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LIMIT THEOREMS FOR MULTITYPE BELLMAN-HARRIS BRANCHING PROCESSES WITH STATE DEPENDENT IMMIGRATION

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In the non-critical cases a multitype age-dependent branching process allowing immigration only in the state zero is studied. The asymptotic behaviour of the first two moments is obtained and limit theorems are also proved.

1. Introduction. A multitype branching process is a mathematical description of th growth of a population consisting of a finite number of types of individuals who pro duce offspring according stochastic laws. The notions of type and offspring may take various forms, depending on the application. For instance, the types in biological models are the different genotypes, in physics — the electrons and photons producing in cosmic-ray cascades, etc.

Multitype branching processes are investigated in details in the monographs [1], [7] and [10]. Foster [3] and Pakes [8, 9] considered a modification of the Galton-Watson processes with one type of particles, allowing immigration whenever the number of particles is zero. The continuous-time analogue of this process was studied by Yamazato [12].

A multitype age-dependent process with state-dependent immigration and $m>1$ types of particles is investigated by Mitov [6] in the critical case. The present paper is closely connected with [11] and [13], where the Bellman-Harris branching processes with state-dependent immigration and one type of particles are studied and limit theorems are proved in the non-critical cases. Here we establish the asymptotic results for the first two factorial moments and limit theorems in the non-critical cases.

2. Definition and equations. The prototype of the branching process to be studied in this paper is the model of the Bellman-Harris branching process with state-dependent immigration defined in [5] and [6].

Denote by $\mathbf{N}^r$ the Euclidean space of all r-dimensional vectors with non-negative integer-valued components, i. e. $\mathbf{N}^r=\{\boldsymbol{\alpha}:\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_r),\ \alpha_i\in\mathbf{N},\ \alpha_i\geqq 0\}$.

We use also the following notations

$$(\boldsymbol{x},\ \boldsymbol{y})=x_1y_1+\dots+x_ry_r,\quad \boldsymbol{xy}=(x_1y_1,\dots,x_ry_r),$$

$$\boldsymbol{s}^{\alpha}=s_1^{\alpha_1}s_2^{\alpha_2}\dots s_r^{\alpha_r}.$$

Let us consider an age-dependent branching process with types $T_1,\dots,T_r$ of particles modified by immigration of new particles at the state zero.

Definition. *Let us have on the probability space* $(\Omega,\mathcal{F},\mathsf{P})$ *three independent sets:*

1) $X=\{X_i,\ i\geq 1\}$ *are independent, identically distributed (i. i. d.) random variables with distribution function* $K(t)=\mathsf{P}\{X_i\leq t\}$, $K(0^+)=0$;

2) $Y=\{Y_i=(Y_i^1, \ldots, Y_i^r), i\geq 1\}$ are i. i. d. random vectors with non-negative integer-valued components and multidimensional probability generating function (p.g.f.)

$$f(s)=\mathsf{E}s^{Y_i}=\mathsf{E}\{s_1^{Y_i^1}\ldots s_r^{Y_i^r}\},\ f(0)=0;$$

3) $Z=\{{}_i Z_{j,k}(t)=(Z_{ij,k}^{(1)}(t), \ldots, Z_{ij,k}^{(r)}(t)),\ i,\ j\geq 1,\ k=1, \ldots, r,\ t\geq 0\}$ are i.i.d. Bellman-Harris branching processes with r types of particles. The lifetime of a type i particle is a random variable, with distribution $G_i(t)$, $G_i(0+)=0$, $i=1, \ldots, r$ and $G(t)=(G_1(t), \ldots, G_r(t))$. The reproduction behaviour is governed by r-dimesional p.g.f. $h(s)=(h^{(1)}(s), \ldots, h^{(r)}(s))$.

The component $Z_{ij,k}^{(m)}(t)$ may be interpreted as the number of particles of type Tm, $m=1, \ldots, r$ in the Bellman-Harris branching process with r types of particles, initiated with one particle of type T_k, $k=1, \ldots, r$.

Then $Z_i(t)=Z_i^{(1)}(t), \ldots, Z_i^{(r)}(t))$, $t\geq 0$, $i\geq 1$, where

$$Z_i^{(j)}(t)=\sum_{k=1}^{r}\sum_{l=1}^{Y_i^k} Z_{il,k}^{(j)}(t),\quad j=1, \ldots, r. \tag{2.1}$$

are i.i.d. Bellman-Harris branching processes starting with a random number $Y_i\neq 0$ of ancestors.

Let τ_i be the life-period of $Z_i(t)$, i. e.

$$\tau_i=\inf\{t: Z_i(t)=0,\ t>0\}. \tag{2.2}$$

Observe that $U_n=\tau n+Xn$, $n\geq 1$ are i.i.d. random variables which form the renewal process $S_0=0$, $S_n=\sum_{i=1}^{n} U_i$ and

$$N(t)=\max\{n\geq 0: S_n\leq t\},\quad t\geq 0. \tag{2.3}$$

Then Bellman-Harris branching processes with state-dependent immigration and r types of particles can be defined as follows

$$Z(t)=Z_{N(t)+1}(t-\xi_t)\mathbf{I}_{\{\xi_t<t\}}, \tag{2.4}$$

where $\xi_t=S_{N(t)}+X_{N(t)+1}$.

Comment. The Foster-Pakes model follows from (2.4) with $r=1$ and

$$G(t)=\begin{cases}0, & t\leq 1,\\ 1, & t>1,\end{cases}\qquad K(t)=\begin{cases}0, & t\leq 0,\\ 1, & t>0.\end{cases}$$

Also, from (2.4) with $r=1$,

$$G(t)=\begin{cases}0, & t\leq 0,\\ 1-e^{-\lambda t}, & t>0,\end{cases}\qquad K(t)=\begin{cases}0, & t\leq 0,\\ 1, & t>0,\end{cases}$$

we obtain the Yamazato process.

