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A TYPE FREE ABSTRACT STRUCTURE FOR THE RECURSION THEORY

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Partially ordered BCI-algebras satisfying some additional conditions were studied by Zashev (1984, 1990) and Skordev (1980) from the viewpoint of abstract algebraic recursion theory as an alternative to Skordev's combinatory spaces and operative spaces of Ivanov (1986). In the present paper we propose a variant of those algebras which provide a simplification of the basic system of operations for producing recursive elements by excluding the operation v (least upper bound of two elements).

1. Preliminaries. By an applicative structure (a. s.) we shall mean a set $\mathscr{F}$ with a binary operation called application and denoted by $\varphi\psi$ (for $\varphi, \psi \in \mathscr{F}$) with the usual convention of association to the left. An a. s. $\mathscr{F}$ is a partially ordered a. s. (p.o.a.s), iff a partial order and a constant O are given, such that $\varphi \leqq \varphi' \ \& \ \psi \leqq \psi' \Rightarrow \varphi\psi \leq \varphi'\psi'$ and $O \leqq \varphi$ for all $\varphi, \psi, \varphi', \psi' \in \mathscr{F}$. An a. s. $\mathscr{F}$ with elements $A, C, K \in \mathscr{F}$, such that for all $\varphi, \psi, \chi \in \mathscr{F}$

$$A\varphi\psi\chi = \varphi(\psi\chi), \tag{1.1}$$

$$C\varphi\psi = \psi\varphi, \quad \text{and} \tag{1.2}$$

$$K\varphi\psi = \varphi, \tag{1.3}$$

will be called an ACK-algebra in the present paper. Finally, a cartesian linear combinatory algebra (c. l. c. a.) is defined as an ACK-algebra $\mathscr{F}$, which is also a p. o. a. s. with respect to its application operation, and two more elements C', D' are given in $\mathscr{F}$, such that $D'OO = O$ and

$$C'\varphi(D'\psi\chi) = \varphi\psi\chi \tag{1.4}$$

for all $\varphi, \psi, \chi \in \mathscr{F}$.

Let $\mathscr{C}$ be a set of operations (including constants considered as O-ary operations) in an a. s. $\mathscr{F}$, and let X be a set of variables for elements of $\mathscr{F}$. Denote by $\mathrm{Term}(\mathscr{C}, X)$ the set of all terms, constructed from variables in X by means of (symbols for) the elements of $\mathscr{C}$ and the application. The same convention of association to the left is adopted in term notations as well. The notion of value of a term $t \in \mathrm{Term}(\mathscr{C}, X)$ under a given evaluation of the variables is defined in an obvious way. Every term $t \in \mathrm{Term}(\mathscr{C}, \{x_1, \ldots, x_n\})$ defines an operation in $\mathscr{F}$ of n arquments corresponding to the different variables $x_1, \ldots, x_n$ and the operations (including elements in the case $n=0$) of that kind will be called $\mathscr{C}$-expressible. If in $t \in \mathrm{Term}(\mathscr{C}, \{x_1, \ldots, x_n\})$ each variable x_i occurs once, then the operation defined by t will be called linear.

Lemma 1.1. *Suppose $\mathscr{F}$ is an AC-algebra (i. e. an a. s. with combinators* (1.1) *and* (1.2) *in it), and $\mathscr{C} \subseteq \mathscr{F}$ is a set of constants, such that $A, C \in \mathscr{C}$, and f is a linear $\mathscr{C}$-expressible n-ary operation in $\mathscr{F}$, where $n \geqq 2$. Then there is a $\mathscr{C}$-expressible $\varphi \in \mathscr{F}$, such that for all $\xi_1, \ldots, \xi_n \in \mathscr{F}$ one has $\varphi\xi_1 \ldots \xi_n = f(\xi_1, \ldots, \xi_n)$.*

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Proof. This is a folkloric result, whose verification may be expressed as 'apply several times identities (1.1) and (.2)'. For every $t \in \mathrm{Term}(\mathscr{C}, X)$ and each variable x, such that x occurs in t exactly once and $t \not\equiv x$ ($\equiv$ means identity of terms) define a term $(\lambda^1 x.t)$ by induction on t considering the cases:

1) $t \equiv xr$; then $(\lambda^1 xt) = Cr$;
2) $t \equiv sx$; then $(\lambda^1 xt) = s$;
3) $t \equiv sr$, x occurs in s, and $x \not\equiv s$; then $(\lambda^1 x.t) = A(Cr)(\lambda^1 x.s)$; and
4) $t \equiv sr$, x occurs in r, and $x \not\equiv r$; then $(\lambda^1 x.t) = As(\lambda^1 x.r)$.

By induction on t we have:

(1.5) If x occurs in t exactly once and $x \not\equiv t$, then $(\lambda^1 x.t) \in \mathrm{Term}(\mathscr{C}, X \setminus \{x\})$, each variable $y \in X \setminus \{x\}$ occurs in $(\lambda^1 x.t)$ as many times as in t, and the equality $t = (\lambda^1 x.t)x$ is true in $\mathscr{F}$ for all evaluations of the variables.

The proof is completed by induction on n, using (1.5). When $n=2$, we have to use the identity $A(CC)(AAC)\xi_1\xi_2 = \xi_1\xi_2$, which follows from (1.1) and (1.2). ∎

Let $\mathscr{F}$ be a p.o.a.s. and $\mathscr{C} \subseteq \mathscr{F}$ be a set of constants in $\mathscr{F}$. Consider a system of inequalities of the form

$$f_i(\xi_0, \ldots, \xi_{n-1}) \leq \xi_i \quad (i<n) \tag{1.6}$$

where f_i $(i<n)$ are $\mathscr{C}$-expressible n-ary operations in $\mathscr{F}$. A solution $\varphi_0, \ldots, \varphi_{n-1}$ of (1.6) will be called minimal, iff for every other solution $\xi_0, \ldots, \xi_{n-1}$ of (1.6) we have $\varphi_i \leq \xi_i$ for each $i<n$. An element $\varphi \in \mathscr{F}$ will be called recursive in $\mathscr{C}$ iff it is a member of a minimal solution of a system of the form (1.6).

