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QUASISIMILARITY OF N-TUPLES OF COMMUTING OPERATORS

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ABSTRACT. Let H be an infinite dimensional Hilbert space and $\mathcal{L}(H)$ be the algebra of all bounded linear operators acting on H. In this article we consider the n-tuples of commuting operators in $\mathcal{L}(H)$ which possess the property double quasisimilarity and we consider some of the corresponding Koszul complexes. Further, we give a definition of a semi-Fredholm spectrum of a n-tuple (Definition 2). If A and B are double quasisimilar n-tuples a theorem of intersection of the semi-Fredholm spectra of A and B is proved.

Introduction. Let H be an infinite dimensional complex Hilbert space and $\mathcal{L}(H)$ be the algebra of all bounded linear operators acting on H.

There are a lot of articles studying spectral properties of quasisimilar operators. Recall that the operators A and B in $\mathcal{L}(H)$ are quasisimilar if there exists operators X and Y in $\mathcal{L}(H)$, both of which are injective and have dense ranges, such that $AX = XB$ and $YA = BY$ [6]. Similar operators are quasisimilar and have equal spectra. It is shown in [6] that quasisimilar operators may have different spectra. But in [3] it is proved that the essential spectra of quasisimilar operator intersect. In [5] this result is sharpened as the following is proved: [5, Theorem 2.1]. *If A and B are quasisimilar operators, then $\sigma_{le}(A)\cap\sigma_{re}(B)\neq\emptyset$ and $\sigma_{re}(A)\cap\sigma_{le}(B)\neq\emptyset$.*

In this article quasisimilar (double quasisimilar) n-tuples of commuting operators in $\mathcal{L}(H)$ are considered (Definition 1.). If $A=(A_1,\dots,A_n)$ and $B=(B_1,\dots,B_n)$ are double quasisimilar n-tuples we have shown some properties of the corresponding Koszul complexes. We introduce the notion of semi-Fredholm spectrum of a n-tuple of commuting operators (Definition 2.). Indeed, this is a family of spectra which are parts of the essential spectrum $\sigma_e(A)$ of the n-tuple A. In the terms of this kind of spectrum we have proved a theorem (Theorem 1) which is a generalization of the theorem given above. Moreover the proof of ours is simpler then this one given in [5].

Through all this article we use the following notations: ∂K will denote the boundary of the set K; by $\bar{K}$ we denote the closure of K; $\ker A$ will be the kernel of the operator A; $\mathrm{Im}A$ will be the range of the operator A.

Quasisimilarity of n-tuples. Let H denote an infinite dimensional Hilbert space and $\mathcal{L}(H)$ denote the algebra of all bounded linear operators acting on H.

Definition 1. *Let $A=(A_1,\dots,A_n)$ and $B=(B_1,\dots,B_n)$ be n-tuples of commuting operators in $\mathcal{L}(H)$. We say that the n-tuples A and B are quasisimilar if there exist operators X and Y in $\mathcal{L}(H)$ such that*

1. $\ker X = \ker Y = \{0\}$;

2. $\overline{\mathrm{Im}X} = \overline{\mathrm{Im}Y} = H$;

3. $A_iX = XB_i$, $YA_i = B_iY$, *for every* $i = 1, 2, \ldots, n$.

4. If $A_i^*X = XB_i^*$, $YA_i^* = B_i^*Y$ *for every* $i = 1, 2, \ldots, n$, *where* A_i^* *and* B_i^* *are adjoint operators of* A_i *and* B_i $(i = 1, 2, \ldots, n)$, *n-tuples* A *and* B *are called double quasisimilar.*

It is evident that if X and Y are operators like in Definition 1. which satisfy conditions 1-3, then Y^* and X^* carry out quasisimilarity between $A^* = (A_1^*, \ldots, A_n^*)$ and $B^* = (B_1^*, \ldots, B_n^*)$. One can guess immediately that if A and B are quasisimilar (double quasisimilar) n-tuples such are $A - z = (A_1 - z_1, \ldots, A_n - z_n)$ and $B - z = (B_1 - z_1, \ldots, B_n - z_n)$, where $z = (z_1, \ldots, z_n) \in C^n$ and if A_i and B_i $(i = 1, \ldots, n)$ are invertible then $A^{-1} = (A_1^{-1}, \ldots, A_n^{-1})$ and $B^{-1} = (B_1^{-1}, \ldots, B_n^{-1})$ are quasisimiliar (double quasisimiliar) too.

