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A CONVERSE THEOREM FOR THE BEST ALGEBRAIC APPROXIMATION IN $L_p[-1, 1]$ $(0<p<1)$

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The problem of characterizing the best algebraic approximation in a finite interval has been solved by K. Ivanov [1, 2, 3], Ditzian and Totik [8] for $1\leqq p\leqq\infty$. In this paper we prove a converse theorem for the best algebraic approximation in $L_p[-1, 1]$, $0<p<1$ which completes the direct theorem proved in [9].

1. Introduction. We shall consider functions belonging to the spaces $L_p[-1, 1]$, $0<p<1$ and as usual we denote

$$\|f\|_{L_p[-1,1]}=\{\int_{-1}^{1}|f(x)|^p dx\}^{1/p},\quad \|f\|_p=\|f\|_{L_g[-1,1]}.$$

Let $H_n(T_n)$ be the set of all algebraic (trigonometric) polynomials of degree at most n. If $w\in C_{[-1,1]}$, $w\geqq 0$, then the best L_p approximation of f by polynomials of degree n with the weight w is given by

$$E_n(w;f)_p=\inf\{\|w.(f-Q)\|_p:Q\in H_n\};\ E_n(1;f)_p=E_n(f)_p.$$

The best L_p approximation of a 2π-periodic function by trigonometric polynomials of degree n is given by

$$E_n^T(f)_p=\inf\{\|f-t\|_{L_p[0,2\pi]}:t\in T_n\}.$$

For $0<p<1$ the following theorem for the best trigonometric approximation was proved in [6, 7].

Theorem A. *If $f\in L_p[0, 2\pi]$, $0<p<1$, $k, n\in N$, N is the set of all natural numbers, then we have*

$$\omega_k(f, n^{-1})_p\leq\frac{c(p,k)}{n^k}\{\sum_{\nu=0}^{n}(\nu+1)^{kp-1}(E_\nu^T(f)_p)^p\}^{1/p},\tag{1.1}$$

where $\omega_k(f,\delta)_p$ is the k-th modulus of L_p continuity of f, i. e.

$$\omega_k(f,\delta)_p=\sup_{|h|\leq\delta}\|\Delta_h^k f(x)\|_{L_p[0,2\pi]}.\tag{1.2}$$

The finite difference $\Delta_h^k f(x)$ is defined as

$$\Delta_h^k f(x)=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}.f(x+ih).\tag{1.3}$$

We denote by $c(A, B, \ldots)$ the positive constants, depending only on A, B etc. The theory of best trigonometric approximation in $L_p[0, 2\pi]$, $0<p<1$ reached a high level of completeness in the works of V. Ivanov, Štorozenko et al. [6, 7]. But some difficulties due to the boundary effect and to the fact that we consider a quasinormed space appear in the solution of the corresponding algebraic problem. This paper contains a theorem, similar to Theorem A for the algebraic problem. We cannot use the usual integral moduli of smoothness $\omega_k(f, \delta)_p$ as a characteristic of the best approximation, again because of the boundary effect. As a characteristic of the best algebraic approximation in $L_p[-1, 1]$, $0<p<1$ we shall use the moduli

$$(1.4) \qquad \tau_k(f, w; \Delta_n)_{p,p}=\| w(\cdot)\,\omega_k(f,\cdot; \Delta_n(\cdot))_p\|_{L_p[-1,1]}, \; 0<p<1,$$

$\tau_k(f, 1; \Delta_n)_{p,p}=\tau_k(f, \Delta_n)_{p,p}$, where the local L_p moduli are defined by

$$(1.5) \qquad \omega_k(f, x; \Delta_n(x))_p=[(2\Delta_n(x))^{-1}\int_{-\Delta_n(x)}^{\Delta_n(x)}|\Delta_v^k f(x)|^p dv]^{1/p}.$$

For $x\in[-1, 1]$ we set $\Delta_n(x)=(1-x^2)^{1/2}/n+1/n^2$ and by $\Delta_v^k f(x)$ we denote the expression (1.3) if $x, x+k.v\in[-1, 1]$ and 0 otherwise. The moduli (1.4) for $1\leq p\leq\infty$ have been introduced for the first time by K. Ivanov [1] and using these moduli he proved in [2] the following converse theorem:

Theorem B. *If* $f\in L_p[-1, 1]$, $1\leq p\leq\infty$ *then*

$$(1.6) \qquad \tau_k(f, \Delta_n)_{p,p}\leq\frac{c(k)}{n^k}\sum_{s=0}^{n}(s+1)^{k-1}E_s(f)_p.$$

Here we shall prove the following converse theorem for the best algebraic approximation in $L_p[-1, 1]$, $0<p<1$.