Suppose $\mathscr{F}$ is a c.l.c.a. and $K' \in \mathscr{F}$ is an $\{A, C, K\}$-expressible element, such that $K'\varphi\psi = K\psi\varphi = \varphi$ for all $\varphi, \psi \in \mathscr{F}$. (The existence of K' follows from Lemma 1.1.) Defining $L = C'K$ and $R = C'K'$, we have $L(D'\varphi\psi) = \varphi$ and $R(D'\varphi\psi) = \psi$. Define also

$$\Delta(\varphi_0, \ldots, \varphi_{n-1}) = D'\varphi_0(D'\varphi_1 \ldots (D'\varphi_{n-1}A) \ldots),$$

$\Delta_0(\varphi) = L\varphi$, and $\Delta_{i+1}(\varphi) = \Delta_i(R\varphi)$ (where $\varphi_0, \ldots, \varphi_{n-1} \in \mathscr{F}$). Then we have $\Delta_i(\Delta(\varphi_0, \ldots, \varphi_{n-1})) = \varphi_i$, $i<n$. Using these notations we may reduce each system of the form (1.6) to one inequality in the following sense: there is a $\mathscr{C} \cup \{A, C, K, C', D'\}$ — expressible $f: \mathscr{F} \to \mathscr{F}$, such that if $\varphi \in F$ is a minimal solution of $f(\xi) \leq \xi$, then $\Delta_0(\varphi), \ldots, \Delta_{n-1}(\varphi)$ is a minimal solution of (1.6). Indeed, define

$$f(\xi) = \Delta(f_0(\Delta_0(\xi), \ldots, \Delta_{n-1}(\xi)), \ldots, f_{n-1}(\Delta_0(\xi), \ldots, \Delta_{n-1}(\xi))),$$

and suppose φ is a minimal solution of $f(\xi) \leq \xi$. Then

$$f_i(\Delta_0(\varphi), \ldots, \Delta_{n-1}(\varphi)) = \Delta_i(f(\varphi)) \leq \Delta_i(\varphi)$$

and if $\xi_0, \ldots, \xi_{n-1}$ is a solution of (1.6) in $\mathscr{F}$, then

$$f(\Delta(\xi_0, \ldots, \xi_{n-1})) = \Delta(f_0(\xi_0, \ldots, \xi_{n-1}), \ldots, f_{n-1}(\xi_0, \ldots, \xi_{n-1}) \leq \Delta(\xi_0, \ldots, \xi_{n-1})$$

whence $\varphi \leq \Delta(\xi_0, \ldots \xi_{n-1})$ and $\Delta_i(\varphi) \leq \xi_i$.

Therefore we have

Proposition 1.2. *Suppose $\mathscr{F}$ is a c.l.c.a., and $A, C, K, C', D' \subseteq \mathscr{C} \subseteq \mathscr{F}$. Then $\varphi \in \mathscr{F}$ is recursive in $\mathscr{C}$ iff $\varphi = L\psi$, where ψ is a least fixed point of a $\mathscr{C}$-expressible mapping $f: \mathscr{F} \to \mathscr{F}$.* ∎

2. Example 1. Let M be a set, and let $*$ be an object such that $* \notin M$. Let $M' = M \cup \{*\}$, and suppose a pairing (i. e. an injective mapping $M'^2 \to M'$, $\langle x, y \rangle$ is given in M', such that $\langle *, * \rangle = *$. Denote by $\mathscr{F}$ the set $\{\varphi \subseteq M' \mid * \in \varphi\}$, and define an application in $\mathscr{F}$ by

$$\varphi\psi = \{x \in M' \mid \exists y \in \psi(\langle y, x \rangle \in \varphi)\}. \tag{2.1}$$

F is closed under this application because $\langle *, *\rangle = *$. Define also $O=\{*\}$; $A=\{\langle\langle x, y\rangle, \langle\langle z, x\rangle, \langle z, y\rangle\rangle\rangle \mid x, y, z \in M'\}$; $C=\{\langle x, \langle\langle x, y\rangle, y\rangle\rangle \mid x, y \in M'\}$; $K=\{\langle x, \langle y, x\rangle\rangle \mid x, y \in M'\}$; $C'=\{\langle\langle z, \langle y, x\rangle\rangle, \langle\langle z, y\rangle, x\rangle\rangle \mid x, y, z \in M'\}$; $D'=\{\langle x, \langle y, \langle x, y\rangle\rangle\rangle \mid x, y \in M'\}$. The sets A, C, K, C', D' belong to $\mathcal{F}$ because $\langle *, *\rangle = *$.

Proposition 2.1. *$\mathcal{F}$ is a c.l.c.a. with respect to $\subseteq$ as partial order, and the application and the constants O, A, C, K, C', D' defined above.*

The proof is a direct calculation. We shall check, for instance, that C', D' satisfy (1.4) and $D'OO=O$. Let $\varphi, \psi, \chi \in \mathcal{F}$ be arbitrary. Then by (2.1) and the definition of D':

$$x \in D'\varphi\psi \Leftrightarrow \exists u, v(\langle u, \langle v, x\rangle\rangle \in D' \,\&\, u \in \varphi \,\&\, v \in \psi)$$

$$\Leftrightarrow \exists u, v(x=\langle u, v\rangle \,\&\, u \in \varphi \,\&\, v \in \psi) \Leftrightarrow x \in \varphi \times \psi,$$

whence $D'\varphi\psi = \varphi \times \psi$ and $D'OO=O$, since $\langle *, *\rangle = *$. Moreover,

$$\times \in C'\chi(D'\varphi\psi) \Leftrightarrow \exists u, v(\langle u, \langle v, x\rangle\rangle \in C' \,\&\, u \in \chi \,\&\, v \in \varphi \times \psi)$$

$$\Leftrightarrow \exists u, y, z(\langle u, \langle\langle z, y\rangle, x\rangle\rangle \in C' \,\&\, u \in \chi \,\&\, z \in \varphi \,\&\, y \in \psi)$$

$$\Leftrightarrow \exists u, y, z(u=\langle z, \langle y, x\rangle\rangle \,\&\, u \in \chi \,\&\, z \in \varphi \,\&\, y \in \psi)$$

$$\Leftrightarrow \exists y, z(\langle z, \langle y, x\rangle\rangle \in \chi \,\&\, z \in \varphi \,\&\, y \in \psi) \Leftrightarrow x \in \chi\varphi\psi.$$

The identities (1.1)—(1.3) for the constants A, C, K are checked in a similar way. ∎

The usual theory of recursive enumerable sets may be treated by this example, with the set of all natural numbers for M. Then the extensions $\alpha \cup \{*\}$ of the recursively enumerable sets $\alpha \subseteq M$ can be described as the recursive in $\mathcal{C}$ elements of the algebra $\mathcal{F}$ for a suitable set $\mathcal{C} \subseteq \mathcal{F}$. More generally, if $\mathcal{M}=\langle M; R_1, \ldots, R_k\rangle$ is an existentially acceptable structure (in the sense of [5]), then for a suitable subset $\mathcal{B} \subseteq \mathcal{F}$ of the algebra $\mathcal{F}$ of the Proposition 2.1 we have: $\alpha \subseteq M$ is existentially inductively definable in $\mathcal{M}$ iff $\alpha \cup \{*\}$ is recursive in $\mathcal{B}$.