Let $A = (A_1, \ldots, A_n)$ be a n-tuple of commuting operators in $\mathcal{L}(H)$. We recall the inductive definition of the Koszul complex associated with the n-tuple A (see also [9,10,1]) which we denote by $K.(A)$. In the case $n = 1$ the Koszul complex of a single operator A is simply $0 \to H \xrightarrow{A} H \to 0$. Let $K.(A')$ be the Koszul complex of the $n-1$-tuple $A' = (A_1, \ldots, A_n)$. Then the operator A_n defines an endomorphism of the complex $K.(A')$. The cone of this morphism is by definition the Koszul complex $K.(A) = \{H_i, d_i\}_{i=0}^n$ of the n-tuple $A = (A', A_n)$. It is easy to see that the i-th term H_i of this complex is a direct sum of $\binom{n}{i}$ copies of the Hilbert space H and the differential of this complex $d_i : H_i \to H_{i+1}$ are bounded linear operators.

Let A and B be n-tuples and $K.(A) = \{H_i, d_i\}_{i=0}^n$ and $K.(B) = \{H_i, \alpha_i\}_{i=0}^n$ the corresponding Koszul complexes. We denote by h_i and χ_i $(i = 0, 1, \ldots, n$ the cohomology groups of this complex, i.e. $h_i = \ker d_i/\overline{\mathrm{Im} d_{i-1}}$, $\chi_i = \ker \alpha_i/\overline{\mathrm{Im}\alpha_{i-1}}$ $(i = 1, 2, \ldots, n)$, $h_0 = \ker d_0$, $\chi_0 = \ker \alpha_0$.

It is not difficult to see that if A and B are quasisimilar then such are d_i and α_i $(i = 0, 1, \ldots, n)$ and

$$d_iX = X\alpha_i, \; Yd_i = \alpha_iY, \text{ for every } i = 0, 1, \ldots, n, \tag{1}$$

but if A and B are double quasisimilar then

$$d_i^*X = X\alpha_i^*, \; Yd_i^* = \alpha_i^*Y, \text{ for every } i = 0, 1, \ldots, n \tag{2}$$

is satisfied too. Here X and Y are the operators from Definition 1.

Lemma 1. *If* $A = (A_1, \ldots, A_n)$ *and* $B = (B_1, \ldots, B_n)$ *are double quasisimilar n-tuples of commuting operators,* h_i *and* χ_i $(i = 0, 1, \ldots, n)$ *are the cohomology groups of the complexes* $K.(A)$ *and* $K.(B)$ *respectively, then* $\dim h_i = \dim \chi_i$ $(i = 0, 1, \ldots, n)$.

Proof. Let $i = 0$ first. Then $h_0 = \ker d_0 = \bigcap_{j=1}^n \ker A_j$ and $\chi_0 = \ker d_0 = \bigcap_{j=1}^n \ker A_j$. As according to Definition 1. $\ker X = \ker Y = \{0\}$, then $\dim Y(h_0) = \dim h_0$ and $\dim Y(\chi_0) = \dim \chi_0$. On the other hand $Y(\ker d_0) \subset \ker \alpha_0$. Indeed, for some $x \in Y(\ker d_0)$ there exists $t \in \ker d_0$ such that $x = Yt$. We obtain by (1)

$$\alpha_0 x = \alpha_0 Yt = Yd_0t = 0.$$

Analogously $X(\chi_0) \subset \ker d_0$ and therefore $\dim Y(\ker d_0) \leq \dim \ker \alpha_0$ and $\dim X(\ker \alpha_0) \leq \dim \ker d_0$. Then

$$\dim h_0 = \dim Y(h_0) \leq \dim \chi_0,$$

$$\dim \chi_0 = \dim X(\chi_0) \leq \dim h_0,$$

hence $\dim h_0 = \dim \chi_0$.