Further, we shall use the following notations:

$$F(t, s)=(F^{(1)}(t, s), \ldots, F^{(r)}(t, s)),\ F^{(k)}(0, s)=s_k,\quad k=1, \ldots, r,$$

$$\mathcal{F}(t, s)=\mathsf{E}s^{Z_i(t)}=f(F(t, s)),\ \mathcal{F}(0, s)=f(s),$$

$$\Phi(t, s)=\mathsf{E}s^{Z(t)}=\mathsf{E}\{s_1^{Z^{(1)}(t)}\ldots s_r^{Z^{(r)}(t)}\},\quad \Phi(0, s)=1,$$

$$R(t, s)=1-\Phi(t, s),\quad Q(t, s)=1-\mathcal{F}(t, s).$$

It is well known (see [10], p. 231) that $\boldsymbol{F}(t, \boldsymbol{s})$ satisfies the following system of integral equations

(2.5)
$$F^{(k)}(t, \boldsymbol{s}) = s_k(1-G_k(t)) + \int_0^t h^{(k)}(\boldsymbol{F}(t-y, \boldsymbol{s}))\,dG_k(y),$$
$$F^{(k)}(0, \boldsymbol{s}) = s_k, \quad k=1, 2, \dots, r.$$

From (2.1) and (2.2) it follows that $V(t) = \mathsf{P}\{\tau_i \leq t\} = \mathsf{P}\{\boldsymbol{Z}_i(0) = \boldsymbol{0}\} = \mathscr{F}(t, \boldsymbol{0})$ and hence

(2.6)
$$L(t) = \mathsf{P}\{U_i \leq t\} = \int_0^t V(t-y)\,dK(y) = \int_0^t f(\boldsymbol{F}(t-y, \boldsymbol{O}))\,dK(y), \quad t \geq 0.$$

Theorem 2.1. *The p.g.f.* $\Phi(t, \boldsymbol{s})$ *satisfies the following multidimensional integral equation*

(2.7)
$$\Phi(t, \boldsymbol{s}) = 1 - K(t) - L(t) + \int_0^t \mathscr{F}(t-y, \boldsymbol{s})\,dK(y) + \int_0^t \Phi(t-y, \boldsymbol{s})\,dL(y),$$

with initial condition $\Phi(0, \boldsymbol{s}) = 1$, *where* $\mathscr{F}(t, \boldsymbol{s}) = f(\boldsymbol{F}(t, \boldsymbol{s}))$ *and* $\boldsymbol{F}(t, \boldsymbol{s})$ *is the unique solution of the system* (2.5).

Theorem 2.2. *The multidimensional p.g.f.*

$$\Phi(t, \tau;\ \boldsymbol{s}_1, \boldsymbol{s}_2) = \mathsf{E}\{\boldsymbol{s}_1^{Z(t)} \boldsymbol{s}_2^{Z(t+\tau)} \mid \boldsymbol{Z}(0) = \boldsymbol{0}\}$$

satisfies the equation

(2.8)
$$\Phi(t, \tau;\ \boldsymbol{s}_1, \boldsymbol{s}_2) = 1 - K(t+\tau) + \int_0^t \Phi(t-u, \tau;\ \boldsymbol{s}_1, \boldsymbol{s}_2)\,dL(u)$$
$$+ \int_0^t [\mathscr{F}(t+\tau-u, \boldsymbol{s}_2) - \mathscr{F}(t+\tau-u, \boldsymbol{0})]\,dK(u)$$
$$+ \int_0^t \Phi(t+\tau-u, \boldsymbol{s}_2)\,dL(u) + \int_0^t \mathscr{F}(t-u, \tau; \boldsymbol{s}_1, \boldsymbol{s}_2)\,dK(u)$$
$$+ \int_0^t \mathscr{F}(t-u, \boldsymbol{s}_1)\Big(\int_{t-u}^{t+\tau-u} \Phi(t+\tau-u-x;\ \boldsymbol{s}_2)\,dV(x)\,dK(u),$$

with initial condition $\Phi(0, \tau;\ \boldsymbol{s}_1, \boldsymbol{s}_2) = \mathsf{E}\,\boldsymbol{s}_2^{Z(\tau)} = \Phi(\tau, \boldsymbol{s}_2)$, *where* $\mathscr{F}(t, \tau;\ \boldsymbol{s}_1, \boldsymbol{s}_2) = \mathsf{E}\{\boldsymbol{s}_1^{Z_1(t)} \boldsymbol{s}_2^{Z_1(t+\tau)} \mid \boldsymbol{Z}_1(0) = Y_1\}$, $\mathscr{F}(0, \tau;\ \boldsymbol{s}_1, \boldsymbol{s}_2) = \mathscr{F}(\tau, \boldsymbol{s}_2)$ *and* $\mathscr{F}(0, 0;\ \boldsymbol{s}_1, \boldsymbol{s}_2) = f(\boldsymbol{s}_2)$.

The proof of Theorem 2.1 follows by the same arguments as in Theorem 1 of [5] and the proof of Theorem 2.2 parallels that of the corresponding result for the one-dimensional processes in [11].

Observe that (2.7) can be given in the following equivalent form

(2.9)
$$R(t, \boldsymbol{s}) = \int_0^t R(t-u, \boldsymbol{s})\,dL(u) + D(t;\ \boldsymbol{s}),$$

where

(2.10)
$$D(t, \boldsymbol{s}) = \int_0^t Q(t-u, \boldsymbol{s})\,dK(u).$$

3. Asymptotic behaviour of the moments. Denote the moments

$$A_l^k(t) = \mathsf{E}\{Z_{ij,k}^{(l)}(t)\} = \frac{\partial F^{(k)}(t, \boldsymbol{s})}{\partial s_e}\Big|_{s=1},$$

$$B^m_{kl}(t)=\begin{cases}\mathsf{E}\{Z^{(k)}_{ij,m}(t)\,Z^{(l)}_{ij,m}(t)=\dfrac{\partial^2F^{(m)}(t,\,s)}{\partial s_k\partial s_l}\big|_{s=1}, & \text{if } k\neq l,\\[2mm] \mathsf{E}\{Z^{(k)}_{ij,m}(t)[Z^{(l)}_{ij,m}(t)-1]\}=\dfrac{\partial^2F^{(m)}(t,\,s)}{\partial s_k^2}\big|_{s=1}, & \text{if } k=l.\end{cases}$$

$$M_k(t)=\mathsf{E}\{Z^{(k)}(t)\}=\frac{\partial\Phi(t,\,s)}{\partial s_k}\big|_{s=1}.$$

$$N_{kl}(t)=\begin{cases}\mathsf{E}\{Z^{(k)}(t)\,Z^{(l)}(t)=\dfrac{\partial^2\Phi(t,\,s)}{\partial s_k\partial s_l}\big|_{s=1}, & \text{if } k\neq l,\\[2mm] \mathsf{E}\{Z^{(k)}(t)[Z^{(l)}(t)-1]=\dfrac{\partial^2\Phi(t,\,s)}{\partial s_k^2}\big|_{s=1}, & \text{if } k=l,\end{cases}$$

where $k,\,l,\,m=1,\,2,\,\dots,\,r$ and further we shall use the notations $\boldsymbol{A}(t)=\|A^k_l(t)\|$, $\boldsymbol{M}(t)=(M_1(t),\,\dots,\,M_r(t))$.