3. Example 2. This example deals with continuous functionals over complete partially ordered sets (c.p.o.s.) in the sense of [6, 1.2.]. Next we shall use some notations and results from [6, 1.2.]. Let M be a c.p.o.s. and define finite types, as usual, by: a) O is a type; b) if a and b are types, then $a \to b$ and $a \times b$ are types. The set $\mathsf{C}_a(M)$ or shortly C_a of the continuous functionals of type a over M are defined inductively, as follows: $\mathsf{C}_0=M$; $\mathsf{C}_{a\to b}=[\mathsf{C}_a \to \mathsf{C}_b]$; $\mathsf{C}_{a\times b}=\mathsf{C}_a \times \mathsf{C}_b$. Denote by C the union $\bigcup_a \mathsf{C}_a$ and by $\leq_a$ and O_a the partial order and the least element in C_a, respectively, (We shall omit a in $\leq_a$ when the type a is clear from the context.) Then for all types a, b, c there are functionals $A_{abc} \in \mathsf{C}_{(a\to b)\to((c\to a)\to(c\to b))}$, $C_{ab} \in \mathsf{C}_{a\to((a\to b)\to b)}$, $K_{ab} \in \mathsf{C}_{a\to(b\to a)}$, $C'_{abc} \in \mathsf{C}_{((a\to(b\to c))\to((a\times b)\to c))}$, and $D'_{ab} \in \mathsf{C}_{a\to(b\to(a\times b))}$, such that for all $f, g, h \in \mathsf{C}$ of proper types

$$A_{abc}(f)(g)(h)=f(g(x)); \quad C_{ab}(f)(g)=g(f); \quad K_{ab}(f)(g)=f;$$

$$C'_{abc}(f)((g, h))=f(g)(h); \quad \text{and} \quad D'_{ab}(f)(g)=\langle f, g\rangle.$$

The existence of those functionals follows easily from the results in [6, 1.2].

A set $\varphi \subseteq \mathsf{C}$ will be called an ideal, iff:

a) $\varphi \cap \mathsf{C}_a$ is a c.p.o.s. with respect to $\leq_a$ for each type a;

b) $f \in \varphi \,\&\, g \leq_a f \Rightarrow g \in \varphi$ for each a and all $f, g \in \mathsf{C}_a$.

Note that if $\varphi \subseteq \mathsf{C}$ is an ideal, then $O_a \in \varphi$ for every type a. Denote by $\mathcal{F}$ the set of all ideals $\varphi \subseteq \mathsf{C}$, and let O be the ideal $\{O_a \mid a \text{ is a type}\}$. The set $\mathcal{F}$ is partially ordered by $\subseteq$ with the least element O. The intersection of an arbitrary family of ideals is an ideal too; denote by $\mathbf{id}Y$ the least ideal $\bigcap\{\varphi\in F \mid Y\subseteq\varphi\}$ containing the subset $Y\subseteq \mathsf{C}$. Define also $A=\mathbf{id}\{A_{abc}\mid a,\ b,\ c$ are types$\}$; $C=\mathbf{id}\{C_{ab}\mid a,\ b$ are types$\}$; $K=\mathbf{id}\{K_{ab}\mid a,\ b$ are types$\}$; $C'=\mathbf{id}\{C'_{abc}\mid a,\ b,\ c$ are types$\}$; $D'=\mathbf{id}\{D'_{ab}\mid a,\ b$ are types$\}$; and

$$\varphi\psi=\mathbf{id}\{f(g)\mid f\in\varphi\ \&\ g\in\psi\ \&\ !f(g)\} \tag{3.1}$$

for all $\varphi,\ \psi\in\mathcal{F}$, where $!f(g)$ means that f and g have proper types, i. e. $f\in \mathsf{C}_{a\to b}$ and $g\in \mathsf{C}_a$ for suitable types $a,\ b$.

Proposition 3.1. *$\mathcal{F}$ is c.l.c.a. with respect to the application* (3.1), *the partial order $\subseteq$, and the constants* O, *A, C, K, C', D' defined above.*

To prove this proposition we need the following lemma:

Lemma 3.1.1. *Let $\varphi,\ \alpha\in\mathcal{F}$. Then the set*

$$(\varphi//\alpha)=\{f\mid \forall g\in\alpha\ (!f(g)\Rightarrow f(g)\in\varphi)\} \tag{3.2}$$

is the greatest solution in $\mathcal{F}$ of the inequality $\xi\alpha\subseteq\varphi$ with respect to ξ. Similarly, if $\alpha O\subseteq\varphi$, then

$$(\varphi/\alpha)=\{f\mid \forall g\in\alpha\ (!\,g(f)\Rightarrow g(f)\in\varphi)\} \tag{3.3}$$

is the greatest solution in F with respect to ξ of $\alpha\xi\subseteq\varphi$.

Proof. We shall prove the second part only, the first being similar. Let $\alpha O\subseteq\varphi$. Suppose $\psi\subseteq \mathsf{C}_a\cap(\varphi/\alpha)$ is directed with respect to $\leq_a$, and $f=\sup\psi$. To prove $f\in(\varphi/\alpha)$ suppose $g\in\alpha$ and $!g(f)$. Then for each $f'\in\psi\ !g(f)$, and since $\psi\subseteq(\varphi/\alpha)$ we have $g(f')\in\varphi$. The set $\{g(f')\mid f'\in\psi\}$ is directed and since φ is an ideal $g(f)=\sup\{g(f')\mid f'\in\psi\}\in\varphi$. So $\mathsf{C}_a\cap(\varphi/\alpha)$ is closed under suprema of directed subsets and since $O_a\in(\varphi/\alpha)$ (because of $\alpha O\subseteq\varphi$ and definition (3.1)), it is a c.p.o.s. with respect to $\leq_a$. Let $f\in(\varphi/\alpha)$ and $f'\leq f$. To prove $f'\in(\varphi/\alpha)$ suppose $g\in a$ and $g(f'')$. Then $!g(f)$ and therefore $g(f)\in\varphi$, but $g(f')\leq g(f)$ because g is monotone as a continuous function. Therefore $g(f')\in\varphi$ since φ is an ideal. So $f'\in(\varphi/\alpha)$, and $(\varphi/\alpha)\subseteq\mathcal{F}$. The inclusion $\alpha(\varphi/\alpha)\subseteq\varphi$ follows directly from definitions (3.1) and (3.3). Also if $\alpha\xi\subseteq\varphi$, $\xi\in\mathcal{F}$, and $f\in\xi$, then from $g\in\alpha$ and $!g(f)$ follows $f\in\varphi$, which proves $f\in(\varphi/\alpha)$ and $\xi\subseteq(\varphi/\alpha)$. ∎

Proof of Proposition 3.1. To prove (1.1) let $\varphi,\ \psi,\ \chi\in\mathcal{F}$ and define $\alpha_0=(\varphi(\psi\chi)//\chi))$, $\alpha_1=(\alpha_0//\psi)$, and $\alpha=(\alpha_1//\varphi)$. Suppose $f\in\varphi$, $g\in\psi$, $h\in\chi$ and $!A_{abc}(f)(g)(h)$. Then $!f(g(h))$ and because $f(g(h))\in\varphi(\psi\chi)$ we have $!A_{abc}(f)(g)\Rightarrow A_{abc}(f)(g)\in\alpha_0$. The same shows that $!A_{abc}(f)\Rightarrow A_{abc}(f)\in\alpha_1$ and $A_{abc}\in\alpha$. Therefore $A\subseteq\alpha$, whence $A\varphi\psi\chi\subseteq\alpha\varphi\psi\chi\subseteq\alpha_1\psi\chi\subseteq\alpha_0\chi\subseteq\varphi(\psi\chi)$, since the operation (3.1) is obviously monotonic. To prove the reverse inclusion notice that $\varphi O\subseteq A\varphi\psi\chi$. Indeed, if $f\in\varphi$ and $!f(O_a)$, then for each type c $f(O_a)=f(O_{c\to a}(O_c))=A_{abc}(f)(O_{c\to a})(O_c)\in A\varphi\psi\chi$. Therefore $(A\varphi\psi\chi/\varphi)$ exists. Then let $g\in\psi$, $h\in\chi$ and $!g(h)$. Suppose $f\in\varphi$ and $!f(g(h))$. Then for suitable $a,\ b,\ c\ f(g(h))=A_{abc}(f)(g)(h)\in A\varphi\psi\chi$, whence $g(h)\in(A\varphi\psi\chi/\varphi)$. By definition (3.1) $\psi\chi\subseteq(A\varphi\psi\chi/\varphi)$ and $\varphi(\psi\chi)\subseteq\varphi(A\varphi\psi\chi/\varphi)\subseteq A\varphi\psi\chi$, which completes the proof of (1.1). Equalities (1.2) and (1.3) are proved in a similar way (it is essential for (1.3) that ψ is nonempty). To prove (1.4) notice that for all $\psi,\ \chi\in\mathcal{F}$