Let now $i = n$. Then $h_n = \text{coker} d_{n-1}$ and $\chi_n = \text{coker}\alpha_{n-1}$. As h_n^* is isomorphic to $\ker d_{n-1}^*$ and χ_{n-1}^* is isomorphic to $\ker \alpha_{n-1}^*$, d_{n-1}^* and α_{n-1}^* are quasisimilar (by Y^* and X^*), then by the previous case we have $\dim h_n^* = \dim \chi_n^*$ and hence $\dim h_n = \dim \chi_n$.

Let i be arbitrary, $0 < i < n$. It is clear that $H_i = \text{Im} d_{i-1} \oplus (\ker d_i)^\perp \oplus L_i$ and $H_i = \text{Im}\alpha_{i-1} \oplus (\ker \alpha_i)^\perp \oplus N_i$ where L_i and h_i are isomorphic all together and such are N_i and χ_i. We show that $Y(L_i) \subset N_i$. So, let $c \in L_i$ is arbitrary. Then $c \in \ker d_i$ and $c \perp \text{Im} d_{i-1}$. But as the following diagram

$$\begin{array}{ccccc} H_{i-1} & \xrightarrow{\alpha_{i-1}} & H_i & \xrightarrow{\alpha_i} & H_{i+1} \\ Y\downarrow & & Y\downarrow & & Y\downarrow \\ H_{i-1} & \xrightarrow{\alpha_{i-1}} & H_i & \xrightarrow{\alpha_i} & H_{i+1} \end{array}$$

is commutative, we obtain that $Yc \in \ker \alpha_i$. Let $z \in \text{Im}\alpha_{i-1}$ is arbitrary, then there exists $t \in H_{i-1}$ such that $z = \alpha_{i-1}t$. Then we have

$$(Yc, z) = (Yc, \alpha_{i-1}t) = (\alpha_{i-1}^* Yc, t) = (Y\alpha_{i-1}^* c, t) = 0$$

according to (2) and because $c \in (\text{Im} d_{i-1})^\perp \subset \ker d_{i-1}^*$. Thus we obtain $Yc \perp \text{Im}\alpha_{i-1}$ and therefore $Yc \in N_i$. Analogously $X(N_i) \subset L_i$. Since $\ker X = \ker Y = \{0\}$ we have

$$\dim L_i = \dim Y(L_i) \leq \dim N_i,$$

$$\dim N_i = \dim X(N_i) \leq \dim L_i,$$

i.e. $\dim L_i = \dim N_i$ and hence $\dim h_i = \dim \chi_i$. The Lemma is proved.

Remark. We notice that in the cases $i = 0$ and $i = n$ of the Lemma 1's proof we did not use the double quasisimilarity of A and B, but quasisimilarity of these n-tuples only.

The following lemma will be useful.

Lemma 2. *Let the complex $\{H_i, d_i\}_{i=0}^n$ be exact. Then there exist linear bounded operators $b_i : H_{i+1} \to H_i$ $(i = 0, 1, \ldots, n)$ from the algebra generated by d_i and d_i^* such that*

$$d_{i-1}b_{i-1} + b_i d_i = I, \quad i = 1, \ldots, n. \tag{3}$$

P r o o f. We determine the operators b_i by induction, beginning from the right side of the complex.

Since $d_n = 0$ we put $b_n = 0$. That's why in the case $i = n$ we have to prove that there exists an operator $b_{n-1} : H_n \to H_{n-1}$ such that $d_{n-1}b_{n-1} = I$. By assumption $\text{coker} d_{n-1} = \{0\}$ therefore $\ker d_{n-1}^* = \{0\}$. Hence the operator $d_{n-1}d_{n-1}^*$ is invertible. Putting $b_{n-1} = d_{n-1}^*(d_{n-1}d_{n-1}^*)^{-1}$ we reach the desired equation $d_{n-1}b_{n-1} = I$.

