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MINIMALLITY OF THE GROUP *AUTOHOMEOM (C)*

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ABSTRACT. The main results are the following:
Theorem 1. The group Autohomeom $(I^n) \in \mathbb{N}$ is a minimal topological group iff $h = 1$.
Theorem 2. The group Autohomeom $(D^{\aleph_0})$ where $D^{\aleph_0}$ is a Cantor cube with a countable weight is minimal.
These results are partly an answer to the general question raised by Prodanov & Stojanov (1984), Dierolf et.al. (1979) & (1977).

A Hausdorff topological group (G, τ_0) is called minimal, if no Hausdorff group topology τ on G is strictly coarser than τ_0.

Let (G, τ) be a topological group and let $N_e(\tau)$ denote the set of all τ -neighbourhoods of the identity element $e \in G$. Suppose we have an action $\alpha : G \times X \to X$ where X is a set with a uniformity $\mathcal{U}$. Then α is called quasibounded if for every $P \in \mathcal{U}$ there exist $Q_P \in \mathcal{U}$ and $U_P \in N_e(\tau)$ such that if $(x, y) \in Q_P$ and $g \in U_P$ then $(gx, gy) \in P$. A Hausdorff group topology τ on the group $\mathrm{Aut}(K)$ of homeomorphisms of a compact set K is called quasibounded if the natural action $\alpha : \mathrm{Aut}(K) \times K \to K$ is quasibounded [5].

Our first step is to prove the following proposition:

Proposition. *Let* $\mathrm{Aut}(K)$ *be the group of homeomorhisms of a compact set* K. *The compact -open topology* τ_0 *is minimal in the class of all quasibounded topologies on* $\mathrm{Aut}(K)$.

First we recall the following characteristic property of compacts.

Lemma 1. *Let* $\mathcal{F}$ *be an ultrafilter on a set* X *and let* φ *be any mapping from* X *into a compact set* K. *Then there exists one and only one point* $\tilde{x} \in K$ *such that for each open set* $O(\tilde{x})$ *containing* $\tilde{x}$ *we have* $\varphi^{-1}(O(\tilde{x})) \in \mathcal{F}$. *In other words,* $\tilde{x}$ *is a limit point of* φ *under* $\mathcal{F}$.

Now suppose τ a quasibounded topology on $\mathrm{Aut}(K)$ that is strictly coarser than the compact open-topology τ_0. It can easily be verified that if an action $\mathrm{Aut}(K) \times K \to K$ is continuous from left, i.e. if the orbit maps $\mathrm{Aut}(K) \to K$ given by $g \to gk_0$ (where $k_0 \in K$) are continuous, then $\tau = \tau_0$. So we admit the existence of $x_0 \in K$ and $P_0 \in \mathcal{U}$ ($\mathcal{U}$ -the natural uniformity on K) such that for every $\mathcal{U} \in N_e(\tau)$ there exists $g_u \in U$ for which $(gx_0, x_0) \notin P_0$. Consider a filter on the set $N_e(\tau)$ with the filter base $\{F(U)\}_{U \in N_e(\tau)}$ where $F(U) = \{V \in N_e(\tau) | v \subset U\}$. Let $\mathcal{F}$ be an ultrafilter containing this one. For each $x \in K$ we consider the map from $N_e(\tau)$ to K given by the rule: $U \to g_u x$. Let $\tilde{x}$ be a point defined as in Lemma 1. Consider the mapping $h : k \to k$ where $h(x) = \tilde{x}$.

Lemma 2. *H is nontrivial homeomorphism of* K.

Proof. Suppose $x, y \in K$ are such that $x \neq y$ and $h(x) = h(y) = z$. Let $P \in \mathcal{U}$ and $(x, y) \notin P$. We may choose $Q_P \in \mathcal{U}$, $U_P \in N_e(\tau)$ from the definition of the quasibounded topology and choose $Q \in U$ such that $Q^2 \subset Q_P$. By the definition of h we have $\{U \in N_e(\tau)|(g_u x, z) \in Q\} \in \mathcal{F}$ and $\{U \in N_e(\tau)|(g_u y, z) \in Q\} \in \mathcal{F}$. But only $F(U_P^{-j}) \in \mathcal{F}$, hence the intersection of these three sets is nonempty. Suppose U_0 is an element of this intersection. Then $g_{u_0} \in U_0 \subset U_P^{-1}$ and $(g_{u_0}x, g_{u_0}y) \in Q^2 \subset Q_P$; but $(x, y) \notin P$ - which is a contradiction.

