random variable determining the number of caps in the process $\mathscr{A}$ on the surface S^2 of a sphere, i. e. $N=\mathrm{card}\,[E^*_{\mathscr{A}}]$. We remind that any spherical closed cap $Q(U(\varphi, \theta), a)$ with an angular (spherical) diameter a and the centre at the point $U(\varphi, \theta)\in S^2$ surrounded by the circumference $\partial Q(U(\varphi, \theta), a)$ has area [2, 12]

$$\mu(Q(U(\varphi, \theta), a))=\mu(a)=4\pi \sin^2 (a/4) \tag{2.1}$$

and perimeter

$$\rho(\partial Q(U(\varphi, \theta), a))= \rho(a)=2\pi \sin (a/2). \tag{2.2}$$

The centres $U_{(i)}(\varphi_{(i)}, \theta_{(i)})$ of the caps $Q_{(i)}$ of the process $\mathscr{A}$ from the realization

$$E_{\widetilde{\mathscr{A}}}=\{U_{(1)}(\varphi_{(1)}, \theta_{(1)}), \ldots, U_{(i)}(\varphi_{(i)}, \theta_{(i)}), \ldots U_{(N)}(\varphi_{(N)}, \theta_{(N)})\}$$

of the point process $\widetilde{\mathscr{A}}=(\mathscr{E}_{\widetilde{\mathscr{A}}}, \mathscr{X}_{\widetilde{\mathscr{A}}}, P_{\widetilde{\mathscr{A}}})$ on the surface S^2 of a sphere (fig. 1). The position of any cap $Q_{(i)}(U_{(i)}(\varphi_{(i)}, \theta_{(i)}), a_{(i)})$ on S^2 is identically determined by the set of parameters $[\varphi_{(i)}, \theta_{(i)}; a_{(i)}]$ where $(\varphi_{(i)}, \theta_{(i)})\in\Delta_{\varphi,\theta}$. Thus a random simple unordered MPP of parameters $\mathscr{D}=(\mathscr{E}^*_{\mathscr{D}}, \mathscr{X}^*_{\mathscr{D}}, P^*_{\mathscr{D}})$ in the bounded space $(Z=\Delta_{\varphi,\theta}\times K$, $\mathfrak{A}_Z=\mathfrak{A}_{\Delta\varphi,\theta}\otimes\mathfrak{A}_K$, $\mathscr{B}_Z=\mathscr{B}_{\Delta\varphi,\theta}\odot\mathscr{B}_K)$ can be associated with the cap process $\mathscr{A}$ on S^2. We interpret $Z=\Delta_{\varphi,\theta}\times K=\{z=(\varphi, \theta, a)\colon (\varphi, \theta)\in\Delta_{\varphi,\theta},\ a\in K\}$ as a phase space, $\Delta_{\varphi,\theta}$ as a position space and K as a mark space [4, 5, 16]. The realization $E^*_{\mathscr{D}}\in\varepsilon^*_{\mathscr{D}}$ has the form

$$E^*_{\mathscr{D}}=\{z_{(1)}, \ldots, z_{(i)}, \ldots, z_{(N)}\}=\{[\varphi_{(1)}, \theta_{(1)}; a_{(1)}], \ldots, [\varphi_{(i)}, \theta_{(i)}; a_{(i)}], \ldots, [\varphi_{(N)}, \theta_{(N)}; a_{(N)}]\}$$

Obviously, every realization $E^*_{\mathscr{D}}$ of the parameter MPP $\mathscr{D}$ in the phase space Z corresponds to the realization $E^*_{\mathscr{A}}$ of the cap process $\mathscr{A}$ on the surface S^2 of a sphere. The space K is considered as a general population of marks $G_{\mathscr{A}}$ with a distribution-function $F(a)$ possessing a probability density $f(a)\in C^{(1)}[0, A]$.

Assume that the processes $\mathscr{A}$, $\widetilde{\mathscr{A}}$, $\mathscr{D}$ possess the following properties:

a) The random variable N has a finite expectation, i. e. $E(N)<\infty$.

b) The process $\mathscr{A}$ is isotropic in mean with respect to the group of rotations of the sphere S^2 around the origin 0.

c) The MPP of parameters $\mathscr{D}=(\mathscr{E}^*_{\mathscr{D}}, \mathscr{X}^*_{\mathscr{D}}, P^*_{\mathscr{D}})$ is a random unordered simple point process (PP) with independent marking in the bounded space $(Z, \mathfrak{A}_Z, \mathscr{B}_Z)$ [5, 6, 14].