$$D'\psi\chi=(\psi\times\chi)\cap O. \tag{3.4}$$

Indeed, the inclusion $(\psi\times\chi)\cup O\subseteq D'\psi\chi$ is trivial from the definitions and $(\psi\times\chi)\cup O$ is an ideal, because if $\alpha\subseteq(\psi\times\chi)\cup O$ is a directed subset of C_d and $d=axb$, then $\alpha\subseteq\psi\times\chi$ and the projections $\alpha_0\subseteq\psi$ and $\alpha_1\subseteq\chi$ are directed subsets of C_a and C_b respectively, whence $\sup\alpha_0\in\psi$ and $\sup\alpha_1\in\chi$, but $\sup\alpha=(\sup\alpha_0,\ \sup\alpha_1)\in\psi\times\chi$; and if d is not of the form $a\times b$, then $\alpha\subseteq O$ and $\sup\alpha=O_d\in(\psi\times\chi)\cup O$. Therefore we may define $((\psi\times\chi)\cup O//\chi)$ and prove $D'\psi\subseteq((\psi\times\chi)\cup O//\chi)$ by definitions (3.3) and (3.1) as above. From (3.4) it follows that $D'OO=O$. Finally, suppose $f\in\varphi$ and $!C'_{abc}(f)$. Then if $k \in D'\psi\chi$ and $!C'_{abc}(f)(k)$, then $k \in \psi \times \chi$, $k = (g, h)$, where $g \in \psi$ and $h \in \chi$, and $C'_{abc}(f)(k) = f(g)(h) \in \varphi\psi\chi$, whence $C'_{abc}(f) \in (\varphi\psi\chi // D'\psi\chi)$ and $C'\varphi \subseteq \varphi\psi\chi // D'\psi\chi)$. Thus we prove $C'\varphi(D'\psi\chi) \subseteq \varphi\psi\chi$, and the reverse inclusion is proved in a similar way. ■

A notion of recursiveness can be defined for the typed algebra C in the same way as that for c.l.c.a. Namely, consider a system of the form (1.6), where ξ_i are typed variables of type a_i and the operations f_i are defined by typed terms of type a_i constructed from the variables ξ_i and constants belonging to a given subset $\mathscr{C}$ of C. Then the members of least solutions in C of typed systems of that kind are called recursive in $\mathscr{C}$ elements of C. Note that typed systems of the form (1.6) are not trivially reducible to one inequality of the form $tx \leq x$ because combinators are not all supposed to belong to $\mathscr{C}$. Then the elements $f \in$ C recursive in $\mathscr{C}$ can be described by means of recursiveness in c.l.c.a. $\mathscr{F}$ in the Example 2 as follows. For $f \in$ C define $\bar{f} = \mathbf{id}\{f\} = \{f' \in \mathrm{C} \mid f' \leq f\} \cup \mathrm{O}$. An element $f \in$ C is recursive in $\mathscr{C}$ iff $\bar{f}$ is recursive in $\bar{\mathscr{C}} = \{\bar{g} \mid g \in \mathscr{C}\}$ in the algebra $\mathscr{F}$, and for every element $\varphi \in F$ recursive in $\bar{\mathscr{C}}$ there is an element $f \in$ C recursive in $\mathscr{C}$, such that $\varphi = \bar{f}$. The proof of this fact will not be treated here.

4. Other examples. Next we shall consider two other examples, which are well known models of the λ-calculus of graph type, but also fall under the scheme considered here.

4.1. Example 3. Let M be an admissible set (some notations from [7] will be used below), and let $\mathscr{F} = \{\varphi \mid \varphi \subseteq M\}$. Define application in $\mathscr{F}$ by

$$\varphi\psi = \{x \in M \mid \exists y(\langle y, x\rangle \in \varphi \ \&\ y \subseteq \psi)\}.$$

It is well known that F is a λ-algebra with respect to this application (see, for instance, [8]), so the combinators (1.1)—(1.3) exist in $\mathscr{F}$. To show that there is a cartesian pair in $\mathscr{F}$ define

$$D' = \{\langle y_0, \langle y_1, x\rangle\rangle \mid \exists i < 2 \exists y \in y_i (x = \langle y, i\rangle)\},$$

and check directly that

$$x \in D'\psi_0\psi_1 \Leftrightarrow \exists y_0, y_1 (\exists i < 2 \exists y \in y_i (x = \langle y, i\rangle) \ \&\ y_0 \subseteq \psi_0 \ \&\ y_1 \subseteq \psi_1)$$
$$\Leftrightarrow \exists i < 2 \exists y \in \psi_i (x = \langle y, i\rangle) \Leftrightarrow x \in (\psi_0 \times \{0\}) \cup (\psi_1 \times \{1\}),$$

where $\psi_0, \psi_1 \in F$. Thence $D'\varnothing\varnothing = \varnothing$ and defining

$$C' = \{\langle z, \langle y, x\rangle\rangle \mid \exists y_0, y_1 (\langle y_0, \langle y_1, x\rangle\rangle \in z \ \&\ y = (y_0 \times \{0\}) \cup (y_1 x \{1\}))\},$$

we may easily check that

$$x \in \varphi\psi_0\psi_1 \Leftrightarrow \exists y_0, y_1 (\langle y_0, \langle y_1, x\rangle\rangle \in \varphi \ \&\ y_0 \subseteq \psi_0 \ \&\ y_1 \subseteq \psi_1)$$
$$\Leftrightarrow \exists y_0, y_1 (\langle y_0, \langle y_1, x\rangle\rangle \in \varphi \ \&\ (y_0 \times \{0\}) \cup (y_1 \times \{1\}) \subseteq (\psi_0 \times \{0\}) \cup (\psi_1 \times \{1\}))$$
$$\Leftrightarrow \exists y, z, y_0, y_1 (\langle y_0, \langle y_1, x\rangle\rangle \in z \ \&\ y = (y_0 \times \{0\}) \cup (y_1 \times \{1\}) \ \&\ z \subseteq \varphi$$
$$\&\ y \subseteq D'\psi_0\psi_1) \Leftrightarrow x \in C'\varphi(D'\psi_0\psi_1).$$

So C', D' is a cartesian pair in $\mathscr{F}$ and $\mathscr{F}$ is c.l.c.a. with $\subseteq$ as partial order and $\varnothing$ as the least element O.