Suppose $y \in K$. Consider the mapping $N_e(\tau) \to k$ given by $U \to g_u^{-1}y$ and take $x = \tilde{x}$ as in Lemma 1. We prove that $h(x) = y$. Suppose P is any element of $\mathcal{U}$ and $Q_P \in \mathcal{U}$, $U_P \in N_e(\tau)$ are taken as above. Then from the definition of $\mathcal{F}$ we have $\{U \in N_e(\tau)|(g_u^{-1}y, x) \in Q_P\} \in \mathcal{F}$. On the other hand $F(U_P) \in \mathcal{F}$. For each U from the intersection of these sets we obtain $U \subset U_P$ and $(g_u^{-1}y, x) \in Q_P$ hence $(y, g_u x) \in P$ and $\{U \in N_e(\tau)|(g_u x, y) \in P\} \in \mathcal{F}$. This means $y = h(x)$. We conclude that h is a bijection. Now we prove that if there exists a pair $(x, y) \in Q_P$ such that $(h(x), h(y)) \in P$, then there exists neighbourhoods O_1, O_2 of points $h(x)$ and $h(y)$ such that for every $t' \in O_1$ and $t'' \in O_2$, $(t', t'') \notin P$. From the definition of h:

$$\{U \in N_e(\tau)|g_u x \in O_1\} \in \mathcal{F} \quad \text{and} \quad \{U \in N_e(\tau)|g_u y \in O_2\} \in \mathcal{F}.$$

Besides this we have $F(U_P) \in \mathcal{F}$. If U_0 is an element of all these sets, then $g_{U_0} \in U_P$ and $(g_{u_0}x, g_{u_0}y) \notin P$, but $(x, y) \in Q_P$. This contradiction proves that $h$is continuous. Hence h is a homeomorphism. For each $U \in N_e(\tau)(g_{u_0}x, x_0,) \notin P_0$. Hence $h(x_0) \neq x_0$ and $h \neq e$. □

The next Lemma shows that the topology τ is not Hausdorff.

Lemma 3. *For every $U \in N_e(\tau)$, $h \in U$.*

Proof. For every $g \in \text{Aut}(K)$ and $P \in \mathcal{U}$ let $\tilde{P}(g' \in \text{Aut}(K)|\forall x \in K(gx, g'x) \in P'$. It is well-known that $\{\tilde{P}(g) : P \in \mathcal{U}, g \in \text{Aut}(K)\}$ form a base for the compact-open topology τ on Aut(U). In order to prove our statement it is enough to show that for every $\mathcal{U}_0 \in N_e(\tau)$ and $P \in \mathcal{U}$, we have $h \in [\tilde{P}^3_{(e)}]^{-1}\mathcal{U}$. Indeed, for any $\mathcal{U} \in N_e(\tau)$ we can find $V \in N_e(\tau), V^2 \subset U, V^{-1} = V$. But $\tau \subset \tau_0$, hence there exists $P \in \mathcal{U}, \tilde{P}^3(e) \subset V$, hence $[\tilde{P}^3(e)]^{-1}V \subset V^{-1}V \subset U$.

Let $Q_P \in \mathcal{U}$ and $U_P \in N_e(\tau)$ be chosen for $P \in \mathcal{U}$ according to the definition of the quasibounded topology. Suppose $x \in K$ and $x' = h^{-1}(x)$. From the definition of $h, A(x) = \{U \in N_e(\tau)|(g_u x', x) \in P\}$ belongs to $\mathcal{F}$. We can choose $R \in \mathcal{U}, R \subset P$ such that if $(t', t'') \in R$, then $(h^{-1}(t'), h^{-1}(t'')) \in Q_P$. Consider an R-net of K, i.e. a finite subset $\{x_1, x_2, \ldots, x_n\} \subset K$ such that for every $x \in K$ there exists an i, $1 \leq i \leq n$ such that $(x, x_i) \in R$. Suppose that $U \in \cap_{i=1}^n A(x_i) \cap F(U_P \cap U_0)$. Thus for any $x \in K$ for some i we obtain $(h^{-1}(x), h^{-1}(x_i)) \in Q_P$ and $(g_u h^{-1}(x), g_u h^{-1}(x_i)) \in P$. But $R \subset P$, hence $(x, x_i) \in P$ and from $U \in A(x_i)$ it follows that $(g_u h^{-1}(x_i), x_i) \in P$. It is not difficult to see that $(g_u h^{-1}(x_i), x_i) \in P^3$, hence $g_u h^{-1} \in \tilde{P}^3(e)$. Since $g_u \in U_0$, we have $h \in [\tilde{P}^3(e)]^{-1}U_0$ and our proposition is proved. □