Corollary 2.1. *The process $\mathscr{A}$ has a constant intensity λ.*

Corollary 2.2. *For the mean value $E(N)$ of the random variable N we have*

$$E(N)=4\pi\lambda. \tag{2.3}$$

Denote by $N^*(E^*_{\mathscr{D}}, \bar{Z})=\mathrm{card}\,[E^*_{\mathscr{D}}\cap\bar{Z}]$, $\bar{Z}\in\mathfrak{A}_Z$ the counting measure of marked point process $\mathscr{D}$. Then the following statement is true [7, 15].

Theorem 2.1. *For any set* $\bar{Z}\in\mathfrak{A}_Z$ *the first order moment measure* $\nu^{(1)}(\bar{Z})=E[N^*(E^*_{\mathcal{D}}, \bar{Z})]$ *of the simple unordered random parameter PP with independent marking* $\mathcal{D}=(\varepsilon^*_{\mathcal{D}}, \mathcal{X}^*_{\mathcal{D}}, P^*_{\mathcal{D}})$ *is calculated by the formula:*

$$\bar{\nu}^{(1)}(\bar{Z})=\int_{z\in\bar{Z}}\nu^{(1)}(dz)=\iiint_{(\varphi,\theta,a)\in\bar{Z}}\lambda\cos\theta f(a)\,d\varphi\,d\theta\,da, \tag{2.4}$$

where $\nu^{(1)}(dz)=\lambda_{\mathcal{D}}(z)dz=\lambda\cos\theta f(a)\,d\varphi\,d\theta\,da$ *and* $\lambda_{\mathcal{D}}(z)=\lambda_{\mathcal{D}}(\varphi, \theta, a)=\lambda\cos\theta f(a)$ *is the intensity function of the MPP* $\mathcal{D}=(\varepsilon^*_{\mathcal{D}}, \mathcal{X}^*_{\mathcal{D}}, P^*_{\mathcal{D}})$.

3. Calculation of the first order moment measure of the random cap process. Let us assume that $R(l)\in\mathfrak{A}_{S^2}$ ($\mathfrak{A}_{S^2}$ is the σ=algebra of Borel sets of the sphere S^2) is a region on sphere S^2 which is bounded by the contour $K=C_1D_1D_2C_1C_2$ (fig. 1) formed by the arcs $L_1=\overset{\frown}{C_1D_1}$, $L_2=\overset{\frown}{C_2D_2}$ of the parallels $\tilde{G}_1$, $\tilde{G}_2$ and by the arcs $L_3=\overset{\frown}{C_1C_2}$, $L_4=\overset{\frown}{D_1D_2}$ of the meridians G_3, G_4. The rectangle (fig. 1)

$$C=\{(\varphi, \theta):\ \varphi_1\le\varphi\le\varphi_2,\ \theta_1\le\theta\le\theta_2\}$$

in the space $\Delta_{\varphi,\theta}$ corresponds to the spherical region $R(l)$. Let $\rho(L_1)$, $\rho(L_2)$, $\rho(L_3)$, $\rho(L_4)$ be the lengths of the arcs L_1, L_2, L_3, L_4, respectively. Moreover, $\rho(L_3)=\rho(L_4)=\theta_2-\theta_1=l$, and hence the area $\mu[R(l)]$ of the region $R(l)$ is of the form [9]

$$\mu[R(l)]=\iint_C\cos\theta\,d\varphi d\theta=2(\varphi_2-\varphi_1)\sin\frac{l}{2}\cos\frac{\theta_1+\theta_2}{2}. \tag{3.1}$$

Define the set $Z[R(l)]\subset Z$ for the quadrangle $R(l)$ as follows [10, 13]:

$$Z[R(l)]=\{z:\ z=(\varphi, \theta, a)\in Z,\ Q(U(\varphi, \theta), a)\cap R(l)\neq\varnothing\}, \tag{3.2}$$

where $Q(U(\varphi, \theta), a)$ is a cap corresponding to the point $z=(\varphi, \theta, a)$ from the parametric space Z. The set $Z[R(l)]$ consists of admissible points $z=(\varphi, \theta, a)$ of the parametric space Z for which the intersection of the caps $Q(U(\varphi, \theta), a)$ corresponding to those ones with region $R(l)$, is not empty.