Set

$$m_{ij}=\frac{\partial h^{(i)}(\boldsymbol{s})}{\partial s_j}\big|_{s=1},\quad b^i_{jk}=\frac{\partial^2h^{(i)}(\boldsymbol{s})}{\partial s_j\partial s_k}\big|_{s=1},\quad \nu_j=\frac{\partial f(\boldsymbol{s})}{\partial s_j}\big|_{s=1},\quad n_{ij}=\frac{\partial^2 f(\boldsymbol{s})}{\partial s_i\partial s_j}\big|_{s=1},$$

$$\mu_j=\int_0^\infty x\,dG_j(x),\quad a=\mathsf{E}X_l=\int_0^\infty x\,dK(x),\quad k,\,i,\,j=1,\,2,\,\dots,\,r,\quad l\geq 1,$$

$$\boldsymbol{M}=\|m_{ij}\|,\quad \boldsymbol{\mu}=(\mu_1,\,\dots,\,\mu_r),\quad \boldsymbol{\nu}=(\nu_1,\,\nu_2,\,\dots,\,\nu_r).$$

From (2.5) and (2.7) by differentiating and setting $\boldsymbol{s}=\boldsymbol{1}$ it follows that:

$$A^l_k(t)=\delta_{kl}(1-G_l(t))+\int_0^t[\sum_{i=1}^r m_{li}A^i_k(t-u)]\,dG_l(u),\tag{3.1}$$

$$B^j_{kl}(t)=\int_0^t[\sum_{i=1}^r\sum_{m=1}^r b^j_{im}A^m_l(t-u)\,A^i_k(t-u)]\,dG_j(u)+\int_0^t[\sum_{i=1}^r m_{ji}B^i_{kl}(t-u)]\,dG_j(u),\tag{3.2}$$

$$M_k(t)=\int_0^t M_k(t-u)\,dL(u)+\int_0^t[\sum_{l=1}^r \nu_lA^l_k(t-u)]\,dK(u),\tag{3.3}$$

$$N_{kl}(t)=\int_0^t[\sum_{i=1}^r\sum_{j=1}^r n_{ij}A^j_l(t-u)\,A^i_k(t-u)]\,dK(u)+\int_0^t[\sum_{i=1}^r \nu_iB^i_{kl}(t-u)]\,dK(u)$$
$$+\int_0^t N_{kl}(t-u)\,dL(u),\quad j,\,k,\,l=1,\,\dots,\,r.\tag{3.4}$$

It is well known (see [10]), that for the Bellman-Harris branching processes with $r>1$ types of particles there is a following definition: the process is subcritical if the Perron root ρ of the mean matrix M associated with $\boldsymbol{h}(\boldsymbol{s})$ is less than one, critical in case $\rho=1$ and $b=\sum_{ij,k=1}^r \nu_i b^i_{jk}u^ju^k>0$, where $\boldsymbol{u}=(u^1,\,\dots,\,u^r)$ and $\boldsymbol{v}=(v_1,\,\dots,\,v_r)$ are left and right eigenvectors of the matrix $\boldsymbol{M}$, corresponding to ρ and supercritical when $\rho>1$.

Also, there is an analog to the concept of a Malthusian parameter of a classical age-dependent process. Let $\boldsymbol{M}^*=\|m_{ij}\int_0^\infty e^{-\alpha t}dG_i(t)\|$.

The Malthusian parameter is that number α (unique, if it exists), such that the maximal eigenvalue of $\boldsymbol{M}^*$ is one.

In the supercritical case $\rho>1$, the Malthusian parameter α always exists, but in the subcritical case the parameter may not exist. From now on it will be assumed that there always exists a unique number, such that the maximal eigenvalue of $\boldsymbol{M}^*$ is one.

From now on it will be supposed that $\boldsymbol{M}$ is irreducible and regular.

Then $\boldsymbol{M}$ has the right and left eigenvectors, corresponding to ρ, such that $\boldsymbol{M}\boldsymbol{u}=\rho\boldsymbol{u}$, $\boldsymbol{v}\boldsymbol{M}=\rho\boldsymbol{v}$, $(\boldsymbol{u}, \mathbf{1})=1$, $(\boldsymbol{u}, \boldsymbol{v})=1$.

It is known (see [10], p. 312), that for the subcritical and supercritical processes hold

$$A^i_j(t)\sim\overline{A}^i_{\alpha j}\exp\{\alpha t\},\quad t\to\infty, \tag{3.5}$$

where

$$\overline{A}^i_{\alpha j}=u^i_\alpha v_{\alpha j}\frac{\int\limits_0^\infty\exp\{-\alpha u\}[1-G_j(u)]\,du}{\sum\limits_{k,l=1}^r M^l_{\alpha k}u^k_\alpha v_{\alpha l}}, \tag{3.6}$$

u^i_α and $v_{\alpha j}$ are the i-th and j-th components of the right and left eigenvectors respectively, corresponding to the Perron root ρ_α of the matrix $\boldsymbol{E}-\boldsymbol{M}^*$ and $M^l_{\alpha k}=m_{kl}\int\limits_0^\infty e^{-\alpha u}dG_k(u)$.