4.2. Example 4. Let M be a set with a pair coding (i. e. an injective mapping $j: M^2 \to M$) and suppose there are $c_0, c_1 \in M$, such that $c_0 \neq c_1$. Denote by M^∞ the set of all finite sequences of elements of M and let **Tr** (M) be the set of all **trees** over M, defined as subsets $\tau \subseteq M^\infty$ satisfying:

1) $\langle\ \rangle \in \tau$ (where $\langle\ \rangle$ is the empty sequence); and

2) $x * y \in \tau \Rightarrow x \in \tau$ $(x, y \in M^\infty)$, where $*$ denotes concatenation of finite sequences.

For $x \in M^\infty$ and $\tau \in \boldsymbol{Tr}(M)$ define $(\tau \mid x) = \{y \mid x * y \in \tau\}$. Then $x \in \tau \Rightarrow (\tau \mid x) \in \boldsymbol{Tr}(M)$. Define also

$$x*\tau=\{x*y \mid y\in\tau\}\cup\{z\in M^\infty \mid \exists y\in M^\infty\ (z*y=x)\},$$

so that $x*\tau\in \boldsymbol{Tr}(M)$; and for $\tau,\sigma\in \boldsymbol{Tr}(M)$ define $\langle\tau,\sigma\rangle=\widehat{c}_0*\tau\cup\widehat{c}_1*\tau$ where $\widehat{x}$ as usual denotes the one-membered sequence $\langle x\rangle$. It is clear that $\langle\tau,\sigma\rangle$ is a pair coding in $\boldsymbol{Tr}(M)$. Then we may define an application in the set $F=\{\varphi \mid \varphi\subseteq \boldsymbol{Tr}(M)\}$ as follows:

$$\varphi\psi=\{\tau\in \boldsymbol{Tr}(M) \mid \exists\sigma\in \boldsymbol{Tr}(M)(\langle\sigma,\tau\rangle\in\varphi\ \&\ \forall x\in M(\widehat{x}\in\sigma\Rightarrow(\sigma\mid\widehat{x})\in\psi))\}.$$

Then $\mathscr{F}$ is a combinatory algebra with respect to this application and there is a cartesian pair in $\mathscr{F}$. This can be shown by the same method as in the previous example. So $\mathscr{F}$ is a c.l.c.a. with respect to $\emptyset$ as O and $\subseteq$ as partial order.

5. Iterative c.l.c.a. Definition 5.1. *Suppose $\mathscr{F}$ is a c.l.c.a. Then*

(a) *a normal initial is a nonempty subset $\mathscr{A}\subseteq\mathscr{F}$ of the form $\bigcap_{i=0}^{\infty}\{\vartheta\in\mathscr{F} \mid \alpha_i\vartheta\leq\beta_i\}$, where $\alpha_i,\beta_i\in\mathscr{F}$ for every natural number i;*

(b) *a subset $\mathscr{A}\subseteq F$ is* **invariant** *with respect to $\varphi\in\mathscr{F}$,* iff $\varphi\mathscr{A}\subseteq\mathscr{A}$, i. e. $\xi\in\mathscr{A}\Rightarrow\varphi\xi\in\mathscr{A}$ *for each* $\xi\in\mathscr{A}$;

(c) c.l.c.a. $\mathscr{F}$ is **iterative** *iff for every $\varphi\in\mathscr{F}$ there is an element $\mathbf{I}(\varphi)\in\mathscr{F}$, such that $\varphi\mathbf{I}(\varphi)\leq\mathbf{I}(\varphi)$ and $\mathbf{I}(\varphi)\in\mathscr{A}$ for every normal initial $\mathscr{A}\subseteq\mathscr{F}$, which is invariant with respect to φ.*

The definition of iterative c.l.c.a. is similar to those of iterative combinatory and iterative operative space in Skordev [3] and Ivanov [4] respectively. Note that if $\mathscr{F}$ is an iterative c.l.c.a., then $\mathbf{I}(\varphi)$ is the least solution of the inequality $\varphi\xi\leq\xi$ with respect to ξ in $\mathscr{F}$, and therefore the least fixed point of the mapping $\xi\longmapsto\varphi\xi$ in $\mathscr{F}$. Indeed, if $\varphi\chi\leq\chi$, then it is easy to see, that $\mathscr{A}=\{\vartheta\in\mathscr{F} \mid \vartheta\leq\chi\}$ is a normal initial which is invariant w.r.t. φ. Therefore $\mathbf{I}(\varphi)\in\mathscr{A}$ i. e. $\mathbf{I}(\varphi)\leq\chi$.

The following proposition is similar to the corresponding criteria of iterativity for combinatory and operative spaces in [3] and [4].

Proposition 5.2. *Suppose $\mathscr{F}$ is a c.l.c.a. and $\varkappa$ is a cardinal number, such that either $\varkappa=\omega$, or $\varkappa>\operatorname{card}\mathscr{F}$. Let for every increasing transfinite sequence $(\varphi_\iota)_{\iota<\varkappa}$, $\varphi_\iota\in\mathscr{F}$, the least upper bound $\sup_{\iota<\varkappa}\varphi_\iota$ exists in $\mathscr{F}$, and for each $\alpha\in\mathscr{F}$*

$$\alpha\sup_{\iota<\varkappa}\varphi_\iota=\sup_{\iota<\varkappa}\alpha\varphi_\iota.$$

Then $\mathscr{F}$ is an iterative c.l.c.a.

The proof is by an usual argument. (See, for instance, Skordev [3], Propositions 1.1.5, 1.2.5.) Namely in the case $\varkappa=\omega$ we define inductively $\varphi_0=\mathrm{O}$ and $\varphi_{n+1}=\varphi\varphi_n$, and $\mathbf{I}(\varphi)=\sup_{n<\omega}\varphi_n$. It the case $\varkappa>\operatorname{card}\mathscr{F}$ we define a transfinite increasing sequence $(\varphi_\iota)_{\iota<\varkappa}$ such that $\varphi_\iota=\sup_{\nu<\iota}\varphi\varphi_\nu$ and $\varphi_\iota\leq\varphi\varphi_\iota$ for each $\iota<\varkappa$, and we define $\mathbf{I}(\varphi)=\varphi_\mu$, where $\mu<\varkappa$ is the first ordinal, such that $\varphi_\mu=\varphi_{\mu+1}$. In both the cases we have to prove that $\varphi_n\in\mathscr{A}$ (respectively $\varphi_\iota\in\mathscr{A}$) for every normal initial $\mathscr{A}\subseteq\mathscr{F}$, such that $\varphi\mathscr{A}\subseteq\mathscr{A}$. The proof is by induction on n (respectively on $\iota<\varkappa$), in the basis of which we use that $\mathrm{O}\in\mathscr{A}$ because $\mathscr{A}$ is nonempty.

c.l.c.a. $\mathscr{F}$ in examples 1 and 2 satisfies the conditions of Proposition 5.2 with $\varkappa=\omega$, and the algebra $\mathscr{F}$ in the examples 3 and 4 satisfies the same conditions with $\varkappa>\operatorname{card}\mathscr{F}$.