Let for $i > k$ the operators b_i be determined. We have to show that there exists an operator $b_k : H_{k+1} \to H_k$ with the property $d_k b_k + b_{k+1}d_{k+1} = I$. It is easy to see that the operator $Q = b_{k+1}d_{k+1}$ is projection of H_{k+1} onto $(\ker d_{k+1})^\perp$. Then the operator $M = Q + d_k d_k^*(I - Q)$ is invertible. Now we put

$$b_k = \begin{cases} d_k^* M^{-1} & \text{on } \text{Im} d_k \\ 0 & \text{on } H_k \ominus \text{Im} d_k \end{cases}.$$

By elementary checking using the exactness of the considering complex one can see that the operator b_k satisfies the equation (3). The proof is completed.

Let A and B be double quasisimilar n-tuples of commuting operators in $\mathcal{L}(H)$ and $K.(A) = \{H_i, d_i\}_{i=0}^n$ and $K.(B) = \{H_i, \alpha_i\}_{i=0}^n$ be their Koszul complexes. Let b_i and β_i be the operators constructed in Lemma 2 for the complexes $K.(A)$ and $K.(B)$ respectively. Then by Lemma 2 and equations (1) and (2) we obtain

$$(4) \qquad b_i X = X\beta_i, \; Y b_i = \beta_i Y \text{ for every } i = 0, 1, \ldots, n.$$

In his fundamental paper [9] and [10] J. L. Taylor introduced the notion of parameterized Koszul complex (we denote it by $K.(A, z) = \{H_i, d_i(z)\}_{i=0}^n$) of a commuting n-tuple A of bounded linear operators. This is the Koszul complex inductive definition which was given above but of the n-tuple $A - z = (A_1 - z_1, \ldots, A_n - z_n)$ where $z = (z_1, \ldots, z_n) \in C^n$. We notice that the differentials $d_i(z) : H_i \to H_{i+1}$ $(i = 0, 1, \ldots, n)$ of the complex $K.(A, z)$ are linear and hence analytic functions of $z \in C^n$. By $h_i(z)$ $(i = 0, 1, \ldots, n)$ we denote the cohomology groups of $K.(A, z)$.

The Taylor spectrum $\sigma(A)$ of the commuting n-tuple A is by definition the set of all points $z \in C^n$ such that the complex $K.(A, z)$ is not exact free [9]. The essential Taylor spectrum $\sigma_e(A)$ of the n-tuple A is the set of all points $z \in c^n$ such that the complex $K.(A, z)$ is not Fredholm, i.e. either there exists i_0 $(0 \leq i_0 \leq n)$ such that $\operatorname{Im} d_{i_0}(z)$ is not closed, or there exists j_0 $(0 \leq j_0 \leq n)$ such that $\dim h_{j_0}(z) = +\infty$ [1], [2]. The Fredholm spectrum $\sigma_F(A)$ of the n-tuple A is the set $\sigma_F(A) = \sigma(A) \backslash \sigma_e(A)$.

J. L. Taylor has constructed an analytic functional calculus for n-tuples of commuting operators [10]. In [7] another construction is given (see also [11] [12]) as follow. Let $f(z)$ be an analytic function in a neighbourhood of $\sigma(A)$ and S be a smooth surface around $\sigma(A)$ such that $S \cap \sigma(A) = \emptyset$. Then the value of the function $f(z)$ for the n-tuple A is given by the formula

$$(5) \qquad f(A) = \frac{1}{(2\pi i)^n} \int_S f(z) R(A, z) \wedge dz_1 \wedge \ldots \wedge dz_n,$$

where $R(A, z) = n! b_0(z) \wedge \bar{\partial} b_1(z) \wedge \ldots \wedge \bar{\partial} b_{n-1}(z)$ is a $\bar{\partial}$-closed differential form of type $(0, n-1)$ in $C^n \backslash \sigma(A)$. Here $b_0(z), \ldots, b_{n-1}(z)$ are the operators constructed in Lemma 2 for the complex $K.(A, z)$ for $z \in C^n \backslash \sigma(A)$.