In the subcritical case the processes $\boldsymbol{Z}_{ij,k}(t)$ degenerate, i. e. $F^{(i)}(t, \mathbf{0})\uparrow 1$, $t\to\infty$, $i=1, 2, \ldots, r$ (see [10], Ch. VIII, Theorem 3.2, p. 238). Hence $V(0)=L(0)=0$ and $V(\infty)=L(\infty)=1$.

For the extinction probability $\boldsymbol{q}=(q_1, q_2, \ldots, q_r)$ in the supercritical case (see [10], p. 238) there exists k, such that

$$q_k=\lim_{t\to\infty}\mathsf{P}\{\boldsymbol{Z}_{ij,k}(t)=\mathbf{0}\}=\lim_{t\to\infty}F^{(k)}(t, \mathbf{0})<1,$$

i. e. $\boldsymbol{q}<\mathbf{1}$.

Therefore $V(t)\to f(\boldsymbol{q})$, $t\to\infty$. $V(0)=\mathsf{P}\{\boldsymbol{Z}_i(0)=\mathbf{0}\}=f(\mathbf{0})=0$ and $L_0=L(\infty)=f(\boldsymbol{q})$.

From (2.10) with $\boldsymbol{s}=\mathbf{0}$ it follows that $\lim\limits_{t\to\infty}D(t, \mathbf{0})=1-f(\boldsymbol{q})$. Then from equation (2.9) with $\boldsymbol{s}=\mathbf{0}$ we obtain (see [2], Section XI.6)

$$\lim_{t\to\infty}R(t, \mathbf{0})=\frac{\lim\limits_{t\to\infty}D(t, \mathbf{0})}{1-L_0}=1.$$

In this section we shall investigate the asymptotic behavior of the first two factorial moments in the subcritical ($\rho<1$) and supercritical ($\rho>1$) cases and of the moment $\mathsf{E}\{\boldsymbol{Z}(t)\boldsymbol{Z}(t+\tau)\}$ in the supercritical case. In the critical case a limit theorem is proved by Mitov [6].

Theorem 3.1. *If the process $\boldsymbol{Z}(t)$ is subcritical and $a<\infty$, $\nu<\infty$, $K(t)$, $L(t)$, $G_i(t)$, $i=1, \ldots, r$ are non-lattice, then there exists*

$$\lim_{t\to\infty}M_k(t)=\frac{\int\limits_0^\infty[\sum\limits_{i=1}^r\nu_i A^i_k(u)]\,du}{\nu_0},$$

where $\nu_0=\int\limits_0^\infty[1-L(t)]\,dt$ and $\alpha<0$ is the Malthusian parameter.

Proof. From equation (3.3) we have

$$M_k(t)=\int_0^t M_k(t-u)\,dL(u)+U_k(t), \tag{3.7}$$

where $U_k(t)=\int_0^t[\sum_{i=1}^r \nu_i A^i_k(t-u)]\,dK(u)$, $k=1, \dots, r$.

On the other hand, it is hold

$$\int_0^\infty U_k(t)\,dt=\int_0^\infty[\sum_{l=1}^r \nu_l \int_0^t A^l_k(t-u)\,dK(u)]\,dt \tag{3.8}$$

$$=\int_0^\infty dK(u)\int_u^\infty[\sum_{l=1}^r \nu_l A^l_k(t-u)]\,dt=\int_0^\infty[\sum_{l=1}^r \nu_l A^l_k(u)]\,du.$$

Now, from (3.5) and (3.6) we obtain

$$\left|\int_0^\infty A^k_l(u)\,du\right|=\left|\int_0^\infty e^{\alpha u}A^k_l(u)\,e^{-\alpha u}du\right| \tag{3.9}$$

$$\le \bar{A}^k_{\alpha l}\left|\int_0^\infty e^{\alpha u}du\right|=\bar{A}^k_{\alpha l}/\alpha<\infty.$$

Applying the basic renewal theorem (see [2], Ch. XI.1) to the equation (3.7) and using the relations (3.8)-(3.9) implies the theorem.

Denote $\tilde{L}(t)=L(t)/L(\infty)$, so that $\tilde{L}(+\infty)=1$.

Theorem 3.2. *If the functions $K(t)$, $G_i(t)$, $i=1, 2, \dots, r$ and $L(t)$ are nonlattice and the process $\mathbf{Z}(t)$ is supercritical, then*

$$\lim_{t\to\infty} M_k(t)\exp\{-\alpha t\}=M_k, \tag{3.10}$$

where $\alpha>0$ is the Malthusian parameter and

$$\bar{M}_k=\frac{\int_0^\infty e^{-\alpha u}dK(u)[\sum_{l=1}^r \nu_l \bar{A}^l_{\alpha k}]}{1-\int_0^\infty e^{-\alpha u}dL(u)}, \tag{3.11}$$

$k=1, \dots, r.$

Proof. Let $\bar{a}=\int_0^\infty e^{-\alpha u}d\tilde{L}(u)$ and $\bar{L}(u)=\int_0^u e^{-\alpha x}d\tilde{L}(x)/\bar{a}$. Applying the substitutions

$$M_k(t)=\bar{a}e^{\alpha t}\bar{M}_k(t),\quad A^j_i(t)=e^{\alpha t}\bar{A}^j_i(t),$$

$i, j, k=1, 2, \dots, r,$
from (3.3) we obtain

$$M_k(t)=c_0\int_0^t \bar{M}_k(t-u)\,d\bar{L}(u)+(1/\bar{a})\int_0^t e^{-\alpha u}\{\sum_{l=1}^r \nu_l \bar{A}^l_k(t-u)\}\,dK(u), \tag{3.12}$$

$k=1, \dots, r$, where $c_0=L_0\bar{a}<1$.

On the other hand, using (3.5) and (3.6) we have

$$\lim_{t\to\infty}\bar{A}^l_k(t)=\bar{A}^l_{\alpha k},\quad k, l=1, \dots, r, \tag{3.13}$$

where $\bar{A}^l_{\alpha k}$ are defined by (3.6).