Consider the random variable $\mathcal{K}(E^*_{\mathcal{A}}, R(l))$={the number of the caps of the realization $E^*_{\mathcal{A}}=\{Q_{(1)}, \dots, Q_{(N)}\}$ of the process $\mathcal{A}$ intersecting the region $R(l)$}=card $[E^*_{\mathcal{A}}\cap R(l)]$ (fig. 1). According to definition (3.2) of the set $Z[R(l)]$, any cap $Q_{(i)}(U_{(i)}(\varphi_{(i)}, \theta_{(i)}), a_{(i)})$ of the realization $E^*_{\mathcal{A}}$ of the process $\mathcal{A}$ intersects the region $R(l)$: $Q_{(i)}(U_{(i)}(\varphi_{(i)}, \theta_{(i)}), a_{(i)})\cap R(l)\neq\varnothing$, iff the point $z_{(i)}=[\varphi_{(i)}, \theta_{(i)}; a_{(i)}]$ corresponding to this cap belongs to the set $Z[R(l)]$. Therefore $\mathcal{K}(E^*_{\mathcal{A}}, (R(l))=N^*(E^*_{\mathcal{D}}, Z[R(l)])$ and $\zeta^{(1)}[R(l)]=E[\mathcal{K}(E^*_{\mathcal{A}}, R(l))]=E[N^*(E^*_{\mathcal{D}}, Z[R(l)])]=\nu^{(1)}(Z[R(l)])$.

For the spherical region $R(l)$ the expectation $\zeta^{(1)}[R(l)]$ of the random variable $\mathcal{K}(E^*_{\mathcal{A}}, R(l))$ is called a first order moment measure of the random cap process $\mathcal{A}$ [19]. It follows from Theorem 2.1 that

$$\zeta^{(1)}[R(l)]=\nu^{(1)}(Z[R(l)])=\int_{z\in Z[R(l)]}\nu^{(1)}(dz). \tag{3.3}$$

Theorem 3.1. *Let* $\mathcal{A}=(\mathcal{E}^*_{\mathcal{A}}, \mathcal{X}^*_{\mathcal{A}}, P^*_{\mathcal{A}})$ *be a random cap process on the surface* S^2 *of a unit sphere for which the corresponding MPP* $\mathcal{D}=(\varepsilon^*_{\mathcal{D}}, \mathcal{X}^*_{\mathcal{D}}, P^*_{\mathcal{D}})$

of the the parameters is a simple unordered random PP with independent marking in the bounded space (Z, $\mathfrak{A}_Z$, $\mathcal{B}_Z$). *Then the first order moment measure* $\zeta^{(1)}[R(l)]$ *of the process is calculated as follows:*

$$\zeta^{(1)}[R(l)]=\lambda\{\mu[R(l)]+\frac{1}{2\pi}\rho[R(l)]\bar{\rho}(\partial Q)+\Omega_1\bar{\mu}(Q)\}. \tag{3.4}$$

Here $\mu[R(l)]$ is area of the region $R(l)$, $\rho[R(l)]$ is a perimeter of the region $R(l)$,

$$\bar{\rho}(\partial Q)=2\pi\int_0^A \sin(a/2)f(a)\,da \tag{3.5}$$

is a mean perimeter of the caps of the process $\mathcal{A}$,

$$\bar{\mu}(Q)=4\pi\int_0^A \sin^2(a/4)f(a)\,da \tag{3.6}$$

is a mean area of the caps of the process $\mathcal{A}$, the quantity Ω_1 has the following value

$$\Omega_1=1+\frac{1}{4\pi\sin(l/2)}[(-1)^m\omega_1+(-1)^n\omega_2],$$

lwhere $m=1$, $n=2$ if $\theta_1>0$, $\theta_2>0$; $m=1$, $n=1$ if $\theta_1<0$, $\theta_2>0$; $m=2$, $n=1$ if $\theta_1<0$. $\theta_2<0$ and the functions ω_1 and ω_2 are of the form:

$$\omega_1=\{\mu^2[R(l)]+[\rho(L_1)-\rho(L_2)]^2-4\sin^2(l/2)\rho^2(L_1)\}^{\frac{1}{2}},$$

$$\omega_2=\{\mu^2[R(l)]+[\rho(L_1)-\rho(L_2)]^2-4\sin^2(l/2)\rho^2(L_2)\}^{\frac{1}{2}}$$

Proof. From relations (2.4) and (3.3)

$$\zeta^{(1)}[R(l)]=\lambda\iiint_{(\varphi,\theta,a)\in Z[R(l)]}\cos\theta\, f(a)\,d\varphi\,d\theta\,da. \tag{3.7}$$

In order to calculate the integral (3.7), we respresent the set $Z[R(l)]$ as follows:

$$Z[R(l)]=\bigcup_{a\in K}[C(a)\times\{a\}], \tag{3.8}$$

where

$$C(a)=\{(\varphi,\theta):(\varphi,\theta,a)\in Z[R(l)],\ Q(U(\varphi,\theta),a)\cap R(l)\neq\emptyset\} \tag{3.9}$$

for any fixed $a\in K$. The set $C(a)$ has the following property: $C\subseteq C(a)$. Therefore $C(a)=C\cup\Delta C(a)$ where the set

$$\Delta C(a)=\{(\varphi,\theta):(\varphi,\theta)\bar{\in}C,\ Q(U(\varphi,\theta),a)\cap R(l)\neq\emptyset\}.$$