Theorem 1. *Suppose $I = [0, 1]$ and $n \in \mathbb{N}$. The group $\text{Aut}(I^h, \tau_0)$ of homeomorphisms of I^n with the compact-open topology τ_0 is minimal iff $n = 1$.*

Proof. Let τ be a group topology on Aut(I) strictly coarser than τ_0. From proposition 1 we see that τ is not quasibounded. Thus there exists $c > 0$ such that for every pair $\alpha = (n, U) \in \mathbb{N} \times N_e(\tau)$ there exists $x_\alpha y_\alpha \in I$ and $g_\alpha \in U$ such that $|x_\alpha - y_\alpha| < 1/n$, but $|g_\alpha x_\alpha - g_\alpha y_\alpha| \geq C$. On the set $\mathbb{N} \times N_e(\tau)$, which we denote by $\mathcal{N}$, we consider an ultrafilter $\mathcal{F}$ containing the filter base $\{F(\alpha)\}_{\alpha \in \mathcal{N}}$, where for $\alpha_0 \in (n_o, U_0) \in \mathcal{N}$ we let $F(\alpha_0) = \{(n, U) \in \mathcal{N}|n \geq n_0, U \subset U_0\}$. We obtain four mappings from $\mathcal{N}$ to I given by the rules: $\alpha \to x_\alpha, \alpha \to y_\alpha, \alpha \to g_\alpha x_\alpha, \alpha \to g_\alpha y_\alpha$ for $\alpha = (n, U) \in \mathcal{N}$. Let z_1, z_2, x_0 and

y_0 be points for these mappings specified as in Lemma 1. It is clear that $z_1 = z_2 = z$ and $|x_0 - y_0| \geq c$. Without loss of generality we may assume $x_0 < y_0$. Let us take $t_1, t_2 \in I$ with $x_0 < t_1 < t_2 < y_0$ and fix a nontrivial homeomorphism $\xi \in \mathrm{Aut}(I)$ which is trivial on the set $I \setminus (t_1, t_2)$, i.e. if $t < t_1$ or $t > t_2$ then $\xi(t) = t$.

Lemma 4. *For every $U \in N_e(\tau)$ we have $\xi \in U$.*

Proof. We will show that for every $U_0 \in N_e(\tau)$ and $\varepsilon > 0$ we have $\xi \in U_0 O_\varepsilon(e) U_0^{-1}$, where $O_\varepsilon(e) = \{g \in \mathrm{Aut}(I) | \forall x \in I, |gx - x| < \varepsilon\}$. Since $[0,1)$ and $(t_2, 1]$ are neighbourhoods of x_0 and y_0 we obtain:

$$\{\alpha \in \mathcal{N} | g_\alpha x_\alpha < t_1\} \in \mathcal{F} \text{ and } \{\alpha \in \mathcal{N} | g_\alpha y_\alpha > t_2\} \in \mathcal{F}$$

Let us choose $n_0 \in \mathbb{N}$ such that $1/n_0 < \varepsilon$ and let us assume that $\alpha_0 = (n_0, U_0)$. Because of $F(\alpha_0) \in \mathcal{F}$ there exists an $\alpha_1(n_1, U_1)$ which belongs to all these sets; hence $|x_{\alpha_1} - y_{\alpha_1}| < 1/n_1 < 1/n_0 < \varepsilon, g_{\alpha_1} \in U_1 \subset U_0$. Because $g_{\alpha_1} x_{\alpha_1} < t_1$ and $g_{\alpha_1} y_{\alpha_1} > t_2$ we obtain: $g_{\alpha_1}^{-1}((t_1, t_2)) \subset (x_{\alpha_1}, y_{\alpha_1})$ or $g_{\alpha_1}^{-1}((t_1, t_2)) \subset (y_{\alpha_1}, x_{\alpha_1})$. Let us consider $g_{\alpha_1}^{-1} \xi g_{\alpha_1}$. If $t \notin (x_{\alpha_1}, y_{\alpha_1})$ or $t \notin (y_{\alpha_1}, x_{\alpha_1})$ then $(g_{\alpha_1}^{-1} \xi g_{\alpha_1})(t) = t$. Hence $g_{\alpha_1}^{-1} \xi g_{\alpha_1} \in O_\varepsilon(e)$; but $g_{\alpha_1} \in U_0$ hence $\xi \in U_0 O_\varepsilon(e) U_0^{-1}$. But $\{O_\varepsilon(e)\}_{\varepsilon > 0}$ is a base of neighbourhoods of the identity in the compact-open topology. We conclude that the τ -topology is not Hausdorff. □