Theorem 1. *Let* $f\in L_p[-1, 1]$, $0<p<1$. *Then we have*

$$(1.7) \qquad \tau_k(f, \Delta_n)_{p,p}\leq\frac{c(k, p)}{n^k}\{\sum_{s=0}^{n}(s+1)^{kp-1}.E_s^p(f)_p\}^{1/p}.$$

In [9] we proved the following direct theorem for the best algebraic approximation in $L_p[-1, 1]$, $0<p<1$.

Theorem 2. *If* $f\in L_p[-1, 1]$, $0<p<1$, *then*

$$E_n(f)_p\leq c(k, p).\tau_k(f, A(k, p).\Delta_n)_{p,p}.$$

We expect that the constant $A(k, p)$ may be replaced by 1. Then from Theorem 1 and 2 we obtain the following characterization of the best algebraic approximation in $L_p[-1, 1]$, $0<p<1$.

Corollary 1: *If* $0<\alpha<k$, $f\in L_p[-1, 1]$, $0<p<1$, *then*

$$E_n(f)_p=0(n^{-\alpha})\Leftrightarrow\tau_k(f, \Delta_n)_{p,p}=0(n^{-\alpha}).$$

2. Some auxiliary results

Lemma 1. *If* $f\in L_p[-1, 1]$, $0<p<1$, *then*

$$(2.1) \qquad \tau_k(f, \Delta_n)_{p,p}\leq c(k, p)\|f\|_p.$$

Proof. We set

$$\tilde{g}(x)=\begin{cases} f(x), & \text{if } |x|\leq 1\\ 0, & \text{if } |x|>1\end{cases}$$

and have

$$\tau_k^p(f, \Delta_n)_{p,p}=\int_{-1}^{1}\frac{1}{2\Delta_n(x)}\int_{-\Delta_n(x)}^{\Delta_n(x)}|\Delta_h^k f(x)|^p dh dx$$

$$=\int_{-1}^{1}\frac{1}{2\Delta_n(x)}\int_{-\Delta_n(x)}^{\Delta_n(x)}|\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}\tilde{g}(x+r.h)|^p dh dx$$

$$\leq \sum_{r=0}^{k} \binom{k}{r}^p \int_{-1}^{1} \frac{1}{2\Delta_n(x)} \int_{-\Delta_n(x)}^{\Delta_n(x)} |\tilde{g}(x+r.h)|^p dh dx.$$

From (4.8) in [1] it follows that

$$(2.2) \qquad \left\{\frac{1}{2\Delta_n(x)} \int_{-\Delta_n(x)}^{\Delta_n(x)} |\tilde{g}(x+r.h)^p dh\right\}^{1/p} \leq \frac{k}{r} \left\{\frac{1}{2k\Delta_n(x)} \int_{-k\Delta_n(x)}^{k\Delta_n(x)} |\tilde{g}(x+t)|^p dt\right\}^{1/p}.$$

Using the above inequalities and (2.5) from [1], we have

$$(2.3) \qquad \tau_k^p(f, \Delta_n)_{p,p} \leq \|\tilde{g}\|_p^p + c(k, p) \int_{-1}^{1} \frac{1}{2k\Delta_n(x)} \int_{-k\Delta_n(x)}^{k\Delta_n(x)} |\tilde{g}(x+t)|^p dt dx$$

$$\leq \|f\|_p^p + c(k, p) \int_{-1}^{1} \int_{-k\Delta_n(x)}^{k\Delta_n(x)} \frac{|\tilde{g}(x+t)|^p}{\Delta_n(x+t)} dt dx$$

$$= \|f\|_p^p + c(k, p) \int_{-1}^{1} \int_{x-k\Delta_n(x)}^{x+k\Delta_n(x)} \frac{|\tilde{g}(y)|^p}{\Delta_n(y)} dy dx.$$

Let $\mu(x)=\{y: x-k.\Delta_n(x)<y<x+k.\Delta_n(x)\}$. From (2.5) in [1] it follows that

$$(2.4) \qquad y-k(4k+2)\Delta_n(y)<y-k.\Delta_n(x)<x<y+k.\Delta_n(x)<y+k(4k+2)\Delta_n(y).$$

Thus

$$(2.5) \qquad |\mu(x)| \leq 2k(4k+2)\Delta_n(y).$$

Using (2.3) and (2.5), we get

$$\tau_k^p(f, \Delta_n)_{p,p} \leq \|f\|_p^p + c'(k, p)\|\tilde{g}\|_p^p = c''(k, p)\|f\|_p^p.$$

Thus lemma 1 is proved.