Let A and B are double quasisimilar n-tuples in $\mathcal{L}(H)$ and X and Y be the operators from Definition 1. Let $b_i(z)$ and $\beta_i(z)$ $(i = 0, 1, \ldots, n)$ be the operators constructed in Lemma 2 for the complexes $K.(A, z)$ and $K.(B, z)$ respectively and $z \in C^n \backslash (\sigma(A) \cup \sigma(B))$. Then from (4) we have $R(A, z)X + XR(B, z)$ and $YR(A, z) + R(B, z)Y$ for every $z \in C^n \backslash (\sigma(A) \cup \sigma(B))$. Now if $f(z)$ and $g(z)$ are functions analytic in a neighbourhood of $\sigma(A) \cup \sigma(B)$ then

$$(6) \qquad f(A)X = Xg(B), \; Yf(A) = g(B)Y.$$

Let $K.(A, z) = \{H_i, d_i(z)\}_{i=0}^n$ be the parameterized Koszul complex of the n-tuple A and $h_i(z)$ $(i = 0, 1, \ldots, n)$ be the cohomology groups of this complex. Let $I = (i_1, i_2, \ldots, i_k)$ be a subset of N with the properties: 1) $i_1 \leq i_2 \leq \ldots i_k$; 2) $i_p - i_{p-1} \geq 2$ for every $p = 2, 3, \ldots, k$; 3) $0 \leq i_p \leq n$ for every $p = 1, \ldots k$.

Definition 2. *We call semi-Fredholm spectrum of n-tuple A corresponding to the set I the set $\sigma^I(A)$ of all points $z \in C^n$ such that or there exists j_0 $(0 \leq j_0 \leq n)$ such that* $\operatorname{Im} d_{j_0}(z)$ *is not closed, either there exists $j_1 \bar{\in} I$ such that* $\dim h_{j_1}(z) = +\infty$.

It is not difficult to prove that $\sigma^I(A)$ is a nonempty compact subset of $\sigma_e(A)$ and

$$\partial\sigma_e(A) \subset \cap_I \sigma^I(A). \tag{7}$$

Let again $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ be n-tuples of commuting operators in $\mathcal{L}(H)$ and $K.(A, z) = \{H_i, d_i(z)\}_{i=0}^n$ and $K.(B, z) = \{H_i, \alpha_i(z)\}_{i=0}^n$ be corresponding Koszul complexes. By $h_i(z)$ and $\chi_i(z)$ $(i = 0, 1, \ldots, n)$ we denote the cohomology groups of these two complexes.

Lemma 3. *If A and B are double quasisimilar n-tuples, then*

$$\sigma_e(A)\backslash\sigma^I(A) \subset \cap_I \sigma^J(B), \tag{8}$$

$$\sigma(B)\backslash \cap_J \sigma^J(B) \subset \sigma(A), \tag{9}$$

where $I \subset N$ and $J \subset N$ are in Definition 2, I is fixed and intersections in (8) and (9) are over all J such that $I \cap J = \emptyset$.

Proof. We prove (8) first. The inclusion (7) implies that the right side of (8) is nonempty. If $\sigma_e(A)\backslash\sigma^I(A) = \emptyset$, then (8) is trivial. That is why there exists $z^0 \in \sigma_e(A)\backslash\sigma^I(A)$. Then $\mathrm{Im} d_i(z^0)$ is closed for every $i = 0, 1, \ldots, n$ and $h_p(z^0)$ is finite dimensional for every $p \bar{\in} I$ but since $z^0 \in \sigma_e(A)$, there is r $(0 \le r \le n)$ such that $\dim h_r(z^0) = +\infty$. It is clear that $r \in I$. Lemma 1 implies that $\dim \chi_p(z^0) < +\infty$ for every $p \bar{\in} I$ and $\dim \chi_r(z^0) = +\infty$. If J is an arbitrary subset of N such that $I \cap J = \emptyset$, then $z^0 \in \sigma^J(B)$ by Definition 2. Thus, (8) is proved.

Let us prove (9). By an argument like above, we suppose that the left side of (9) is nonempty. Let $z^1 \in \sigma(B)\backslash \cap_J \sigma^J(B)$, where the intersection is over all $J \subset N$ like in definition 2 such that $I \cap J = \emptyset$. If $z^1 \in \sigma_F(B)$, then there exists i_0 $(0 \le i_0 \le n)$ such that $\chi_{i_0}(z^1) \ne \{0\}$ By Lemma 1 we have $h_{i_0}(z^1) \ne \{0\}$, hence $z^1 \in \sigma(A)$. If $z^1 \in \sigma_e(B)\backslash \cap_J \sigma^J(B)$, then $\mathrm{Im}\alpha_i(z^1)$ in closed for every $i = 0, 1, \ldots, n$ and there exists r $(0 \le r \le n)$ such that $\dim \chi_r(z^1) = +\infty$. By Lemma 1 again we have $\dim h_r(z^1) = +\infty$, i.e. $z^1 \in \sigma(A)$. The lemma is proved.