By using the representations (3.8) and (3.9), we find

$$\zeta^{(1)}[R(l)]=\lambda\{\mu[R(l)]+\int_K f(a)[\iint_{\Delta C(a)}\cos\theta\,d\varphi\,d\theta]\,da\}$$

$$=\lambda\{\mu[R(l)]+\frac{1}{2\pi}\rho[R(l)]\bar{\rho}(\partial Q)+\Omega_0\bar{\mu}(Q)\},$$

where

$$(3.10)\qquad \Omega_0 = 1 + \frac{1}{2\pi}\left[(-1)^m\{(\varphi_2-\varphi_1)^2-\rho^2(L_1)\}^{\frac{1}{2}} + (-1)^n\{(\varphi_2-\varphi_1)^2-\rho^2(L_2)\}^{\frac{1}{2}}\right].$$

Since

$$(3.11)\qquad \rho(L_1)-\rho(L_2) = 2(\varphi_2-\varphi_1)\sin\frac{l}{2}\sin\frac{\theta_1+\theta_2}{2},$$

then from the system of equations (3.1) and (3.11) we get

$$(3.12)\qquad (\varphi_2-\varphi_1)^2 = \frac{\mu^2[R(l)]+[\rho(L_1)-\rho(L_2)]^2}{4\sin^2(l/2)}.$$

By substituting the value of $(\varphi_2-\varphi_1)^2$ from the formula (3.12), in the relation (3.10) we obtain the formula (3.4).

The following statements are consequences of Theorem 3.1.

Corollary 3.1. *The measure* $\zeta^{(1)}R[(l)]$ *is invariant with respect to the rotation group of the unit sphere* S^2.

Corollary 3.2. *Let* $\mathcal{K}(E^*_{\mathcal{A}}, L_1)$ *be a random variable which is equal to the number of the caps of the process realization* $E^*_{\mathcal{A}}$ *intersecting the arc* L_1 *of the circumference* $\widetilde{G}_1$ *of the small circle* ($\mathcal{K}(E^*_{\mathcal{A}}, L_1) = \operatorname{card}[E^*_{\mathcal{A}} \cap L_1]$, fig. 1). *If* $l\to 0$, *the following limit relations will be satisfied: the region* $R(l)\to L_1$ *in the Hausdorff metric* [1] (i. e. $d(R(l)L_1)\to 0$ *when* $l\to 0$ *where* $d(R(l), L_1)$ *is the Hausdorff distance between the sets* $R(l)$ *and* L_1), $\mu[R(l)\to 0, \rho(L_2)\to\rho(L_1), \Omega_1\to 1$ *and for almost all realizations* $E^*_{\mathcal{A}}$ *of the process* $\mathcal{A}$ *we have* $\mathcal{K}(E^*_{\mathcal{A}}, R(l))\to\mathcal{K}(E^*_{\mathcal{A}}, L_1)$. *On the basis of Lebesgue's theorem on passage to the limit in the integral we have*

$$E[\mathcal{K}(E^*_{\mathcal{A}}\ R(l))] \to E[\mathcal{K}(E^*_{\mathcal{A}}, L_1)] \quad l\to 0.$$

In addition

$$(3.13)\qquad E[\mathcal{K}(E^*_{\mathcal{A}}, L_1] = \lambda\{\frac{1}{\pi}\rho(L_1)\bar{\rho}(\partial Q)+\bar{\mu}(Q)\}.$$

Corollary 3.3. *Denote* $\mathcal{K}(E^*_{\mathcal{A}}, L)=\{$*the number of the caps of the process realization* $E^*_{\mathcal{A}}$ *intersecting the arc* L *of the circumference* G *of the big circle*$\}$ $=\operatorname{card}[E^*_{\mathcal{A}}\cap L]$ (fig. 1). *For almost all realizations* $E^*_{\mathcal{A}}$ *of the process* $\mathcal{A}$ *the random variable* $\mathcal{K}(E^*_{\mathcal{A}}, R(l))\to\mathcal{K}(E^*_{\mathcal{A}}\ L)$ *as* $l\to 0$. *In addition* $E[\mathcal{K}(E^*_{\mathcal{A}}, R(l))]\to E[\mathcal{K}(E^*_{\mathcal{A}}, L)]$ *as* $l\to 0$ *and*

$$(3.14)\qquad E[\mathcal{K}(E^*_{\mathcal{A}}, L] = \lambda\{\frac{1}{\pi}\rho(L)\bar{\rho}(\partial Q)+\bar{\mu}(Q)\}.$$

The expectation $E[\mathcal{K}(E^*_{\mathcal{A}}, L)]$ *is invariant with respect to the rotation group of the unit sphere* S^2.