Since

$$\int_0^\infty e^{-\alpha u}\{\sum_{l=1}^r \nu_l \overline{A}_k^l(t-u)\}\, dK(u) < \int_0^\infty e^{-\alpha u} dK(u)\{\sum_{l=1}^r \nu_l \overline{A}_{\alpha k}^l\} < \infty,$$

then using (3.13), we obtain

$$\lim_{t\to\infty}\int_0^\infty e^{-\alpha u}\{\sum_{l=1}^r \nu_l \overline{A}_k^l(t-u)\}\, dK(u) = \{\sum_{l=1}^r \nu_l \overline{A}_{\alpha k}^l\}\int_0^\infty e^{-\alpha u} dK(u) = \overline{A}_k, \quad k=1, \ldots, r.$$

Therefore by Lemma 4 (see [4], Ch. VI)

$$\lim_{t\to\infty} \overline{M}_k(t) = \overline{A}_k/(1-c_0), \quad k=1, \ldots, r,$$

which proves the theorem.

Denote

$$\mu_0 = \int_0^\infty e^{-2\alpha u} d\widetilde{L}(u) < 1,$$

$$\widetilde{L}_1(t) = (1/\mu_0)\int_0^\infty e^{-2\alpha u} d\widetilde{L}(u), \tag{3.14}$$

$$\overline{B}_{jk}^i = \frac{\sum_{l,s,m=1}^r D_{2\alpha,l}^i \int_0^\infty b_{sm}^l \exp\{-2\alpha u\} \overline{A}_{\alpha j}^s \overline{A}_{\alpha k}^m dG_i(u)}{|\boldsymbol{E}-\boldsymbol{G}_{2\alpha}|},$$

where $D_{2\alpha,j}^i$ is the complementary minor of the (i, j)-th element in $\boldsymbol{D}_{2\alpha} = \| \delta_{ij} - m_{ij}\int_0^\infty e^{-2\alpha u} dG_i(u)\|$, $\overline{A}_{\alpha j}^k$ are defined by (3.6), $\boldsymbol{G}_{2\alpha} = \| m_{ij}\int_0^\infty e^{-2\alpha u} dG_i(u)\|$ and $\alpha > 0$ is the Malthusian parameter.

Theorem 3.3. *Under conditions of Theorem 3.2 and if* $n_{ij} < \infty$, $b_{jk}^i < \infty$, $i, j, k = 1, \ldots, r$, *then*

$$\lim_{t\to\infty} N_{kl}(t) e^{-2\alpha t} = \overline{N}_{kl}, \quad k, l = 1, \ldots, r. \tag{3.15}$$

where

$$\overline{N}_{kl} = \frac{\int_0^\infty e^{-2\alpha u} dK(u)[\sum_{i=1}^r \nu_i \overline{B}_{kl}^i + \sum_{i=1}^r \sum_{j=1}^r n_{ij} \overline{A}_{\alpha k}^i \overline{A}_{\alpha l}^j]}{1-\int_0^\infty e^{-2\alpha u} dL(u)} \tag{3.16}$$

and $\overline{B}_{kl}^i$ *are defined by* (3.14).

Proof. Applying the substitutions

$$N_{kl}(t) = \mu_0 e^{2\alpha t} \overline{N}_{kl}(t),$$

$$B_{kl}^m(t) = e^{2\alpha t} \overline{B}_{kl}^m(t),$$

$$A_k^l(t) = e^{\alpha t} \overline{A}_k^l(t), \quad k, l, m = 1, \ldots, r$$

in equation (3.4) we obtain

$$\overline{N}_{kl}(t) = c_0 \int_0^t \overline{N}_{kl}(t-u)\, d\widetilde{L}_1(u) + S_{kl}(t), \tag{3.17}$$

where

$$S_{kl}(t)=(1/\mu_0)\int_0^t\{\sum_{i=1}^r \nu_i\bar{B}^i_{kl}(t-u)\}\,e^{-2\alpha u}dK(u)$$

$$+(1/\mu_0)\int_0^t\{\sum_{i=1}^r\sum_{j=1}^r n_{ij}\bar{A}^i_k(t-u)\,\bar{A}^j_l(t-u)\}\,e^{-2\alpha u}dK(u),$$

$c_0=L_0\mu_0$ and μ_0, L_0 are introduced in (3.14).

It is known (see [10], p. 314) that for supercritical Bellman-Harris branching processes with $r>1$ types of particles

$$\lim_{t\to\infty}\bar{B}^i_{jk}(t)=\bar{B}^i_{jk}, \tag{3.18}$$

where $\bar{B}^i_{jk}$ are defined by (3.14).

The relations (3.5), (3.6) and (3.18) show that as $t\to\infty$

$$\int_0^t[\sum_{i=1}^r \nu_i\bar{B}^i_{kl}(t-u)]\,e^{-2\alpha u}dK(u)\to\int_0^\infty e^{-2\alpha u}dK(u)[\sum_{j=1}^r \nu_i\bar{B}^i_{kl}],$$

$$\int_0^t[\sum_{i=1}^r\sum_{j=1}^r n_{ij}\bar{A}^i_k(t-u)\,\bar{A}^j_l(t-u)]\,e^{-2\alpha u}dK(u)\to\int_0^\infty e^{-2\alpha u}dK(u)[\sum_{i=1}^r\sum_{j=1}^r n_{ij}\bar{A}^i_{\alpha k}\bar{A}^j_{\alpha l}].$$

Therefore

$$\lim_{t\to\infty}S_{kl}(t)=\frac{\{\sum_{i=1}^r\nu_i\,\bar{B}^i_{kl}+\sum_{i=1}^r\sum_{j=1}^r n_{ij}\bar{A}^i_{\alpha k}\bar{A}^j_{\alpha l}\}\int_0^\infty e^{-2\alpha u}dK(u)}{\mu_0}=S_{kl}<\infty.$$

Applying Lemma 4 (see [4], Ch. VI) to Equation (3.17), we obtain that $\lim_{t\to\infty}\bar{N}_{kl}(t)=\bar{N}_{kl}/(1-L_0\mu_0)$, $k, l=1,\ldots,r$.

The theorem is proved.