We need the following lemma which is a version of Lemma 4.3 in [8].

Lemma 4. *Let $X.(z) = \{X_i, d_i(z)\}_{i=0}^n$ be a complex of Hilbert spaces and $d_i(z)$ are analytic functions in a neighbourhood of $z^0 \in C^n$. Let $\mathrm{Im} d_i(z^0)$ be closed for every $i = 0, 1, \ldots, n$ and $h_i(z)$ be the cohomology groups of the complex $X.(z)$. Then in a sufficiently small neighbourhood of z^0 there exists a complex of Hilbert space $L.(z) = \{L_i, a_i(z)\}_{i=0}^n$ with* $\dim L_i = \dim h_i(z^0)$ *for every $i = 0, 1, \ldots, n$, where $a_i(z)$ are analytic operator-valued functions and there exists an analytic quasi-isomorphism of complexes $\phi.(z) :\ L(z) \to X.(z)$.*

The proof of Lemma 4 is very similar to this one given in [8], so we omit it.

The following theorem is a generalization of Theorem 2.1 in [5].

Theorem 1. *If $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ are double quasisimilar n-tuples of commuting operators in $\mathcal{L}(H)$, then*

$$\sigma^I(A) \cap (\bigcap_J \sigma^J(B)) \ne \emptyset, \tag{10}$$

where $I \subset N$ and $J \subset N$ are in Definition 2, I is a fixed and intersection is over all J such that $I \cap J = \emptyset$.

Proof. For the sake of convenience we put

$$M = \sigma_e(A)\backslash\sigma^I(A), \; N = \bigcap_{\substack{J \\ I\cap J=\emptyset}} \sigma^J(B).$$

It is clear by Lemma 3 that $N \neq \emptyset$.

We consider some cases.

I. $M \neq \emptyset$. Since M is open but N is closed the inclusion (8) gives us $\partial M \subset N$. If $z \in \partial M$, then for every neighbourhood U_z of z we have $U_z \cap M \neq \emptyset$. Hence $\partial M \subset \sigma_e(A)$.

Let $z^0 \in \partial M$, i.e. $z^0 \bar{\in} M$ and $z^0 \in \sigma_e(A)$. Hence $z^0 \in \sigma^I(A)$. But $\partial M \subset N$ and thus $z^0 \in \sigma^I(A) \cap N$. This case is completed.

II. $M = \emptyset$. In this case $\sigma_e(A) = \sigma^I(A)$. We suppose that (10) is wrong, i.e. through this case we suppose that $\sigma^I(A) \cap N = \emptyset$.

1. Let first $\sigma_F(A) = \emptyset$. Now $\sigma(A) = \sigma_e(A) = \sigma^I(A)$. We show that N is a closed and open subset of $\sigma(B)$. It is enough to be proved that N is open. If N is not open there exists a sequence $\{z^m\}_{m=1}^{\infty} \subset \sigma(B)\backslash N$ such that $z^m \overrightarrow{m} \infty\, z^0 \in N$. By (9) we have $\{z^m\} \subset \sigma(A)$, so that $z^0 \in \sigma(A) = \sigma^I(A)$, i.e. $z^0 \in \sigma^I(A) \cap N$ which is impossible by assumption. Hence N is a closed and open subset of $\sigma(B)$.

Let $f(z)$ be an analytic function in a neighbourhood of $N \cup \sigma(A)$ and S is a smooth surface around N such that $S \cap \sigma(A) = \emptyset$. We have by (4) that

$$f(A) = \frac{1}{(2\pi i)^n} \int_S f(z)R(A,z) \wedge dz_1 \wedge \ldots \wedge dz_n = 0,$$

and

$$f(B) = \frac{1}{(2\pi i)^n} \int_S f(z)R(B,z) \wedge dz_1 \wedge \ldots \wedge dz_n \neq 0.$$

But (6) implies $0 = f(A)X = Xf(B)$ and $0 = Yf(A) = f(B)Y$, therefore $f(B) = 0$. This contradiction shows that this situation is impossible.

