Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

ON COMPACTLY DETERMINED SPACES

B. D. DOITCHINOV, V. I. MALYHIN

In this paper we introduce the notion of a compactly determined space and investigate some natural questions which arise in this connection. We prove that a space is hereditarily compactly determined iff it is a Frechet-Urysohn space. We show that the assertion that the product $Q \times 2^{\omega_1}$ is compactly determined is independent of the axioms ZFC.

Describing topological spaces* in terms of their compact subspaces turned out to be useful. In such a way one can obtain k-spaces, k_1-spaces. Frechet-Urysohn spaces, for example. On the other hand, a space may be supposed to possess some properties with respect to its dense subsets. This is the way, for instance, to obtain extensions of Wallman type.

In this paper we consider a concept which combines both approaches.

Definition (D. Doitchinov [1]). *The space Y is said to be a compactly determined extension of its dense subspace X, iff for each point $y \in Y$ there exists a $K \subset X$ such that $y \in [K]_Y$ and $[K]_Y$ is compact.*

So, one sees, that every compact space is a compactly determined extension of each its dense subset. This is probably the reason for the next definition**:

Definition (G. Choquet). A space is called compactly determined, iff it is a compactly determined extension of each its dense subset.

The class of compactly determined spaces is large enough. In particular, such are all k_1-spaces, all locally compact spaces and all the spaces with a countable character. The property of a space to be compactly determined is hereditary by open subspaces; open mappings conserve it. In fact, the following more general result holds:

Proposition 1. *Let $\varphi: X \to Y = \varphi(X)$ be a mapping such that the image of every non-empty open set has a non-empty interior, and let X be a compactly determined space. Then the space Y is also a compactly determined space.*

Proof. Let S be dense in Y and let $y \in Y$. Then $E = \varphi^{-1}(S)$ is dense in X. In virtue of the fact that X is compactly determined, for each point $x \in \varphi^{-1}(y)$ one can find a $K \subset E$, such that $x \in [K]$ and $[K]_X$ is compact. It is clear, that $\varphi(K) \subset S$, $y \in [\varphi(K)]$ and $[\varphi(K)]$ is compact.

Corollary. *If the product of a family of spaces is compactly determined, then every sub-product, and, in particular, every single space of the family, is compactly determined.*

The following result is also of a certain interest:

Theorem (D. Doitchinov). *Let X be a compactly determined space, M — a dense subset of X, and $\varphi: M \to Y$. A continuous extension $\tilde{\varphi}: X \to Y$ of φ exists iff the following two conditions are fulfilled*:

*All topological spaces in this paper are supposed to be Tychonoff, and all the mappings to be continuous. Sometimes the subscript of the closure operator is omitted, if this would not lead to misunderstanding.

** Arisen in a private conversation between G. Choquet and D. Doitchinov, as the latter informed us.

SERDICA Bulgaricae mathematicae publicationes, Vol. 17, 1991, p. 157-160.

a) if $A \subset M$ and $[A]_X$ is compact, then $[\varphi(A)]_Y$ is compact.

b) if $B, C \subset Y$, $B \cap C = \emptyset$, B is compact and C is closed, then $[\varphi^{-1}(B)]_X \cap [\varphi^{-1}(C)]_X = \emptyset$.

This theorem is very similar to the well-known Taimanov theorem (see e. g. [2]). It can be proved analogously (but it can also be obtained directly from Theorem 2 of [1]).

In this paper we answer some natural questions, concerning the notion of compactly determined spaces. We are grateful to prof. D. Doitchinov for posing these questions and for his help in preparing this paper.

1. It is easily seen that Frechet-Urysohn spaces are hereditarily compactly determined. V. I. Ponomarev has made the conjecture that this property characterizes that kind of spaces. We answer positively this conjecture.

Theorem 1. *A space is hereditarily compactly determined if and only if it is a Frechet-Urysohn space.*

Proof. Let X be a hereditarily compactly determined space. According to a theorem of A. Arhangelskii (of [3, p. 160]), if X is a non-Frechet-Urysohn space, then there exists a $Y \subset X$ and $y \in Y$ such that y is non-isolated in Y but $y \notin [B \setminus \{y\}]$ for any compact $B \subset Y$. So Y is not a compact extension of its dense subspace $Y \setminus \{y\}$, and therefore Y is non-compactly determined.