Up to the end of the paper we shall suppose that $\mathscr{F}$ is an iterative c.l.c.a. and $\mathscr{C}\subseteq\mathscr{F}$ is a set of constants. Denote by $\mathscr{C}_1$ the set $\mathscr{C}\cup\{A, C, K, C', D'\}$.

Theorem 5.3 ("first recursion theorem"). *For every $\mathscr{C}$-expressible mapping $\Gamma:\mathscr{F}\to\mathscr{F}$ there is a least fixed point of Γ in $\mathscr{F}$, which is $\mathscr{C}_1$-expressible.*

P r o o f. For every $\varphi \in \mathscr{F}$ define an operation $\varphi^n: \mathscr{F} \to \mathscr{F}$ by induction on n: $\varphi^0(\xi) = \xi$; $\varphi^{n+1}(\xi) = \varphi\varphi^n(\xi)$. Suppose the term defining $\Gamma(\xi)$ contains exactly k occurrences of the variable (for) ξ, and $k > 0$ (if $k = 0$ the assertion of the theorem is trivial). Using Lemma 1.1 and (1.4) we may find an $\{A, C, C', D'\} \cup \mathscr{C}$-expressible $c_\Gamma \in \mathscr{F}$, such that for all φ, ψ, $\vartheta \in \mathscr{F}$

$$c_\Gamma \vartheta((D'\varphi)^k(\psi)) = D'\Gamma(\varphi)(\vartheta\psi). \tag{5.1}$$

Let $\gamma = \mathbf{I}(c_\Gamma)$ and for $\varphi \in \mathscr{F}$ define $\nabla(\varphi) = \mathbf{I}(D'\varphi)$. Then we shall show that for every $\varphi \in \mathscr{F}$

$$\nabla(\Gamma(\varphi)) = \gamma\nabla(\varphi). \tag{5.2}$$

By the definition of $\nabla(\varphi)$ we have $D'\varphi\nabla(\varphi) = \nabla(\varphi)$, and therefore

$$D'\Gamma(\varphi)(\gamma\nabla(\varphi)) = c_\Gamma\gamma((D'\varphi)^k(\nabla\varphi))) = c_\Gamma\gamma\nabla(\varphi) = \gamma\nabla(\varphi),$$

whence $\nabla(\Gamma(\varphi)) \leq \gamma\nabla(\varphi)$. To prove the reverse inequality consider $\mathscr{A} = \{\vartheta \in \mathscr{F} \mid \vartheta\nabla(\varphi) \leq \nabla(\Gamma(\varphi))\}$. $\mathscr{A}$ is a normal initial because $\vartheta\psi = C\psi\vartheta$ and $O\psi \leq KO\psi = O$ for all ψ, $\vartheta \in \mathscr{F}$, whence $O \in \mathscr{A}$. Suppose $\vartheta \in \mathscr{A}$. Then

$$c_\Gamma\vartheta\nabla(\varphi) = c_\Gamma\vartheta((D'\varphi)^k(\nabla(\varphi))) = D'\Gamma(\varphi)(\vartheta\nabla(\varphi)) \leq D'\Gamma(\varphi)\nabla(\Gamma(\varphi)) = \nabla(\Gamma(\varphi)),$$

i. e. $c_\Gamma\vartheta \in \mathscr{A}$. Therefore $c_\Gamma\mathscr{A} \subseteq \mathscr{A}$, whence $\gamma \in \mathscr{A}$ and the proof of (5.2) is complete. Now suppose $\Gamma(\xi) \leq \xi$, $\xi \in \mathscr{F}$. Then by (5.2) $\gamma\nabla(\xi) \leq \nabla(\xi)$, whence $\mathbf{I}(\gamma) \leq \nabla(\xi)$ and $L\mathbf{I}(\gamma) \leq L\nabla(\xi) = L(D'\xi\nabla(\xi)) = \xi$. (Where $L = C'K$.) Therefore to complete the proof of the theorem it is enough to show that $\Gamma(L\mathbf{I}(\gamma)) \leq L\mathbf{I}(\gamma)$. To do that we shall prove that

$$\gamma_n = D'\gamma'_n\gamma_n$$

by an induction on n, where $\gamma_n = \gamma^n(O)$ and $\gamma'_n = \Gamma^n(O)$. Indeed, $\gamma_0 = O = D'OO = D'\gamma'_0\gamma_0$, and by the hypothesis of the induction for n and (5.1)

$$\gamma_{n+1} = \gamma\gamma_n = \gamma((D'\gamma'_n)^k(\gamma_n)) = c_\Gamma\gamma((D'\gamma'_n)^k(\gamma_n))$$
$$= D'\Gamma(\gamma'_n)(\gamma\gamma_n) = D'\gamma'_{n+1}\gamma_{n+1}.$$

Therefore $L\gamma_n = \gamma'_n$ and since $\gamma_n \leq \mathbf{I}(\gamma)$ is obvious by induction on n, we have $D'(L\gamma_n)\gamma_n \leq \mathbf{I}(\gamma)$, and because $\gamma_n \leq \gamma_{n+1}$ (by induction on n) we have as well

$$D'(L\gamma_m)\gamma_n \leq \mathbf{I}(\gamma) \tag{5.3}$$

for all natural numbers m, n. Then fix m and define

$$\mathscr{A}_1 = \{\vartheta \in \mathscr{F} \mid D'(L\gamma_m)(\gamma^n(\vartheta)) \leq \mathbf{I}(\gamma) \text{ for all } n\}.$$

By (5.3) $O \in \mathscr{A}_1$ and $\mathscr{A}_1$ is a normal initial, and obviously $\gamma\mathscr{A}_1 \subseteq \mathscr{A}_1$. Therefore $\mathbf{I}(\gamma) \in \mathscr{A}_1$, whence

$$D'(L\gamma_m)\mathbf{I}(\gamma) = D'(L\gamma_m)(\gamma^0(\mathbf{I}(\gamma)) \leq \mathbf{I}(\gamma).$$

Using this inequality, we see that

$$\mathscr{A}_2 = \{\vartheta \in F \mid D'(L\gamma^m(\vartheta))\mathbf{I}(\gamma) \leq \mathbf{I}(\gamma) \text{ for all } m\}$$

is a normal initial, and $\mathscr{A}_2$ is invariant with respect to γ. Therefore $\mathbf{I}(\gamma) \in \mathscr{A}_2$, whence $D'(L\mathbf{I}(\gamma))\mathbf{I}(\gamma) \leq \mathbf{I}(\gamma)$ which by the definition of the operator ∇ shows that

$$\nabla(L\mathbf{I}(\gamma)) \leq \mathbf{I}(\gamma),$$

whence by (5.2)

$$\nabla(\Gamma(L\mathbf{I}(\gamma))) = \gamma\nabla(L\mathbf{I}(\gamma)) \leq \gamma\mathbf{I}(\gamma) = \mathbf{I}(\gamma),$$

and

$$\Gamma(L\mathbf{I}(\gamma))=L\nabla(\Gamma(L\mathbf{I}(\gamma)))\leq L\mathbf{I}(\gamma). \blacksquare$$

From this proof and Proposition 1.2 we have also:

Theorem 5.4 (*"normal form theorem"*). *For every recursive in $\mathscr{C}$ element $\varphi\in\mathscr{F}$ there is an $\mathscr{C}\cup\{A, C, C', D'\}$-expressible $c\in\mathscr{F}$ such that $\varphi=L_2\mathbf{I}^2(c)=L(L\mathbf{I}(\mathbf{I}(c)))$, where $L_2=ALL$.* $\blacksquare$

In order to prove a parametrization theorem for iterative c.l.c.a. we shall consider some properties of the operator $\nabla(\varphi)=\mathbf{I}(D'\varphi)$ introduced in the proof of Theorem 5.3.