Corollary. 3.4. *Let* $R(l)$ *be a spherical belt* $B(l)$ *bounded by the circumferences* $\widetilde{G}_1$ *and* $\widetilde{G}_2$ *of the small circles for which* $\varphi_2-\varphi_1=2\pi$ (fig. 1). *Consider the random* *variable*$(\mathscr{K}E^*_{\mathscr{A}}, B(l))=$*{the number of the caps of the process realization* $E^*_{\mathscr{A}}$ *intersecting the spherical belt* $B(l)\}=\text{card}\,[E^*_{\mathscr{A}}\cap B(l)]$. *Then the following equality holds:*

$$(3.15)\qquad E[\mathscr{K}(E^*_{\mathscr{A}}, B(l))]=\lambda\{\mu[B(l)]+\frac{1}{2\pi}[\rho(\widetilde{G}_1)+\rho(\widetilde{G}_2)]\,\bar{\rho}(\partial Q)+\Omega_2\,\bar{\mu}(\Omega)\},$$

where $\mu[B(l)]$ *is the area of the spherical belt* $B(l)$, $\rho(\widetilde{G}_1)$ *and* $\rho(\widetilde{G}_2)$ *are the lengths of the parallels* $\widetilde{G}_1$ *and* $\widetilde{G}_2$,

$$\Omega_2=\frac{1}{2\pi}[(-1)^m\{4\pi^2-\rho^2(\widetilde{G}_1)\}^{\frac{1}{2}}+(-1)^n\{4\pi^2-\rho^2(\widetilde{G}_2)\}^{\frac{1}{2}}],$$

where $m=1$, $n=2$ if $\theta_1>0$, $\theta_2>0$; $m=1$, $n=1$ if $\theta_1<0$, $\theta_2>0$; $m=2$, $n=1$ if $\theta_1<0$, $\theta_2<0$. *A magnitude* $E[\mathscr{K}(E^*_{\mathscr{A}}, (B(l))]$ *is invariant with respect to the rotation group of the sphere* S^2.

Corollary 3.5. *Let* $\mathscr{K}(E^*_{\mathscr{A}}, \widetilde{G}_1)$ *be the random variable which is equal to a number of caps of the process realization* $E^*_{\mathscr{A}}$ *intersecting the circumference* $\widetilde{G}_1$ *of the small circle of the sphere* S^2: $\mathscr{K}(E^*_{\mathscr{A}}, \widetilde{G}_1)=\text{card}\,[E^*_{\mathscr{A}}\cap\widetilde{G}_1]$ (fig. 1). *For almost all realizations* $E^*_{\mathscr{A}}$ *of the process* $\mathscr{A}$ *we have* $\mathscr{K}(E^*_{\mathscr{A}}, B(l))\to\mathscr{K}(E^*_{\mathscr{A}}, \widetilde{G}_1)$ *as* $l\to 0$. *In addition* $E[\mathscr{K}(E^*_{\mathscr{A}}, B(l))]\to E[\mathscr{K}(E^*_{\mathscr{A}}, \widetilde{G}_1)]$ *as* $l\to 0$ *and*

$$(3.16)\qquad E[\mathscr{K}(E^*_{\mathscr{A}}, \widetilde{G}_1)]=\frac{\lambda}{\pi}\rho(\widetilde{G}_1)\,\bar{\rho}(\partial Q).$$

The expectation $E[\mathscr{K}(E^*_{\mathscr{A}}, \widetilde{G}_1)]$ *is invariant with respect to the rotation group of the sphere* S^2.

Corollary 3.6. *Let* $\mathscr{K}(E^*_{\mathscr{A}}, G)$ *be the random variable which is equal to the number of the caps of the process realization* $E^*_{\mathscr{A}}$ *intersecting the circumference of the big circle of the sphere* S^2: $\mathscr{K}(E^*_{\mathscr{A}}, G)=\text{card}\,[E^*_{\mathscr{A}}\cap G]$ (fig. 1). *Then for almost all realizations* $E^*_{\mathscr{A}}$ *of the process* $\mathscr{A}$ *we have* $\mathscr{K}(E^*_{\mathscr{A}}, B(l))\to\mathscr{K}(E^*_{\mathscr{A}}, G)$ *as* $l\to 0$. *In addition* $E[\mathscr{K}(E^*_{\mathscr{A}}, B(l))]\to E[\mathscr{K}(E^*_{\mathscr{A}}, G)]$ *as* $l\to 0$ *and*

$$(3.17)\qquad E[\mathscr{K}(E^*_{\mathscr{A}}, G)]=2\lambda\bar{\rho}(\partial Q).$$