Now let us prove the second part of our theorem for $n > 1$. Suppose F is a boundary of I^n, and $|\cdot|$ is a natural norm in I^n. For $\varepsilon > 0$ define
$O_\varepsilon(e) = \{g \in \mathrm{Aut}(I^n) | \forall x \in I^n |gx - x|, |g^{-1}x - x| < \varepsilon\}$
$F_\varepsilon = \{x \in I^n | \exists y \in F |x - y| < \varepsilon\}$
$O_\varepsilon(e) = \{g \in \mathrm{Aut}(I^n) | \forall x \in I^n \setminus F_\varepsilon |gx - x|, |g^{-1}x - x| < \varepsilon\}$

Lemma 5. *The family $\{\tilde{O}_\varepsilon(e)\}_{\varepsilon > 0}$ is a neighbourhood base of the identity in some group topology τ that is strictly coarser than compact-open topology τ.*

Proof. Note the following facts:
(i) For every $\varepsilon_1, \varepsilon_2 > 0, \tilde{O}_{\varepsilon_1}(e) \cap \tilde{O}_{\varepsilon_2}(e) \supset \tilde{O}_\delta(e)$ where $\delta < min\{\varepsilon_1, \varepsilon_2\}$.
(ii) Suppose $\varepsilon > 0$, $\delta < \varepsilon/2$ and $g_1, g_2 \in \tilde{O}_\delta(e)$. If $x \notin F_\varepsilon$, then $x \notin F_\delta$ i.e. $|g_1 x - x| < \delta$ and $g_1 x \in F_\delta$. This means $|g_2 g_1 x - g_0 x| < \delta$, i.e.$|g_2 g_1 x - x| < \varepsilon$. The same reasoning shows that $|g_1^{-1} g_2^{-1} x - x| < \varepsilon$ if $x \notin F_\varepsilon$, and we obtain $\tilde{O}_\delta(e) \tilde{O}_\delta(e) \subset \tilde{O}_\varepsilon(e)$.
(iii) From the definition we have $\tilde{O}_\varepsilon^{-1}(e) = \tilde{O}_\varepsilon(e)$.
(iv) If $g \in \tilde{O}_\varepsilon(e)$ and $\varepsilon_1 = \sup\limits_{x \notin F_\varepsilon} \max\{|gx - x|, |g^{-1}x - x|$, then $\varepsilon_1 < \varepsilon$. For $\varepsilon_2 = \varepsilon - \varepsilon_1$, we can take $\delta < \varepsilon_2$ such that from $|t' - t''| < \delta$ it follows that $|gt' - gt''| < \varepsilon_2$ and $|g^{-1}t' - g^{-1}t''| < \varepsilon_2$. For $g' \in \tilde{O}_\delta(e)$ and $x \notin F_\varepsilon$ we obtain $x \notin F_\delta$ and $|g'x - x| < \delta$. Consequently, $|gg'x - x| < |gg'x - gx| + |gx - x| < \varepsilon$. Besides this, from $x \notin F_\varepsilon$ it follows that $|g^{-1}x - x| < \varepsilon$ and $g^{-1}x \notin F_{\varepsilon_2}$, i.e. $g^{-1}x \in F_\delta$. Then we get $|g_1^{-1} g^{-1} - g^{-1}x| < \delta < \varepsilon_2$ and $|g_1^{-1} g^{-1} x - x| < \varepsilon_2 + \varepsilon_1 = \varepsilon$. Hence $g\tilde{O}_\delta(e) \subset \tilde{O}_\varepsilon(e)$.
(v) Suppose $\varepsilon > 0$ and $g_0 \in \mathrm{Aut}(I^n)$. There exists $\delta > 0$ such that $F_\delta \subset g_0(f_\varepsilon)$, and from $|t' - t''| < \delta$ it follows that $|g_0 t' - g_0 t''| < \varepsilon$. If $x \notin F_\varepsilon$ then $gx_0 \notin F_\delta$. Thus for each g from $\tilde{O}_\delta(e)$ we conclude that $|gg_0 x - g_0 x| < \delta$, $|g^{-1} g_0 x - g_0 x| < \delta$, and hence that $|g_0^{-1} g g_0 x - x| < \varepsilon$, $|g_0^{-1} g^{-1} g_0 x - x| < \varepsilon$ i.e. $g_0^{-1} \tilde{O}_\delta(e) d_0 \subset \tilde{O}_\varepsilon(e)$.
Suppose $g \neq e$. The set $U = \{x \in I^n | gx \neq x\}$ is open in I^n. Hence $U \setminus F \neq \emptyset$. Suppose $x_0 \in U \setminus F$, $\varepsilon < |gx_0 - x_0|$ and $\varepsilon < |x_0 - x|$ for every $x \in F$. Then we have $x_0 \notin F_\varepsilon$ and $g \in \tilde{O}_\varepsilon(e)$.This proves that τ is a Hausdorff topology.