2. Let now $\sigma_F(A) \neq \emptyset$. It is easy to see that there exists a component L_0 of $C^n\backslash\sigma_e(A)$ such that $N \subset L_0$.

Let first $L_0 \subset \sigma(A)$, i.e. $L_0 \subset \sigma_F(A)$. Then for every $z \in L_0$ there is i $(0 \leq i \leq n)$ such that $h_i(z) \neq \{0\}$ and by Lemma 1 we have $z \in \sigma(B)$, hence $\bar{L_0} \subset \sigma(B)$. We denote by P the boundary of unbounded component of $C^n\backslash L_0$. If $\dim \sigma_F(A) < n$, P will be an arbitrary component of the boundary of L_0. Since $P \subset \sigma(A) = \sigma^I(A)$, then $P \cap N = \emptyset$. We show that $P \cap \sigma_e(B) = \emptyset$. It is enough to show that $P \cap (\sigma_e(B)\backslash N) = \emptyset$. Let us assume that there is $z^0 \in P \cap (\sigma_e(B)\backslash N)$. Then there is a subset J_0 of N as in Definition 2 with the property $J_0 \cap I = \emptyset$ and such that $z^0 \in \sigma_e(B)\backslash\sigma^{J_0}(B)$. That is why $\mathrm{Im}\alpha_i(z^0)$ is closed for every $i = 0, 1, \ldots, n$ and for every $j \bar{\in} J_0$ we have $\dim \chi_j(z^0) < +\infty$ but exists $p \in J_0$ such that $\dim \chi_p(z^0) = +\infty$. According to Lemma 4 there is neighbourhood U_{z^0} of the point z^0 such that for every $z \in U_{z^0}$ we have $\dim \chi_j(z) < +\infty$ for every $j \bar{\in} J_0$ and $\dim \chi_p(z) = +\infty$. It follows by Lemma 1 that $\dim h_p(z) = +\infty$ for every $z \in U_{z^0}$. Since $z^0 \in P$, then $U_{z^0} \cap \sigma_F(A) \neq \emptyset$. Then for every $z \in U_{z^0} \cap \sigma_F(A)$ we have $\dim h_p(z) = +\infty$ which is impossible. Thus we proved that $P \cap \sigma_e(B) = \emptyset$. Since $P \subset \sigma(B)$ we have $P \subset \sigma_F(B)$. The set $\sigma_F(B)$ is open but P is compact. That is why there exists an open set $K_0 \subset \sigma_F(B)$ such that $P \subset K_0$. Let Q be the boundary of the unbounded component of $C^n \backslash K_0$. It is clear that $Q \subset \partial K_0$ and hence $Q \cap \bar{L_0} = \infty$. Then $Q \cap N = \infty$. But by (7) we have

$$Q \subset \partial K_0 \subset \partial\sigma_e(B)\partial \bigcap_J \sigma^J(B) \subset \bigcap_{\substack{J \\ I \cap J = \emptyset}} \sigma^J(B) = N$$

We obtained a contradiction, so the situation is impossible.

Let now $L_0 \not\subset \sigma(A)$. It is clear that L_0 intersects $\sigma(A)$ in a finite number of points from $\sigma_F(A)$ only. We have by (9) that there is a smooth surface S around N and $S \cap \sigma(A) = \infty$. Then by (6) the operators

$$T_A = \frac{1}{(2\pi i)^n} \int_S R(A, z) \wedge dz_1 \wedge \ldots \wedge dz_n$$

and

$$T_B = \frac{1}{(2\pi i)^n} \int_S R(B, z) \wedge dz_1 \wedge \ldots \wedge dz_n$$

are double quasisimilar. But T_A is a finite rank operator while T_B is not. This contradiction completes the proof of the theorem.

Remark. In the case of single operators A and B Theorem 1 reduces to the Theorem 2.1 from [5]. In this case, as the remark after Lemma 1 shows, it is not necessary for A and B to be double quasisimilar but similar only.

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