We need further the following notions (A. Arhangelskii). A family of compact subspaces of a space X is called a k-base for X if every compact subspace of X is contained in some element of the family. The least among the cardinalities of all possible k-bases is called the k-weight of the space X.

Let us recall also that a space X is called nowhere compact if no closure of a non-empty open set in it is compact.

Theorem 2. *If a space X is nowhere compact, and its k-weight is λ, and if $|Y| \geq 2$, then $X \times Y$ is not compactly determined.*

Proof. Let us fix a point $x_0 \in X$, a k-base (for X) $\{K_\alpha \; \alpha \in \lambda\}$ of cardinality λ, a point $y_0 \in Y$ and a neighbourhood $V_0 \neq Y$ of y_0. We define the set $Z \subset X \times Y^\lambda$ in the following way: $\langle x, \tilde{y} \rangle \in Z$ if $x \neq x_0$ and $\tilde{y}(\alpha) \notin Y_0$ whenever $x \in K_\alpha$. We will show that Z is dense in $X \times Y^\lambda$. Let $U \subset X$, $W \subset Y^\lambda$ be open and non-empty. One can suppose that $W = W_{\alpha_1} \times \ldots \times W_{\alpha_n} \times Y^{\lambda \setminus \{\alpha_1, \ldots, \alpha_n\}}$ where $W_{\alpha_i} \subset Y$ are open $(i = 1, \ldots, n)$. Since X is nowhere compact, there is an $x \neq x_0$ with $x \in U \setminus (\cup \{K_{\alpha_i} : i = 1, \ldots, n\})$. We pick out a point $\tilde{y} \in Y^\lambda$ such that $\tilde{y}(\alpha_i) \in W_{\alpha_i}$ for $i = 1, \ldots, n$ and, in the case when $x \in K_\alpha$, $\tilde{y}(\alpha) \notin V_0$ for $\alpha \notin \{\alpha_1 \ldots, \alpha_n\}$. Then $\langle x, \tilde{y} \rangle \in Z \cap (U \times W)$. So Z is dense in $X \times Y^\lambda$.

We will verify that $X \times Y^\lambda$ is not a compactly determined extension of X.

Denote by $\bar{y}_0$ a point of $X \times Y^\lambda$ such that $\bar{y}_0(\alpha) = y_0$ for $\alpha \in \lambda$. Suppose there exists a set $M \subset Z$ such that $\langle x_0, \bar{y}_0 \rangle \in [M]$ and that $T = [M]_{X \times Y^\lambda}$ is compact. Then $\pi_X T$ is a compact subset of X and therefore there is a $K_\alpha \supseteq \pi_X T$. By the definition of the set Z we have $\pi_{(\alpha)} M \cap Y_0 = \emptyset$ and hence $\bar{y}_0 \notin [\pi_{Y^\lambda} M]$ what contradicts to the fact that $\langle x_0, \bar{y}_0 \rangle \in [M]$. This achieves the proof.

Let us note that, as it is seen from considerations in the proof it follows that $X \times Y^\lambda$ is a non-k_1-space.

Corollary 1. *The product of a family of compactly determined spaces can be non-compactly determined (even in the case of two factors only).*

Indeed, the space of rational numbers $\mathbf{Q}$ is nowhere compact and its k-weight does not exceed $\mathfrak{c}$, hence, by Theorem 2, the space $\mathbf{Q} \times 2^{\mathfrak{c}}$ is not compactly determined, although $\mathbf{Q}$ and $2^{\mathfrak{c}}$ are.

Corollary 2. *A space X is compact if and only if X^λ is compactly determined for any λ.*

Indeed, if X is not compact, then X^{ω_0} is nowhere compact and the k-weight of X^{ω_0} does not exceed $\lambda = 2^{|X|}$. Therefore, by Theorem 2, the space $X^{\omega_0} \times X^\lambda = X^\lambda$ is non-compactly determined.

Corollary 3. *The product $\mathbf{I} \times 2^{\mathfrak{c}}$, where $\mathbf{I}$ is the space of irrationals, is not compactly determined.*

Indeed, $\mathbf{I}$ is nowhere compact and its k-weight is less than $\mathfrak{c}$.

Let us recall that the following assertion, sometimes called Booth Lemma, is an important consequence of Martin's Axiom (see [4]).

BL. If ξ is a collection of subsets of ω with $|\xi| < \mathfrak{c}$ and if $|\cap \xi'| = \omega_0$ for any finite subcollection $\xi' \subset \xi$, then there exists an infinite set $K \subset \omega$ such that $|K \setminus A| < \omega_0$ whenever $A \in \xi$.