Suppose a and b are two distinct points of F. It can easily be verified that for every $\varepsilon > 0$ there exists $g \in \tilde{O}_\varepsilon(e)$ such that $ga = b$. Thus if $\varepsilon_0 = |a-b|$, the condition $\tilde{O}_\varepsilon(e) \subset O_{\varepsilon_0}(e)$ does not hold for every $\varepsilon > 0$ and $\tau \neq \tau_0$. □

Let $K = D^{\aleph_0}$ be a Cantor cube with a countable weight and a metric d and let τ be a Hausdorff topology on $\mathrm{Aut}(K)$ that is strictly coarser than the compact-open topology τ_0.

Lemma 6. *For every $U \in N_e(\tau)$ and $x, y \in K$ there exists $\delta > 0$ such that every element $g \in \mathrm{Aut}(K)$ that is trivial on $K \setminus B(x,\delta) \cup B(y,\delta)$ satisfies $g \in U$.*

Proof. From Theorem 1 we know that the topology τ is not quasibounded on $\mathrm{Aut}(K)$, i.e. for each $\alpha = (n, U), n \in \mathbb{N}, U \in N_e(\tau)$ there exist $x_\alpha, y_\alpha \in U$ and $g_\alpha \in \mathrm{Aut}(K)$ such that $d(x_\alpha, y_\alpha) < g_\alpha/n \in U$. But $d(g_\alpha x_\alpha, g_\alpha y_\alpha) \geq c$ where c is a positive constant. Suppose $\mathcal{F}$ is an ultrafilter containing the natural filter on the set $\{(n,U)|n \in \mathbb{N}, U \in N_e(\tau)\} \equiv \mathcal{N}$ and that z_1, z_2, x_0, y_0 are limits for the following mappings from $\mathcal{N}$ to $K : \alpha \to x_\alpha, \alpha \to y_\alpha, \alpha \to g_\alpha, x_\alpha$ and $\alpha \to g_\alpha, y_\alpha$. Then $z_1 = z_2 = z$ and $d(x_0, y_o) \geq c$.

First we shall prove this lemma for $x = x_0, y = y_0$. Suppose $U_0 \in N_e(\tau)$ and $\varepsilon > 0$ are fixed. It is enough to find a $\delta > 0$ such that every element $g \in \mathrm{Aut}(K)$ that it trivial on $K \setminus B(x_0,\delta) \cup B(y_0,\delta)$ belongs to $V_\varepsilon(e)U_0V_\varepsilon(e)U_0V_\varepsilon(e)$. Suppose $n_0 \in \mathbb{N}$ and $\frac{1}{n_0} < \frac{\varepsilon}{3}$. For x_0, y_0 and $\alpha_0 = (n_0, U_0)$ we obtain:

$$\{\alpha \in \mathcal{N} | d(g_\alpha x_\alpha, x_0) < \varepsilon\} \in \mathcal{F} \quad \text{and} \quad \{\alpha \in \mathcal{N} | d(g_\alpha y_\alpha, y_0) < \varepsilon\} \in \mathcal{F}.$$