4. Generalized Wicksell problem on the surface of a sphere. Now consider the non-overlapping balls randomly positioned in the opaque medium in R^3. Suppose that the centers of the balls are from a Poisson process of constant intensity λ. Ball diameters $r\in[0, R]$ are random variables independently drawn from an unknown distribution function $F(r)$. Always the distribution function $F(r)$ has a density function $f(r)$. The following quantities can be determined by the planar section of the opaque medium:

$\varphi(x)$ is the density function of diameters $x\in[0, R]$ of section circles;

λ_0 is the intensity of the center PP of non-overlapping circles.
What is the relation between $f(r)$, λ and $\varphi(x)$, λ_0? This problem was posed and solved analytically by Wicksell [11, 21]. In this connection C. D. Wicksell has found an integral equation

$$\varphi(x)=\frac{x}{r_0}\int_x^R \frac{f(r)}{(r^2-x^2)^{1/2}}dr, \tag{4.1}$$

where r_0 is the mean diameter of spheres. The solution of this equation enables us to find the probability density $f(r)$ and the intensity λ:

$$f(r)=-\frac{2r\,r_0}{\pi}\int_r^R (x^2-r^2)^{-\frac{1}{2}}\frac{d}{dx}\left\{\frac{\varphi(x)}{x}\right\}dx, \tag{4.2}$$

$$\lambda=\frac{2}{\pi}\lambda_0\int_0^R\frac{\varphi(x)}{x}dx, \tag{4.3}$$

where

$$r_0=\frac{\pi}{2}\left\{\int_0^R\frac{\varphi(x)}{x}dx\right\}^{-1}. \tag{4.4}$$

At present equations (4.1)—(4.4) are the main equations of stereology. They have been dealt with by a number of well-known scientists namely Kendall and Moran [3], Santalo [8], Weibel [20], Serra [18], Ripley [17].

Next we proceed with explaining the generalized Wicksell problem for the two-dimensional Euclidean sphere.

As a result of the intersection of the process realization $E^*_{\mathscr{A}}=\{Q_{(1)},\ldots,Q_{(N)}\}$ with oriented circumference $G\subset S^2$ of the big circle we have a realization $E^*_\Gamma=(\Gamma_1(V_1(\varphi_1),\gamma_1),\ldots,\Gamma_H(V_H(\varphi_H),\gamma_H))$ of some random ordered process $\Gamma=(\varepsilon^*_\Gamma,\mathscr{X}^*_\Gamma,P^*_\Gamma)$ of closed mutually disjoint spherical chords $\Gamma_j=\overset{\frown}{M_jN_j}$, $j=\overline{1,H}$, $H=\text{card}\,[E^*_\Gamma]$ (fig. 1). The centers $V_j(\varphi_j)$, $j=\overline{1,H}$ of the chords Γ_j of the process Γ form a realization $E_{\tilde\Gamma}=(V_1(\varphi_1),\ldots,V_H(\varphi_H))$ of some point process $\tilde\Gamma=(\varepsilon_{\tilde\Gamma},\mathscr{X}_{\tilde\Gamma},P_{\tilde\Gamma})$ on the circumference G of the big circle. The lengths γ_j, $j=\overline{1,H}$ of the chords Γ_j, $j=\overline{1,H}$ take values in the interval $K=[0,A]$. The chord length γ_j generated by spherical cap $Q(U(\varphi,\theta),a)$ may be calculated through Pythagorean formula for Euler rectangular spherical triangles [1, 2]:

$$\gamma_j=\gamma(\varphi,\theta)=2\arccos\left\{\frac{\cos(a/2)}{\cos\theta}\right\}.$$

Any chord $\Gamma_j(V_j(\varphi_j),\gamma_j)$ of the realization E^*_Γ is identically determined by a pair of numbers $[\varphi_j;\gamma_j]$. This is the reason to let simple ordered MPP of parameters $\psi=(\mathscr{E}^*_\Gamma,\mathscr{X}^*_\Gamma,P^*_\Gamma)$ in bounded space $(\mathscr{S}_\varphi=\Delta_\varphi\times K,\ \mathfrak{A}_Y=\mathfrak{A}_{\Delta_\varphi}\otimes\mathfrak{A}_K,\ \mathscr{B}_Y=\mathscr{B}_{\Delta_\varphi}\odot\mathscr{B}_K)$ correspond to a process of chords Γ. We interpret $\mathscr{S}$ as a phase space, $\Delta_\varphi=\{\varphi:\ 0\leq\varphi<2\pi\}$ as a position space and K as a mark space. Every trajectory $E^*_\varphi\in\mathscr{E}^*_\varphi$ has a form

$$E^*_\varphi=([\varphi_1;\gamma_1],\ldots,[\varphi_H;\gamma_H]).$$

Moreover, the marks (lengths of chords) of the process ψ take values in the general population $G_\Gamma=K$ with distribution function $\Phi(\gamma)$ and probability density $\varphi(\gamma)$. The processes Γ, $\tilde\Gamma$, ψ possess the following properties:

(α) A realization E^*_Γ of process Γ is finite with probability one;

(β) A point process $\tilde{\Gamma}$ is isotropic on the average with respect to a group of rotations of the circumference G of the big circle around of its center;

(γ) A process γ is a simple ordered random point process with independent marking [5, 6, 14].

Corollary 4.1. *A process $\bar{\Gamma}$ possesses constant intensity λ_0.*

Let $H^*(E^*_\psi, \bar{\mathscr{S}})=\text{card}[E^*_\psi \cap \mathscr{S}]$ ($\mathscr{S} \in \mathfrak{A}_{\mathscr{S}}$) be a counting measure of marked point process ψ. The properties (α)—(γ) of the processes Γ, $\tilde{\Gamma}$, ψ enables us to calculate the first order moment measure $\eta^{(1)}(\bar{Y})=E[H^*(E^*_\psi, \bar{\mathscr{S}})]$ of the point process $\psi=(\mathscr{E}^*_\psi, \mathscr{X}^*_\psi, P^*_\psi)$ [15].

Theorem 4.1. *For any set $\bar{\mathscr{S}} \in \mathfrak{A}_{\mathscr{S}}$ the first order moment measure $\eta^{(1)}(\bar{\mathscr{S}})$ of the simple ordered random PP with independent marking of parameters $\psi=(\mathscr{E}^*_\psi, \mathscr{X}^*_\psi, P^*_\psi)$ is calculated by the following formula:*

$$\eta^{(1)}(\bar{\mathscr{S}})=\iint_{(\varphi,\gamma)\in\bar{\mathscr{S}}} \lambda_0\,\varphi(\gamma)\,d\varphi\,d\gamma.$$

Generalize Wicksell problem to two-dimensional sphere S^2 with unit radius let us define a probability density $f(a)$ of diameters and intensity λ of random cap process $\mathscr{A}$ on the sphere S^2 if the probability density $\varphi(\gamma)$ of lengths and intensity λ_0 of the random chord process Γ induced by process $\mathscr{A}$ on a circumference G of the big circle of the sphere S^2, are known.

Theorem 4.2. *Let $f(a)$ be the probability density of the cap diameters of the process $\mathscr{A}$ satisfying the following conditions: 1. $f(a)$ is a differentiable function on the segment* $[0, A]$; *2. The derivative $f'(a)$ is a bounded function on* $[0, A]$; *3. The diameters of the caps have the finite harmonic mean.*

Then

$$f(a)=-\frac{K^*}{\pi}\,\text{tg}\,\frac{a}{2}\int_0^A \left\{\cos^2\frac{a}{2}-\cos^2\frac{t}{2}\right\}^{-\frac{1}{2}}\frac{d}{dt}\left\{\varphi(t)\frac{\cos^2\frac{t}{2}}{\sin\frac{t}{2}}\right\}dt,$$

$$\lambda=\frac{\lambda_0}{2\pi}\left\{\gamma_0+2\int_0^A \varphi(t)\,\text{ctg}\,\frac{t}{2}\,dt\right\},$$

where $K^*=\lambda_0/\lambda$, $\gamma_0=E(\gamma)$ is the mean length of chords of the process Γ.

Proof. Consider the measurable rectangle

$$\bar{\mathscr{S}}=[\alpha,\beta]\times[t, A]\in\mathfrak{A}_{\mathscr{S}}\quad(\alpha,\beta\in\Delta_\varphi,\ \beta-\alpha=1)$$

for any $t\in(0, A)$. From Theorem 4.1 we have

(4.5) $$\eta^{(1)}(\bar{\mathscr{S}})=\lambda_0[1-\Phi(t)].$$

Let

$$\gamma^*(\theta, a)=\begin{cases}\gamma(\theta, a) & \text{if } |\theta|\leq a/2\\ 0 & \text{if } |\theta|>a/2.\end{cases}$$