Lemma 2. *Let* $q_n \in T_n$, $\gamma>0$, $0<p<1$. *Then we have*

$$(2.6) \qquad \|q_n'(x)|\sin x|^\gamma\|_{L_p[-\pi,\pi]} \leq c(\gamma, p)n\|q_n(x)|\sin x|^\gamma\|_{L_p[-\pi,\pi]},$$

$$(2.7) \qquad \|q_n(x)|\sin x|^{\gamma-1}\|_{L_p[-\pi,\pi]} \leq c(\gamma, p)n\|q_n(x)|\sin x|^\gamma\|_{L_p[-\pi,\pi]}.$$

Proof. The inequality (2.6) follows from Corollary 1.2 in [5] for $\rho=\sigma=0$, $\mu=\gamma$, $r=1$, $p=q$ and (2.7) is obtained from the same Corollary for $\rho=\sigma=0$, $r=0$, $q=p$.

Lemma 3. (Markov's inequality). *If* $g(x) \in H_m$, $0<p<1$, $m, r \in N$, *then*

$$(2.8) \qquad \|g^{(r)}.m^{-2r}\|_p \leq c^r(p)\|g\|_p.$$

Proof. The assertion follows from (2.17) in [4] for $\rho=\sigma=0$, $q=p$.

Lemma 4. *Let* $g(x) \in H_m$, $i \in N$, $1 \leq i \leq m$, $0<p<1$. *Then we have*

$$(2.9) \qquad \|g^{(i)}(\Delta_m)^i\|_p \leq c(p)c_1^i(p)i!\|g\|_p.$$

Proof. We have $(\Delta_m(x))^i \leq 2^i \max\{m^{-2i}(m^{-1}(\sqrt{1-x^2})^i\}$. Lemma 3 gives that

$$(2.10) \qquad \|g^{(i)}m^{-2i}\|_p \leq c^i(p)\|g\|_p.$$

Now we shall prove the following inequality for $f \in H_m$

$$(2.11) \qquad \|f'(x)(m^{-1}\sqrt{1-x^2})^{i+1}\|_p \leq c(p),\ i\|f(x)(m^{-1}\sqrt{1-x^2})^i\|_p.$$

Setting $x=\cos y$, the function $f(x)=f(\cos y)=t_m(y)$ is an even trigonometric polynomial. Then (2.11) is equivalent to

$$(2.12) \qquad \int_0^\pi |t_m'(y)\sin^i y|^p \sin y\, dy \leq c^p(p)m^p i^p \int_0^\pi |t_m(y)\sin^i y|^p \sin y\, dy.$$

We have

$$(2.13)\qquad t'_m(y)(\sin y)^{i+1/p}=(t_m(y)(\sin y)^i)'(\sin y)^{1/p}-it_m(y)(\sin y)^{i-1+1/p}\cos y.$$

From (2.6) with $\gamma=1/p$, $n=m+i$ we get

$$(2.14)\qquad \|(t_m(y)(\sin y)^i)'(\sin y)^{1/p}\|_{L_p[0,\pi]}$$
$$=2^{-1/p}\|(t_m(y)(\sin y)^i)'\,|\sin y|^{1/p}\|_{L_p[0,2\pi]}$$
$$\leq 2^{-1/p}c(p)(m+i)\|t_m(y)(\sin y)^i\,|\sin y|^{1/p}\|_{L_p[0,2\pi]}$$
$$=c(p)(m+i)\|t_m(y)(\sin y)^{i+1/p}\|_{L_p[0,\pi]}.$$

From (2.7) with $\gamma=1/p+1$, $n=m+i-1$ we get

$$(2.15)\qquad \|t_m(y)(\sin y)^{i-1+1/p}\|_{L_p[0,\pi]}=2^{-1/p}\|t_m(y)(\sin y)^{i-1}\,|\sin y|^{1/p}\|_{L_p[0,2\pi]}$$
$$\leq 2^{-1/p}c(p)(m+i-1)\|t_m(y)(\sin y)^{i-1}\,|\sin y|^{1/p+1}\|_{L_p[0,2\pi]}$$
$$=c(p)(m+i-1)\|t_m(y)(\sin y)^{i+1/p}\|_{L_p[0,\pi]}.$$

From $i\leq m$, (2.13), (2.14) and (2.15) follows the inequality (2.12). So (2.11) is proved. Let $f=g^{(i)}$. From (2.11) we obtain

$$(2.16)\qquad \|g^{(i+1)}(x)(m^{-1}\sqrt{1-x^2})^{i+1}\|_p\leq c(p)\,i\,\|g^{(i)}(x)(m^{-1}\sqrt{1-x^2})^i\|_p$$
$$\leq\dots\leq c^i(p)\,i!\,\|g\|_p.$$

From (2.10) and (2.16) we get (2.9).

Now we are ready to prove the following basic Bernstein type inequality.

Lemma 5. *If $g\in H_m$, $m\leq n$, $0<p<1$, then*

$$(2.17)\qquad \tau_k(g,\Delta_n)_{p,p}\leq c(k,p)\Big(\frac{m}{n}\Big)^k\|g\|_p.$$

Proof. We shall consider two cases.