There exists $\alpha_1 = (n_1, U_1)$ which belongs to both these sets and $F = (\alpha_0)$, i.e. $n_1 \geq n_0, U_1 \subset U_0$. Because $d(g_\alpha x_\alpha, x_0)$, $d(g_\alpha y_\alpha y_0) < \varepsilon$, there exists a $\overline{g} \in V_\varepsilon(e)$ such that $\overline{g}(g_\alpha, x_\alpha) = x_0$ and $\overline{g}(g_\alpha, y_\alpha) = y_0$. Let $\tilde{g} = \overline{g} g_\alpha$, and let $\delta > 0$ be chosen such that if $d(t', t'') < \delta$, then $d(\tilde{g}^{-1}t', \tilde{g}^{1}t'') < \frac{\varepsilon}{3}$. Let us consider the element $g \in \mathrm{Aut}(K)$ trivial on $K \setminus B(x_0,\delta) \cup B(y_0,\delta)$. From $\tilde{d}^{-1}x_0 = x_{\alpha_1}$, $\tilde{g}^{-1}y_0 = y_{\alpha_1}$, we obtain $\tilde{g}^{-1}(B(x_0,\delta)) \subset B(x_\alpha, \frac{\varepsilon}{3})$ and $\tilde{g}^{-1}(B(y_0,\delta) \subset B(y_\alpha, \frac{\varepsilon}{3})$; but $d(x_\alpha y_{\alpha_1}) < \frac{1}{n_1} \leq \frac{1}{n_0} < \frac{\varepsilon}{3}$, so there exists a ball B with diameter ε such that $\tilde{g}^{-1}(B(x_0,\delta)) \cup \tilde{g}^{-1}(B(y_0,\delta)) \subset B$. From the definition of g it follows that $\tilde{g}^{-1}g\tilde{g} \in V_\varepsilon(e)$. But $\tilde{g} = \overline{g} g_{\alpha_1} \in V_\varepsilon(e)U_0$, which means that $g \in V_\varepsilon(e)U_0V_\varepsilon(e)U_0V_\varepsilon(e)$, what was to be proved.

Now suppose $x, y \in K$ and $U \in N_e(\tau)$. We can find $g_0 \in \mathrm{Aut}(K)$ such that $gy_0 = y, gx_0 = x$, and $V \in N_e(\tau)$ such that $g_0Vg_0^{-1} \subset U$. Let $\varepsilon > 0$ be chosen for x_0, y_0 and V as above. We can choose $\delta > 0$ such that $d(t', t'') < \delta$ implies $d(g_0^{-1}t', g_0^{-1}t'') < \varepsilon$. Suppose $g \in \mathrm{Aut}(K)$ is trivial on $K \setminus B(x,\delta) \cup B(y,\delta)$; then $g_0^{-1}gg_0$ is trivial on $K \setminus B(x_0,\varepsilon) \cup B(y_0,\varepsilon)$, i.e. $g_0^{-1}gg_0 \in V$ and $g \in g_0Vg_0^{1-} \subset U$.

Theorem 2. *The group $\mathrm{Aut}(D^{\aleph_0})$ is minimal.*

Proof. We represent $K = D^{\aleph_0}$ as a set of sequences $(x_1, x_2, \ldots, x_n, \ldots)$ containing only zeros and units. For $a = 0 (a = 1)$ we define $\overline{a} = 1$ $(\overline{a} = 0)$. Suppose $x = (x_1, \ldots x_n, \ldots) \in K$, and define $\overline{x} = (\overline{x}_1, x_2, \ldots x_n, \ldots)$. Let us consider $\Delta : K \to K$ defined by $\Delta : x \to \overline{x}$ and let τ be any topology strictly coarser than the compact-open topology τ_0 on $\mathrm{Aut}(K)$.

The following Lemma shows that τ-topology is not Hausdorff which proves Theorem 2.