Then

(4.6) $$H^*(E^*_\psi, \bar{\mathscr{S}})=\sum_{[\varphi,\gamma](\theta,a)]\in E^*_\psi} J_{\bar{\mathscr{S}}}([\varphi;\gamma(\theta,a)])=\iiint_{(\varphi,\theta,a)\in Z} J_{\bar{\mathscr{S}}}([\varphi,\gamma^*(\theta,a)])N^*(E^*_{\mathfrak{D}}, d\varphi\,d\theta\,da)$$

From formula (4.6) it follows that H^* is a measurable function defined on the realizations $E^*_{\mathscr{D}}$ of the MPP $\mathscr{D}$. Therefore by using Theorem 2.1 we get

$$(4.7)\quad \eta^{(1)}(\bar{\mathscr{S}}) = \int\limits_{E^*_{\mathscr{D}} \in \mathscr{E}^*_{\mathscr{D}}} H^*(E^*_\psi, \mathscr{S}) P^*_{\mathscr{D}}(dE^*_{\mathscr{D}}) = \iiint\limits_{(\varphi,\theta,\alpha) \in Z} J_{\bar{\mathscr{S}}}(\varphi;\ \gamma^*(\theta,\ a)]) \nu^{(1)}(d\varphi\, d\theta\, da)$$

$$= 2\lambda \int\limits_t^A f(a)\{1 - \frac{\cos^2(a/2)}{\cos^2(t/2)}\}^{\frac{1}{2}} da.$$

From formula (4.5) and (4.7) we have

$$(4.8)\qquad \lambda_0[1-\Phi(t)] = 2\lambda \int\limits_t^A f(a)\{1 - \frac{\cos^2(a/2)}{\cos^2(t/2)}\}^{\frac{1}{2}} da.$$

Differentiating (4.8) in t we obtain that the probability density $\varphi(t)$ and $f(a)$ are connected by the relation:

$$(4.9)\qquad \varphi(t) = \frac{\sin(t/2)}{K^* \cos^2(t/2)} \int\limits_t^A f(a) \frac{\cos^2(a/2)}{[\cos^2(t/2) - \cos^2(a/2)]^{1/2}} da,$$

$$K^* = \lambda_0/\lambda.$$

Obviously, (4.9) is a linear integral Volterra equation of the first kind with a singular kernel

$$K(a,\ t) = \frac{\cos^2(a/2)}{[\cos^2(t/2) - \cos^2(a/2)]^{1/2}}$$

with respect to the unknown function $f(a)$. The solution (4.9) is obtained by its inverting to the following Abel type integral equation

$$(4.10)\qquad \varphi_1(\mu) = \int\limits_{h_0}^{\mu} f_1(h)(\mu - h)^{-\frac{1}{2}} dh,$$

where

$$\mu = \cos^2(t/2),\quad h = \cos^2(a/2),\quad h_0 = \cos^2(A/2),$$

$$\varphi_1(\mu) = K^* \mu (1-\mu)^{-\frac{1}{2}} \varphi(2 \arc\cos \mu^{\frac{1}{2}}),$$

$$h_0 \leq \mu \leq 1,$$

$$f_1(h) = \mathrm{ctg}(a/2) f(a)$$

According to [9] the solution $f_1(h)$ of (4.10) is given by the following expression:

$$(4.11)\qquad f_1(h) = \frac{1}{\pi}\{\frac{\varphi_1(h_0+0)}{(h-h_0)^{1/2}} + \int\limits_{h_0}^{h} \frac{\varphi_1'(\mu)}{(h-\mu)^{1/2}} d\mu\}.$$

From (4.11) we easily find that

$$(4.12)\qquad f(a) = -\frac{K^*}{\pi} \mathrm{tg}\frac{a}{2} \int\limits_a^A \{\cos^2(a/2) - \cos^2(t/2)\}^{-\frac{1}{2}} \frac{d}{dt}\{\varphi(t) \frac{\cos^2(t/2)}{\sin(t/2)}\} dt.$$

If we integrate relation (4.12) with respect to a for $a\in[0, A]$ we get

$$\lambda=\frac{\lambda_0}{2\pi}\{\gamma_0+2\int_0^A\varphi(t)\operatorname{ctg}(t/2)\,dt\}. \tag{4.13}$$

We can consider equations (4.9), (4.12), (4.13) as generalizations of Wicksell equations (4.1)-(4.4) for a two-dimensional Euclidean sphere with a unit radius.

Formulas (4.12), (4.13) enable us to find the main characteristics of the cap process $\mathscr{A}$ on the sphere S^2 using the main characteristics of the induced process of the spherical chords on a circumference $G\subset S^2$ of the big circle. For example, from formulas (2.3) and (4.13) we have

$$E(N)=2\lambda_0\{\gamma_0+2\int_0^A\varphi(t)\operatorname{ctg}(t/2)\,dt\}.$$

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Received on 20.09.1989