Lemma 5.5. *There are elements ι, $\mu\in\mathscr{F}$ recursive in $\{A, C, C'\}$ and $\{A, C, C'$ $D'\}$ respectively, and such that for all φ, $\psi\in\mathscr{F}$*

(a) $\iota\nabla(\varphi)=\mathbf{I}(\varphi)$, *and*

(b) $\mu\nabla(\varphi)\nabla(\psi)=\nabla(\varphi\psi)$.

Proof. Define by means of Lemma 1.1 and (1.4) an $\{A, C, C'\}$-expressible $a\in\mathscr{F}$, such that $a\eta(D'\xi\xi)=\xi(\eta\zeta)$ for all ξ, η, $\zeta\in\mathscr{F}$, and let $\iota=\mathbf{I}(a)$. Then

$$\varphi(\iota\nabla(\varphi))=a\iota(D'\varphi\nabla(\varphi))=a\iota\nabla(\varphi)=\iota\nabla(\varphi),$$

whence $\mathbf{I}(\varphi)\leq\iota\nabla(\varphi)$. To prove the reverse inequality consider the normal initia $\mathscr{A}=\{\vartheta\in\mathscr{F}\mid\vartheta\nabla(\varphi)\leq\mathbf{I}(\varphi)\}$. Suppose $\vartheta\in\mathscr{A}$. Then $\alpha\vartheta\nabla(\varphi)=\alpha\vartheta(D'\varphi\nabla(\varphi))=\varphi(\vartheta\nabla(\varphi))\leq\varphi\mathbf{I}(\varphi)$ $=\mathbf{I}(\varphi)$, i. e $a\vartheta\in\mathscr{A}$. Therefore $\iota=\mathbf{I}(a)\in\mathscr{A}$, whence $\mathbf{I}\nabla(\varphi)=\mathbf{I}(\varphi)$. Similarly, define $\mu=\mathbf{I}(b)$, where b as an $\{A, C, C', D'\}$-expressible element, such that for all ξ, η, ζ, η_1, $\zeta_1\in\mathscr{F}$

$$b\xi(D'\eta\eta_1)(D'\zeta\zeta_1)=D'(\eta\zeta)(\xi\eta_1\zeta_1).$$

Then

$$D'(\varphi\psi)(\mu\nabla(\varphi)\nabla(\psi))=b\mu(D'\varphi\nabla(\varphi))(D'\psi\nabla\psi(\psi))=\mu\nabla(\varphi)\nabla(\psi),$$

whence $\nabla(\varphi\psi)\leq\mu\nabla(\varphi)\nabla(\psi)$, and to prove the reverse inequality consider the normal initial $\mathscr{B}=\{\vartheta\in\mathscr{F}\mid\vartheta\nabla(\varphi)\nabla(\psi)\leq\nabla(\varphi\psi)\}$. Let $\vartheta\in\mathscr{B}$. Then

$$b\vartheta\nabla(\varphi)\nabla(\psi)=b\vartheta(D'\varphi\nabla(\varphi))(D'\psi\nabla(\psi))=D'(\varphi\psi)(\vartheta\nabla(\varphi)\nabla(\psi))$$
$$\leq D'(\varphi\psi)\nabla(\varphi\psi)=\nabla(\varphi\psi)$$

i. e. $b\vartheta\in\mathscr{B}$, whence $\mu=\mathbf{I}(b)\in B$ and

$$\mu\nabla(\varphi)\nabla(\psi)\leq\nabla(\varphi\psi). \blacksquare$$

Now suppose there is a representation $n\mapsto\bar{n}\in\mathscr{F}$ of natural numbers n in $\mathscr{F}$, which is **normal** in the sense: there are $\mathscr{C}_1\cup\{\mathbf{I}\}$-expressible element $S\in\mathscr{F}$ and operation $\mathsf{R}_0:\mathscr{F}^2\to\mathscr{F}$, such that for all φ, $\psi\in\mathscr{F}$ and natural numbers n, $\overline{n+1}=S\bar{n}$, $\mathsf{R}_0(\varphi,\psi)\bar{o}$ $=\varphi$, and $\mathsf{R}_0(\varphi,\psi)\overline{n+1}=\psi\bar{n}$. (See [9] where the operation R_0 is called "primitive recursive branching".) For instance, we may define a normal representation in $\mathscr{F}$ as follows: $\bar{o}=CK$; $S=DK'$, where K' is defined in section 1, and D is the combinator defined by $D\xi\eta\zeta=\zeta\xi\eta$ (Lemma 1.1); $\overline{n+1}=S\bar{n}$ $\mathsf{R}_0(\varphi,\psi)=C(D\varphi\psi)$. Using Theorem 5.3, we may define a primitive recursive iteration (in the sense of [9]) $\mathsf{R}_1(\varphi,\psi)$ as the least fixed point of the operation $\Gamma(\xi)=\mathsf{R}_0(\varphi, A\psi\xi)$. Then R_1 satisfies the equalities

$$\mathsf{R}_1(\varphi,\psi)\bar{o}=\varphi \quad\text{and}$$
$$\mathsf{R}_1(\varphi,\psi)\overline{n+1}=\psi(\mathsf{R}_1(\varphi,\psi)\bar{n}).$$

Let $\mathscr{C}_2$ be the set $\mathscr{C}_1\cup\{\bar{o}, S\}$ and let $\mathscr{R}=\{A, C, \bar{o}, S, \mathsf{R}_0, \mathsf{R}_1\}$. By Theorem 1 in [9] all primitive recursive functions f are representable by a $\mathscr{R}$-expressible element $\varphi\in\mathscr{F}$, i. e. $\varphi\bar{n}=\overline{f(n)}$ for all natural numbers n.

Lemma 5.6. (a). *There is* $\delta \in \mathscr{F}$, *which is recursive in* $\mathscr{C}_1$, *and* $\delta \bar{n} = \nabla(\bar{n})$ *for all* n.

(b) *For every* $\varphi \in \mathscr{F}$ *there is* $\varphi^* \in F$, *which is recursive in* $\{A, C, C', D', \varphi\}$ *and* $\varphi^* \bar{n} = \mathsf{I}(\varphi \bar{n})$ *for all* n.