Lemma 7. *For each $U \in N_e(\tau)$ we have $\Delta \in U$.*

Proof. Let U_0 be any element of $N_e(\tau), U^6 \subset U_0, U \in N_e(\tau)$ and choose $\varepsilon > 0$ so that $V_\varepsilon^2(e) \subset U$. For $\varepsilon > 0$ we can choose $n \in \mathbb{N}$ such that $\sum_{k=n}^{\infty} \frac{1}{2^k} < \frac{1}{\varepsilon}$. Let $A = \{a_1, \ldots, a_m\} = \{x \in K | x_{n+1} = x_{n+2} = \ldots = 0\}$ i.e. A is a finite ε-net in K, which consists of $m = 2^n$ elements. Let $W \in N_e(\tau)$ be such that $W^m \subset U$ and choose $\delta > 0$ so that the condition of Lemma 6 holds for pairs $(a_1, \overline{a}_1), \ldots (a_m, \overline{a}_m)$ and W. Take $n \in \mathbb{N}$ so that $\sum_{k=N}^{\infty} \frac{1}{2^k} < \delta$. For every $i, 1 \leq i \leq m$, we consider $\psi_i \in \mathrm{Aut}(K)$ defined by following rule: if coordinates of x coincide with a_i or $\overline{a}_i$

till N and particularly $x_{n+1} = x_{n+2} = \ldots = x_N = O$, then $\psi_i(x) = \overline{x}$ for all the other elements $x \in K$ we put $\psi_i(x) = x$. Then $\psi_i \in W$ and $\psi = \psi_1 \cdot \ldots \cdot \psi_m \in W^m \subset U$.

On the set K we consider four subsets:
$K_1 = \{x \in K | x_{n+1} = 0$ and there exists an $i, n+2 \leq i \leq N$ such that $x_i = 1\}$,
$K_2 = \{x \in K | x_{n+1} = x_{n+2} = \ldots = x_N = 0\}$,
$K_3 = \{x \in K | x_{n+1} = 1$ and there exists an $i, n+2 \leq i \leq N$ such that $x_i = 1\}$,
$K_4 = \{x \in K | x_{n+1} = x_{n+2} = \ldots = x_N = 0\}$.

Let us consider the mappings:
$\varphi_1 : K_1 \cup K_2 \to K_2$ if $x \in K_1 \cup K_2$, then $\varphi_1(x) = y$, where $y_i = x_i$ for $1 \leq i \leq n, y_{n+1} = \ldots = y_N = 0$, and $y_{N+i} = x_{n+1+i}$ for $i \geq 1$,
$\varphi_2 : K_3 \to K_1$ if $x \in K_3$ then $\varphi_2(x) = y$, where $y_i = x_i$ for $i \neq n+1$ and $y_{n+1} = 0$,
$\varphi_3 : K_4 \to K_3 \cup K_4$ if $x \in K_4$ then $\varphi_3(x) = y$, where $y_i = x_i$ for $1 \leq i \leq n$ and $y_{n+1+i} = x_{N+i}$ for $i \geq 1$.

We obtain $\varphi = \varphi_3 \Delta \varphi_2 \Delta \varphi_1 \in \mathrm{Aut}(K)$ and because the coordinates of the points for $i \leq n$ do not change it follows that $\varphi \in V_\varepsilon(e)$. Let us consider $\varphi^{-1}\psi\varphi(x)$. If $x \in K$ and $x_{n+1} = 0$, then it is not difficult to see that $\varphi_1\psi\varphi(x) = \overline{x}$, and if $x_{n+1} = 1$, then $\varphi_1\psi\varphi(x) = x$. On the other hand we have $\varphi_1\psi\varphi \in V_\varepsilon(e) U V_\varepsilon(e) \subset U^3$.

Let $\tilde{\varphi}$ be the element of $\mathrm{Aut}(K)$ such that if $x \in K$ then $\tilde{\varphi}(x) = x'$, where $x'_i = x_i$ for $i \neq n+1$ and $x'_{n+1} = \overline{x}_{n+1}$. Then $\tilde{\varphi}^* \in V_3(e)$ and $\varphi_0 = (\varphi\tilde{\varphi})^{-1}\psi(\varphi\tilde{\varphi}) \in V_\varepsilon^2(e) U V_\varepsilon^2(e) \subset U^3$. But from the definition of φ_0 it follows that if $x \in K$ and $x_{n+1} = 1$, then $\varphi_0(x) = \overline{x}$; if $x_{n+1} = 0$, then $\varphi_0(x) = x$. Finally we see that the composition $\varphi_0(\varphi^{-1}\psi\varphi) \in U^3U^3 \subset U_0$; but $\varphi_0(\varphi^{-1}\psi\varphi) = \Delta$, i.e. $\Delta \in U_0$. ■

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