Denote

$$N_{kl}(t,\tau)=\mathsf{E}\{Z^{(k)}(t)\,Z^{(k)}(t+\tau)\}=\left.\frac{\partial^2\Phi(t,\tau;\,s_1,s_2)}{\partial s_{1,k}\partial s_{2,l}}\right|_{s_1=s_2=1},$$

$$B_{kl}(t,\tau)=\mathsf{E}\{Z_{ij,m}(t)\,Z_{ij,m}(t+\tau)\}=\left.\frac{\partial^2F^{(m)}(t,\tau;\,s_1,s_2)}{\partial s_{1,k}\partial s_{2,l}}\right|_{s_1=s_2=1},$$

$k, l, m=1,\ldots,r$.

It is well known (see [10], p. 154, Theorem 8) that for the supercritical Bellman-Harris branching processes holds

$$\mathsf{E}\{Z_{ij,m}(t)Z_{ij,m}(t+\tau)\}=\bar{B}^m_{kl}\exp\{\alpha(2t+\tau)\}[1+o(1)],$$

where $\bar{B}^m_{kl}$, $k, l, m=1,\ldots,r$ are defined by (3.14).

Theorem 3.4. *Under the conditions of Theorem 3.3*

$$N_{kl}(t,\tau)=\bar{N}_{kl}(t,\tau)\,e^{\alpha(2t+\tau)}(1+o(1)) \tag{3.19}$$

uniformly for $\tau\geq 0$, *where* $\bar{N}_{kl}$, $k, l=1,\ldots,r$ *are defined by* (3.16).

Proof. From (2.8) by differentiating and setting $s_1=s_2=1$ we obtain

$$N_{kl}(t,\tau)=\int_0^t N_{kl}(t-u,\tau)\,dL(u)+\int_0^t[\sum_{i=1}^r\nu_iB^i_{kl}(t-u,\tau)]\,dK(u) \tag{3.20}$$

$$+\int_0^t [\sum_{i=1}^r \sum_{j=1}^r n_{ij} A_k^i (t-u)\, A_l^j (t+\tau-u)]\, dK(u)$$

$$+\int_0^t [\sum_{i=1}^r \nu_i A_k^i (t-u)](\int_{t-u}^{t+\tau-u} M_l (t+\tau-u-x)\, dV(x))\, dK(u).$$

Now applying the substitutions

$$N_{kl}(t, \tau)=\mu_0 e^{\alpha(2t+\tau)}\overline{N}_{kl}(t, \tau),$$

$$B_{kl}^m(t, \tau)=e^{\alpha(2t+\tau)}\overline{B}_{kl}^m(t, \tau),$$

$$M_k(t)=e^{\alpha t}\overline{M}_k(t),$$

in equation (3.20) we have

$$\overline{N}_{kl}(t, \tau)=c_1\int_0^t \overline{N}_{kl}(t-u, \tau)\, d\widetilde{L}_1(u)+S_{kl}(t, \tau), \tag{3.21}$$

where $c_1=L_0\mu_0$ and

$$S_{kl}(t, \tau)=(1/\mu_0)\int_0^t [\sum_{i=1}^r \sum_{j=1}^r n_{ij}\overline{A}_k^i(t-u)\overline{A}_l^j(t+\tau-u)]\, e^{-2\alpha u}dK(u) \tag{3.22}$$

$$+(1/\mu_0)\int_0^t [\sum_{i=1}^r \nu_i \overline{B}_{kl}^i(t-u, \tau)]\, e^{-2\alpha u}dK(u)$$

$$+(1/\mu_0)\int_0^t [\sum_{i=1}^r \nu_i \overline{A}_k^i(t-u)](\int_{t-u}^{t+\tau-u} e^{-\alpha x}\overline{M}_l(t+\tau-u-x)\, dV(x))\, e^{-2\alpha u}dK(u).$$

Let us denote

$$J_{kl}(t, \tau)=\int_0^t [\sum_{i=1}^r \nu_i \overline{A}_k^i(t-u)](\int_{t-u}^{t+\tau-u} \overline{M}_l(t+\tau-u-x)\, e^{-\alpha x}dV(x))\, e^{-2\alpha u}dK(u).$$

Using relations (3.5), (3.6), (3.10) and (3.11), we obtain

$$|J_{kl}(t, \tau|\leq c_{kl}\int_0^t e^{-2\alpha u}(\int_{t-u}^{t+\tau-u} e^{-\alpha x}\, dV(x))\, dK(u)$$

$$\leq c_{kl}\int_0^t e^{-2\alpha u}[V(t+\tau-u)-V(t-u)]\, dK(u)$$

$$\leq c_{kl}[\int_0^{t+\tau} e^{-2\alpha u}V(t+\tau-u)\, dK(u)-\int_0^t e^{-2\alpha u}V(t-u)\, dK(u)],$$

where $c_{kl}=\sum_{i=1}^r \nu_i\overline{A}_k^i\overline{M}_l>0$, $k, l=1, \ldots, r$.

Since $\lim_{t\to\infty}\int_0^t e^{-2\alpha u}dK(u)<\infty$ and $\lim_{t\to\infty} V(t)=f(q)$, then $\lim_{t\to\infty} J_{kl}(t, \tau)=0$, uniformly for $\tau\geq 0$, $k, l=1, \ldots, r$.

Using relations (3.22), (3.5) and (3.6) and applying Theorems 3.1, 3.2 and 3.3, we obtain

$$\lim_{t\to\infty} S_{kl}(t)=\frac{\{\sum_{i=1}^r \nu_i \overline{B}_{kl}^i+\sum_{i=1}^r \sum_{j=1}^r n_{ij}\overline{A}_{\alpha k}^i\overline{A}_{\alpha l}^j\}\int_0^\infty e^{-2\alpha u}dK(u)}{\mu_i}<\infty.$$

From (3.21) using Lemma 4 (see [4], Ch. VI) we finally obtain (3.19), which completes the proof.