Theorem 3. [**BL**]. *Let X be a space with a countable network of compacts and such that its character is less than $\mathfrak{c}$ in any point. Then the space $X \times B$ is compactly determined for any compact space B with character less than $\mathfrak{c}$ in any point.*

Proof. Let $\{X_n : n \in \omega\}$ be a countable network of compacts in X, Z be dense in $X \times B$, $\langle x_0, b_0 \rangle$ be an arbitrary point of $X \times B$, and $\mathscr{V}$ be a base in this point with cardinality less than $\mathfrak{c}$. We set $Z_n \subset Z \cap (X_n \times B)$. If there are finitely many sets Z_n such that $\langle x_0, b_0 \rangle$ is a limit point of their union, then the theorem evidently holds. Otherwise, for any $V \in \mathscr{V}$, let us set $A(V) = \{n \in \omega : Z_n \cap V \neq \emptyset$ and $X_n \subset \pi X V\}$ and $\xi = \{A(V) : V \in \mathscr{V}\}$. One easily checks the conditions of Booth Lemma. Denote by K the subset of ω, whose existence is guaranteed by this Lemma. Then one verifies that the family $\{X_n : n \in K\}$ converges to x_0 in the sense that the set $\{n : X_n \notin 0_{x_0}\}$ is finite for any neighbourhood 0_{x_0} of x_0. It follows that the set $[\cup \{X_n : n \in K\}_{X \times B}$ is compact. On the other hand the closure $[M]_{X \times B}$ of the set $M = \cup \{Z_n : n \in K\}$ is also compact and $\langle x_0, b_0 \rangle \in [M]_{X \times B}$. Thus the theorem is proved.

Corollary 1 [**BL** & $\neg$ **CH**]. *The product $\mathbf{Q} \times 2^{\omega_1}$ is compactly determined.*

Corollary 2. *The assertion of Corollary 1 is independent of the axioms ZFC.*

Indeed, assuming CH, the k-weight of the space $\mathbf{Q}$ of rational numbers is equal to ω_1, hence, by Theorem 2, the space $\mathbf{Q} \times 2^{\omega_1}$ is not compactly determined, which contradicts to Corollary 1.

Using considerations similar to those in the proof of Theorem 3, as well as in the proof of Theorem 5 (which follows) one establishes Theorem 4.

Theorem 4. [BL]. *If Y is a σ-compact space with a countable character and B is a compact whose character in every point is less than $\mathfrak{c}$, then $Y \times B$ is compactly determined.*

Theorem 5 [BL]. *If B is a compact whose character in every point is less than $\mathfrak{c}$ and $\mathbf{I}$ is the space of the irrationals, then the product $\mathbf{I} \times B$ is compactly determined.*

Proof. Let Z be a dense subspace of the space $\mathbf{I} \times B$, and $\langle j_0, b_0 \rangle$ be a point of $\mathbf{I} \times B$. Since B is compact and $t(b_0, B) \leq \chi(b_0, B) < \mathfrak{c}$, there exists a family $\mathscr{L}$ of countable subsets $l \subset Z$ such that the following holds: $\langle j_0, b_0 \rangle \in [\cup \mathscr{L}]$, $|\mathscr{L}| \leq t(b_0, B) < \mathfrak{c}$ and all limit points of each l are contained in $\{j_0\} \times B$. Therefore the cardinality of the set $E = \pi_1(\cup \mathscr{L})$ is also less than $\mathfrak{c}$. On the other hand, the space $\mathbf{I}$ is homeomorphic to the space ω^ω considered with Tychonoff topology on it. By BL every subset of ω^ω with cardinality less than $\mathfrak{c}$ is bounded with respect to the natural ordering of ω^ω, i. e. $f \leq g$ if and only if there is some $k \in \omega$ such that $f(n) \leq g(n)$ for each $n \in \omega \setminus k$. It follows that every subset of $\mathbf{I}$ with cardinality less than $\mathfrak{c}$ is contained in some union of countably many compacts. As $|E| < \mathfrak{c}$, this is true for E. So we have $\cup \mathscr{L} \subset \cup \{K_n : n \in \omega\}$, where K_n are compacts. We can assume that $(\bigcup\mathscr{L})\cap K_n$ is dense in K_n for each $n\in\omega$. Therefore $\langle j_0, b_0\rangle\in[\bigcup_n Z_n]$, where $Z_n=K_n\cap Z$. One can suppose that $\langle j_0, b_0\rangle\notin[Z_n]$ for each $n\in\omega$. Let $\mathscr{V}$ be a neighbourhood base for the point $\langle j_0, b_0\rangle$ with cardinality less than $\mathfrak{c}$, and let $\mathscr{W}$ be a countable neighbourhood base for the point j_0 consisting of closed-open subsets. Consider the countable family of sets $\mathscr{E}=\{Z_n\cap(W\times B): n\in\omega, W\in\mathscr{W}\}$ and set, for any $V\in\mathscr{V}$, $A(V)=\{E\in\mathscr{E}: E\cap V\neq\emptyset$ and $\pi_I E\subset\pi_I V\}$ and $\xi=\{A(V): V\in\mathscr{V}\}$. It is easy to check that one can apply Booth Lemma (replacing ω by $\mathscr{E}$). Let K be that subfamily of $\mathscr{E}$ which exists by Booth Lemma. One verifies that the family $\pi_I K$ (consisting of subsets of $\mathbf{I}$) converges to the point j_0 in the sense explained in the proof of Theorem 3. It follows that the set $[\bigcup K]_X$ is compact. One checks also that $\langle j_0, b_0\in[\bigcup K]$. This achieves the proof.