1) $m>n/(2kc_1)$, where c_1 is the constant $c_1(p)$ in (2.9). Then (2.17) follows from lemma 1.

2) $m\leq n/(2kc_1)$.

It is obvious that $\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^j=0$ for $j=0,1,\dots,k-1$ and

$$\Delta_h^k g(x)=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}g(x+ih)=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\sum_{j=k}^{m}\frac{(ih)^j}{j!}g^{(j)}(x).$$

Using the semi-additivity of the function $\varphi(x)=x^p$ for $0<p<1$, we get from the above equality

$$(2.13)\qquad |\Delta_h^k g(x)|^p\leq\sum_{i=0}^{k}\binom{k}{i}^p\sum_{j=k}^{m}\frac{(i)^{jp}|h|^{jp}}{(j!)^p}|g^{(j)}(x)|^p\leq c(k,p)\sum_{j=k}^{m}\frac{k^{jp}|h|^{jp}}{(j!)^p}|g^{(j)}(x)|^p.$$

From (2.18) and (1.4) we obtain

$$\tau_k^p(g,\Delta_n)_{p,p}=\int_{-1}^{1}\frac{1}{2\Delta_n(x)}\int_{-\Delta_n(x)}^{\Delta_n(x)}|\Delta_h^k g(x)|^p\,dh\,dx$$
$$\leq c(k,p)\sum_{j=k-1}^{m}\int_{-1}^{1}\frac{1}{2\Delta_n(x)}\int_{-\Delta(x)}^{\Delta_n(x)}\frac{k^{jp}|h|^{jp}|g^{(j)}(x)|^p}{(j!)^p}\,dh\,dx$$

$$= \sum_{i=k}^{m} \frac{k^{jp}}{(jp+1)(j!)^p} \int_{-1}^{1} (\Delta_n(x))^{jp} |g^{(j)}(x)|^p dx.$$

From the inequality $\Delta_n(x) \leqq (m/n)\Delta_m(x)$ and Lemma 4 we get

$$\tau_k^p(g, \Delta_n)_{p,p} \leqq c(k, p)(m/n)^{kp} \sum_{j=k}^{m} \frac{k^{jp}}{(jp+1)(j!)^p} (m/n)^{(j-k)\cdot p} \|(\Delta_m)^j g^{(j)}\|_p^p$$

$$\leqq c_1(k, p)(m/n)^{kp} \sum_{j=k}^{m} \frac{(j!)^p}{(jp+1)(j!)^p} \left(\frac{kmc_1(p)}{n}\right)^{(j-k)p} \|g\|_p^p \leqq c_2(k, p)(m/n)^{kp} \|g\|_p^p.$$

Thus Lemma 5 is proved.

3. Proof of Theorem 1. We follow the scheme used in [6] for obtaining the converse theorem for best trigonometric approximation in $L_p[0, 2\pi]$, $0<p<1$. We choose m such that $2^m < n \leqq 2^{m+1}$ and let $g_i(x) \in H_{2^i}$ be the polynomial of best L_p-approximation of f, $i=0, 1, \dots, m+1$, $g_{-1} \in H_0$

$$\|f-g_i\|_p = E_{2^i}(f)_p, \quad E_{1/2} := E_0.$$

From Lemmas 1 and 5 we get

$$\tau_k^p(f, \Delta_n)_{p,p} \leqq \tau_k^p(f-g_{m+1}, \Delta_n)_{p,p} + \tau_k^p(g_{m+1}, \Delta_n)_{p,p} \tag{3.1}$$

$$\leqq c(k, p) E_{2^{m+1}}^p(f)_p + \tau_k^p(g_0-g_{-1}, \Delta_n)_{p,p} + \sum_{\nu=0}^{m} \tau_k^p(g_{\nu+1}-g_\nu, \Delta_n)_{p,p}$$

$$\leqq c(k, p) E_{2^{m+1}}^p(f)_p + \frac{c(k,p)}{n^{kp}} \{E_0^p(f)_p + \sum_{\nu=0}^{m} 2^{(\nu+1)kp} E_{2^\nu}^p(f)_p\}.$$

Using

$$2^{(\nu+1)kp} E_{2^\nu}^p(f)_p \leqq c(k,p) \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{kp-1} E_\mu^p(f)_p,$$

we obtain

$$\tau_k^p(g_{m+1}, \Delta_n)_{p,p} \leqq \frac{c(k, p)}{n^{pk}} \sum_{\nu=0}^{2^m} (\nu+1)^{kp-1} E_\nu^p(f)_p. \tag{3.2}$$

The inequality (1.7) follows from (3.1) and (3.2). So, Theorem 1 is proved.

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