4. Limit theorems. Theorem 4.1. *Suppose that the process* $\boldsymbol{Z}(t)$ *is subcritical,* $\mu<\infty$, $\nu<\infty$, $a=\int_0^\infty x\,dK(x)<\infty$ *and* $G_i(t)$, $i=1, \ldots, r$, $L(t)$ *and* $K(t)$ *are non-lattice. Then*

$$\lim_{t\to\infty} \mathsf{P}\{\boldsymbol{Z}(t)=\boldsymbol{\beta}\}=\Phi_\beta, \qquad \sum_{\beta\in \mathbf{N}^r}\Phi_\beta=1,$$

where

$$\Phi(\boldsymbol{s})=1-\frac{Q(\boldsymbol{s})}{\nu_0}, \qquad \nu_0=\int_0^\infty [1-L(t)]\,dt<\infty, \tag{4.1}$$

and

$$Q(\boldsymbol{s})=\int_0^\infty Q(t, \boldsymbol{s})\,dt, \quad |\boldsymbol{s}|\leq 1. \tag{4.2}$$

Proof. From equation (2.7) we have

$$\Phi(t, \boldsymbol{s})=\int_0^t \Phi(t-u, \boldsymbol{s})\,dL(u)+I(t, \boldsymbol{s}), \tag{4.3}$$

where $I(t, \boldsymbol{s})=1-L(t)-k(t)+\int_0^t \mathscr{F}(t-u, \boldsymbol{s})\,dK(u)$.

On the other hand,

$$I(t, \boldsymbol{s})=1-L(t)-\int_0^t Q(t-u, \boldsymbol{s})\,dK(u)=1-L(t)-D(t, \boldsymbol{s}).$$

It is well known (see [7], Theorem 6.1, p. 186) that there exists a vector $\boldsymbol{c}>\boldsymbol{0}$ such that

$$|1-\mathscr{F}(t, \boldsymbol{s})|\leq f'(\mathbf{1})\,|1-\boldsymbol{F}(t, \boldsymbol{s})|\leq \tilde{c}e^{\alpha t}, \tag{4.4}$$

where $\tilde{c}=\boldsymbol{\nu c}>0$, $\alpha<0$ is the Malthusian parameter.

Using (4.4), we obtain

$$\left|\int_0^\infty D(t, \boldsymbol{s})\,dt\right|\leq \left|\int_0^\infty\left(\int_0^t Q(t-u, \boldsymbol{s})\,dt\right)dK(u)\right|$$

$$=\left|\int_0^\infty dK(u)\int_u^\infty Q(t-u, \boldsymbol{s})\,dt\right|=\left|\int_0^\infty Q(x, \boldsymbol{s})\,dx\right|\leq \tilde{c}\left|\int_0^\infty e^{\alpha x}dx\right|<\infty.$$

Applying the basic renewal theorem (see [2], Ch. XI. 4) to equation (4.3), we conclude that

$$\lim_{t\to\infty}\Phi(t, \boldsymbol{s})=\frac{\int_0^\infty I(t, \boldsymbol{s})\,dt}{\nu_0}=\frac{\nu_0-\int_0^\infty Q(t, \boldsymbol{s})\,dt}{\nu_0}=1-\frac{Q(\boldsymbol{s})}{\nu_0}=\Phi(\boldsymbol{s}). \tag{4.5}$$

From (3.5), (3.6) and (4.4) it follows that

$$|\Phi(\boldsymbol{s})|\leq\left|\int_0^\infty I(t, \boldsymbol{s})\,dt\right|\leq\left|\int_0^\infty Q(x)\,dx\right|\leq \nu\,|\mathbf{1}-\boldsymbol{s}|\left|\int_0^\infty \boldsymbol{A}(t)\,dt\right|<\infty,$$

where $\boldsymbol{A}(t)=\| A_k^i(t)\|$.

Obviously, $\Phi(\boldsymbol{s})\to 1$ as $\boldsymbol{s}\to\mathbf{1}$, which completes the theorem.

Corollary 4.1. *Under the conditions of Theorem 4.1*

$$\mathsf{E}Z_k(\infty)=\frac{\partial\Phi(s)}{\partial s_k}\Big|_{s=1}=\frac{\int_0^\infty[\sum_{l=1}^r \nu_l A_k^l(t)]\,dt}{\nu_0},\quad k,\ l=1,\ \dots,\ r,$$

Proof. From (4.1) and (4.2) by differentiating and using (3.8) we have

$$\frac{\partial Q(s)}{\partial s_k}\Big|_{s=1}=-\int_0^\infty[\sum_{l=1}^r \frac{\partial f(s)}{\partial s_l}\Big|_{s=1}A_k^l(t)]\,dt=-\int_0^\infty\sum_{l=1}^r \nu_l A_k^l(t)\,dt.$$

Finally, we obtain

$$\frac{\partial\Phi(s)}{\partial s_k}\Big|_{s=1}=\frac{\sum_{l=1}^r\int_0^\infty \nu_l A_k^l(t)\,dt}{\nu_0},\quad k=1,\ \dots,\ r,$$

which proves the assertion.

Theorem 4.2. *Under the conditions of Theorem 3.3, if $\alpha>0$ is the Malthusian parameter, then the process $\boldsymbol{W}(t)=\boldsymbol{Z}(t)e^{-\alpha t}$ converges in mean square to a random vector $\boldsymbol{W}>\boldsymbol{0}$ whose Laplace transform $\varphi(\boldsymbol{\lambda})=\mathsf{E}\{e^{-\boldsymbol{\lambda}\boldsymbol{W}}\}$, $\boldsymbol{\lambda}=(\lambda_1,\ \dots,\ \lambda_r)$ satisfies the equation:*

$$(4.6)\qquad \varphi(\boldsymbol{\lambda})=\int_0^\infty\varphi(\boldsymbol{\lambda}e^{-\alpha u})\,dL(u)+\int_0^\infty f(\psi_1(\lambda_1e^{-\alpha u}),\ \dots,\ \psi_r(\lambda_re^{-\alpha u}))\,dK(u)-f(\boldsymbol{q}),$$

where $\psi_i(\theta)$, $i=1,\ \dots,\ r$ satisfy the following system of integral equations:

$$(4.7)\qquad \psi_i(\theta)=\int_0^\infty h^{(i)}(\psi_1(\theta e^{-\alpha u}),\ \dots,\ \psi_r(\theta e^{-\alpha u}))\,dG_i(u),\quad i=1,\ \dots,\ r.$$

Proof. It is not difficult to show that

$$(4.8)\qquad \mathsf{E}\{W^{(k)}(t+\tau)-W^{(k)}(t)\}^2=N_{kk}(t+\tau)e^{-2\alpha(t+\tau)}+M_k(t+\tau)e^{-2\alpha(t+\tau)}+N_{kk}(t)e^{-2\alpha t}+M_k(t)e^{-2\alpha t}-2e^{-\alpha(2t+\tau)}N_{kk}(t,\ \tau),$$

$k=1,\ \dots,\ r.$

Now, applying Theorems 3.2, 3.3, 3.4 for the right side of (4.8) we obtain that $\lim_{t\to\infty}\mathsf{E}\{W^{(k)}(t+\tau)-W^{(k)}(t)\}^2=0$, uniformly for $\tau>0$, which is equivalent to the mean square convergence to a random vector $\boldsymbol{W}=(W_1,\ \dots,\ W_r)>\boldsymbol{0}$.