Proof. (a) Using μ from Lemma 5.5 define $\delta = \mathsf{R}_1(\nabla(\bar{o}), \mu\nabla(S))$. Then (a) follows by induction on n:

$$\delta \overline{n+1} = \mu\nabla(S)(\delta\bar{n}) = \mu\nabla(S)\nabla(\bar{n}) = \nabla(S\bar{n}) = \nabla(\overline{n+1}).$$

(b) By Lemma 5.5 and (a) we have

$$\mathsf{I}(\varphi\bar{n}) = \iota\nabla(\varphi\bar{n}) = \iota(\mu\nabla(\varphi)\nabla(\bar{n})) = A\,\iota(\mu\nabla(\varphi))(\delta\bar{n}) = \varphi^*\bar{n},$$

where $\varphi^* = A(A\,\iota(\mu\nabla(\varphi)))\delta$. ∎

We need some additional constructions. Suppose $p(n, m)$ is a primitive recursive pairing function of natural numbers with primitive recursive projection functions p_0 and p_1, i. e. $p_i(p(n_0, n_1)) = n_i$, $i<2$. Then there is a R-expressible $\beta \in \mathscr{F}$, such that for all $\xi, \eta \in F$

$$\beta\xi\eta\bar{n} = \xi(\overline{p_0(n)})(\eta\overline{p_1(n)}). \tag{5.4}$$

Indeed, by Theorem 1 in [9] there are R-expressible $\pi_0, \pi_1 \in \mathscr{F}$, such that $\pi_i\bar{n} = \overline{p_i(n)}$, $i<2$, and by Lemma 1.1 and the identity $\varphi\psi\chi = C\varphi(D\psi\chi)$ we have

$$\xi\,\overline{p_0(n)}\,(\eta\,\overline{p_1(n)}) = \xi(\pi_0\bar{n})(\eta(\pi_1\bar{n})) = \alpha\xi\eta(D\bar{n}\,\bar{n}) \tag{5.5}$$

for suitable $\mathscr{R}$-expressible $\alpha \in F$. Then defining $\partial = \mathsf{R}_1(D\bar{o}\,\bar{o}, d_0)$, where $d_0 \in \mathscr{F}$ is an $\{A, C, S\}$-expressible element, such that

$$d_0(D\bar{n}\,\bar{n}) = D(S\bar{n})(S\bar{n}),$$

we may prove by induction on n that

$$\partial\bar{n} = D\bar{n}\,\bar{n}, \tag{5.6}$$

whence by (5.5) and Lemma 1.1 we may find $\beta \in \mathscr{F}$, which is $\mathscr{R}$-expressible and satisfies (5.4).

Theorem 5.7. *There is a recursive in* $\mathscr{C}_2$ *element* $\omega \in \mathscr{F}$, *such that:*

(a) *for every recursive in* $\mathscr{C}_2$ *element* $\varphi \in \mathscr{F}$ *there is a natural number* n, *such that* $\omega\bar{n} = \varphi$; *and*

(b) *there is a primitive recursive function* s *of two arguments, such that* $\omega\,\overline{s(n, m)} = \omega\bar{n}\,\bar{m}$ *for all natural numbers* n, m.

Proof. Let us denote $c_0 = A$; $c_1 = C$: $c_2 = C'$; $c_3 = D'$; $c_4 = K$; $c_5 = \bar{o}$; $c_6 = S$; and suppose $\mathscr{C} = \{c_7, \ldots, c_{6+k}\}$. Then define a numeration $\ulcorner t \urcorner$ of terms $t \in \mathrm{Term}(\mathscr{C}_2, \varnothing)$ as follows: $\ulcorner c_i \urcorner = i$ for $i < 7+k$; $\ulcorner ts \urcorner = p(\ulcorner t \urcorner, \ulcorner s \urcorner) + 7 + k$. There is an element $\gamma \in \mathscr{F}$ recursive in $\mathscr{C}_2$, such that $\gamma\,\overline{\ulcorner t \urcorner} = \tilde{t}$ for every $t \in \mathrm{Term}(\mathscr{C}_2, \varnothing)$, where $\tilde{t}$ is the value of t in $\mathscr{F}$. Indeed we may define γ by Theorem 5.3 as the least fixed point of the operator

$$\Gamma(\xi) = \mathsf{R}_0(c_0, \ldots, \mathsf{R}_0(c_{6+k}, \beta\xi\xi)\ldots).$$

Then γ satisfies $\gamma\bar{i} = c_i$ for $i < 7+k$, and $\gamma\overline{n+7+k} = \beta\gamma\gamma\bar{n} = \gamma\,\overline{p_0(n)}(\gamma\,\overline{p_1(n)})$; whence by induction on the construction of $t \in \mathrm{Term}(\mathscr{C}_2, \varnothing)$ we have $\gamma\overline{\ulcorner t \urcorner} = \tilde{t}$. Thence by Theorem 5.4 for every recursive in $\mathscr{C}_2$ element $\varphi \in \mathscr{F}$ there is a natural number n such that $\varphi = L_2\mathsf{I}^2(\gamma\bar{n})$, and by Lemma 5.6 (b)

$$\varphi = L_2\mathsf{I}(\gamma^*\bar{n}) = L_2(\gamma^{**}\bar{n}) = AL_2\gamma^{**}\bar{n}.$$

Therefore to complete the proof it will be enough to find a recursive in $\mathscr{C}_2$ element $\omega \in \mathscr{F}$, such that for all numbers n

$$\omega\overline{2n} = AL_2\gamma^{**}\overline{n} \quad \text{and} \quad \omega\overline{2n+1} = \omega\overline{p_0(n)}\,\overline{p_1(n)}.$$

To do that define by the theorem of representation of primitive recursive functions (Theorem 1 in [9]) two recursive in $\mathscr{C}_2$ elements $\sigma_0, \sigma_1 \in \mathscr{F}$, such that $\sigma_0\overline{2n} = \overline{0}$; $\sigma_0\overline{2n+1} = \overline{1}$; $\sigma_1\overline{2n} = \overline{n}$; and $\sigma_1\overline{2n+1} = \overline{n}$; and then define ω by Theorem 5.3 as the least fixed point of the mapping

$$\Gamma(\xi) = AAC(A\mathrm{R}_0(A(AL_2\gamma^{**})\sigma_1,\ \mathrm{R}_0(A(\beta\xi I)\sigma_1,\ A))\sigma_0)\partial,$$

where $I = A(CA)K$, $I\varphi = \varphi$ for all $\varphi \in \mathscr{F}$. Then for every number m

$$\omega\overline{m} = \Gamma(\omega)\overline{m} = C(A\mathrm{R}_0(A(AL_2\gamma^{**})\sigma_1,\ \mathrm{R}_0(A(\beta\xi I)\sigma_1,\ A))\sigma_0)(\partial\overline{m})$$
$$= \mathrm{R}_0(A(AL_2\gamma^{**})\sigma_1,\ \mathrm{R}_0(A(\beta\omega I)\sigma_1,\ A))(\sigma_0\overline{m})\,\overline{m},$$

whence

$$\omega\overline{2n} = A(AL_2\gamma^{**})\sigma_1\overline{2n} = AL_2\gamma^{**}(\sigma_1\overline{2n}) = AL_2\gamma^{**}\overline{n},$$

and

$$\omega\overline{2n+1} = A(\beta\omega I)\sigma_1\overline{2n+1} = \beta\omega I\overline{n} = \omega\overline{p_0(n)}\,\overline{p_1(n)}. \blacksquare$$

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