Theorem 6. *Let $|A|>\omega_0$ and let, for each $\alpha\in\mathscr{A}$, X_α be a space which is not countably compact and contains a point with a countable character. Then $X=\Pi\{X_\alpha:\alpha\in\mathscr{A}\}$ is not compactly determined.*

Proof. Let us pick out, for each $\alpha\in\mathscr{A}$, a countable discrete closed subset $K_\alpha=\{x_n^\alpha: n\in\omega\}$ of X_α. Select a point x_α with a countable character and countable decreasing base $\{V_n^\alpha: n\in\omega\}$ for x_α, such that $K_\alpha\cap V_0^\alpha=\emptyset$. We define a set $Z\subset\Pi\{X_\alpha:\alpha\in\mathscr{A}\}$ in the following manner. A point $\tilde{z}\in\Pi\{X_\alpha:\alpha\in\mathscr{A}\}$ belongs to Z if there exists a finite subfamily $a(\tilde{z})\subset\mathscr{A}$ with the property: for each $\beta\in\mathscr{A}\setminus a(\tilde{z})$ we have $\tilde{z}(\beta)\in K_\beta$ and, if $\tilde{z}(\alpha)\in V_n^\alpha$ for some $\alpha\in a(\tilde{z})$, then $\tilde{z}(\beta)\in K_\beta\setminus\{x_i^\beta: i\leq n\}$. It is clear that the set Z is dense in X. Let $\bar{x}=\{x_\alpha:\alpha\in\mathscr{A}\}$. Suppose that $\bar{x}\in[M]$ for some $M\subset Z$. Fix a $\alpha_0\in\mathscr{A}$. For each n the set $V_n^{\alpha_0}\times\Pi\{X_\beta:\beta\in\mathscr{A}, \beta\neq\alpha_0\}$ is open and non-empty. Therefore it contains a point $\tilde{z}^n$ of M. The family $\bigcup\{a(\tilde{z}^n): n\in\omega\}$ being countable, there exists a $\beta_0\in\mathscr{A}$ not belonging to it. It is clear that the sequence $\{\tilde{z}^n(\beta_0): n=1, 2, \ldots\}$ contains infinitely many different points, and so the set $\pi_{\{\beta_0\}}([M]_X)$ is not compact. Consequently the set $[M]_X$ is not compact either. Therefore the space X is not compactly determined.

Corollary. $\mathbf{Q}^{\omega_1}$, $\mathbf{I}^{\omega_1}$ are not compactly determined.

REFERENCES

1. D. B. Doitchinov. Compactly determined extensions of topological spaces. *Serdica*, **11**, 1985, 269-286.
2. R. Engelking. General Topology. Warszawa, 1977.
3. A. V. Arhangelskii, V. I. Ponomarev. Foundations of General Topology by Problems and Examples. M., 1974 (in Russian).
4. J. Barwise (Ed.). Handbook of Mathematical Logic., 1977.

Institute of Mathematics
P. O. Box 373
Sofia 1090
Bulgaria

Received 18. 04. 1991