From here it follows that

$$(4.9)\qquad \lim_{t\to\infty}\Phi(t,\ \exp\{-\lambda_1e^{-\alpha t}\},\ \dots,\ \exp\{-\lambda_re^{-\alpha t}\})=\varphi(\lambda_1,\dots,\ \lambda_r),$$

where $\varphi(\lambda_1,\ \dots,\ \lambda_r)=\prod_{i=1}^r\mathsf{E}\{e^{-\lambda_iW_i}\}$, $\lambda_i>0,\ i=1,\ \dots,\ r.$

It is known (see [1], p. 226) that under the conditions of the theorem we have

$$(4.10)\qquad \boldsymbol{Z}_{ij,k}(t)\,e^{-\alpha t}\xrightarrow{L_2}\widetilde{W}\boldsymbol{v},$$

where $\boldsymbol{v}$ is the left eigenvector, corresponding to the maximal eigenvalue of the matrix $\boldsymbol{M}$ and the random variable $\widetilde{W}$ has Laplace transforms $\varphi_l(\theta)=\mathsf{E}\{\exp(-\theta\widetilde{W}\mid\boldsymbol{Z}_{ij,l}(0)=\boldsymbol{e}_l\}$, $l=1,\ \dots,\ r$ which satisfy the system (4.7).

Setting $(s_1,\ \dots,\ s_r)=(\exp\{-\lambda_1e^{-\alpha t}\},\ \dots,\ \exp\{-\lambda_re^{-\alpha t}\})$ in equation (2.7), we obtain

$$\Phi(t, \exp\{-\lambda_1 e^{-\alpha t}\},\dots, \exp\{-\lambda_r e^{-\alpha t}\}) = 1-K(t)-L(t) \tag{4.11}$$
$$+\int_0^t \Phi(t-u, \exp\{-\lambda_1 e^{-\alpha(t-u)}e^{-\alpha u}\}, \dots, \exp\{-\lambda_r e^{-\alpha(t-u)}e^{-\alpha u}\})\, dL(u)$$
$$+\int_0^t f(\boldsymbol{F}(t-u, \exp\{-\lambda_1 e^{-(t-u)}e^{-\alpha u}\}, \dots, \exp\{-\lambda_r e^{-(t-u)}e^{-\alpha u}\}))dK(u).$$

As $t\to\infty$ from (4.11) using (4.9) and (4.10), we obtain (4.6), which proves the theorem.

Denote $\widetilde{\boldsymbol{W}}_{ij,k}(t)=\boldsymbol{Z}_{ij,k}(t)e^{-\alpha t}$, $k=1, \dots, r$.

Theorem 4.3. *Assume conditions of Theorem* 3.3 *hold and*

$$\int_0^\infty \mathsf{E}\{\widetilde{W}^{(l)}_{ij,k}(t)-\widetilde{W}vl\}^2<\infty, \tag{4.12}$$

$k, l=1, \dots, r.$

Then $\lim_{t\to\infty} \boldsymbol{Z}(t)/e^{\alpha t}=\boldsymbol{W}$ *a. s.*

Proof. In the case $h_i(\mathbf{0})=0$, $i=1, \dots, r$ it follows that $\boldsymbol{Z}(t)$ is a Bellman-Harris process with $r>1$ types of particles and it is known (see Mode [7], p. 143) that (4.12) is a sufficient condition for a.s. convergence.

For the Bellman-Harris process $\boldsymbol{Z}_i(t)$ let $\boldsymbol{\xi}_{i,1}=(\xi^{(1)}_{i,1}(t), \dots, \xi^{(r)}_{i,1}(t))$ be the number of particles which are born up to time t and $\boldsymbol{\xi}_{i,2}(t)=(\xi^{(1)}_{i,2}(t), \dots, \xi^{(r)}_{i,2})$ be the number of particles which are dead up to time t.

Denote $S^{(k)}_l(t) = \sum_{j=1}^{N(t)} \xi^{(k)}_{j,l}(t)$, $k=1, \dots, r$, $l=1, 2$, where $N(t)$ is defined in (2.3) and the process $N(t)$ is independent from $\boldsymbol{\xi}_{i,k}(t)$, $k=1, 2$.

If there exists $i\in\{1, \dots, r\}$, such that $h^{(i)}(\mathbf{0})>0$, then we have the representation $Z^{(k)}(t)=S^{(k)}_1(t)-S^{(k)}_2(t)$, $k=1, \dots, r$.

Under the conditions of the theorem $N(t)\to\nu^*$ a.s. as $t\to\infty$ and $\mathsf{E}\nu^*=1/(1-L_0)<\infty$ (see Feller [2], Section XI.2).

On the other hand, we have

$$\eta^{(k)}_{i,l}=\lim_{t\to\infty}[\xi^{(k)}_{i,l}e^{-\alpha t}], \quad l=1, 2, \quad k=1, \dots, r, \ a.s.$$

(see Mode [7], p. 143).

Therefore

$$\frac{S^{(k)}_l(t)}{e^{\alpha t}}=\sum_{i=1}^{N(t)} \xi^{(k)}_{il}(t)\, e^{-\alpha t}\to\sum_{i=1}^{\nu^*} \eta^{(k)}_{i,l}, \text{ a. s. as } t\to\infty, \quad l=1, 2,$$

for $k=1, 2, \dots, r$.

Hence $\lim_{t\to\infty} \boldsymbol{W}(t)=\boldsymbol{W}$ a. s. which completes the theorem.

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