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SELECTIONS OF GRAPHVALUED MAPS

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ABSTRACT. In this article we shall investigate question about importance of convexity at one of the most known theorems of E.Michael about existence of singlevalued continuous selections for manyvalued, see [1].

Introduction.

Theorem M. *For T_2-space X the next conditions are equivalent:*

a) X is paracompact;

b) Any lower semicontinuous (l.s.c.) map from X to Banach space with closed convex values has a singlevalued continuous selection.

One of the possible versions of refuse a convexity in this theorem was proposed by E. Michael in [2], where he introduced the notion of paraconvexity (for definition see sect.1, below). In this article we in fact investigate question about paraconvexity of graphs of continuous functions of one variable.

Let Γ be a class of all such subsets P of Euclidean plane which is a closed connected graph of some continuous function in some orthogonal coordinate system (which depends from P). Let ΓMon and $\Gamma Lip(k)$ be a subclasses of Γ which consist of a graphs of monotone and, accordingly, graphs of Lipshitz (with constant k) functions.

One of the main results of this article is the following theorem.

Theorem 5. *Let $k \geq 0$. Then any l.s.c. map from any paracompact into a Euclidean plane with values in $\Gamma Mon \cup \Gamma Lip(k)$ has a singlevalued continuous selection.*

In section 1 besides the main technical notion of α-paraconvexity we introduce a notion of boundly continuous map which is stronger than lower semicontinuity and weaker than continuity of manyvalued map. Boundly continuity of map F guaranteed an existence of selections in the cases when the index of paraconvexity of $F(x)$, $x \in X$, depends on x. In Michael's result the index of paraconvexity is constant, [2].

In section 2 in fact we show that any element P from $\Gamma Mon \cup \Gamma Lip(k)$ is a α–paraconvex subset of plane with

$$2\alpha = \max\{1 + \sqrt{2}/2; 1 + \sin(\arctan k)\}.$$

In section 3 results about existence of selections are received in the case of fixed coordinate system.

We would remind that manyvalued map $F: X \to Y$ is lower semicontinuous iff for any open U in Y the set

$$F^{-1}(U) = \{x \in X:\ F(x) \cap U \neq \emptyset\}$$

is open in X and that map G: $X \to Y$ is a selection of map F iff $G(x) \subset F(x)$, for any $x \in X$.

1. Let B be a Banach space and $x_1, x_2, \ldots, x_n \in B$. Denote an infimum of set of radius of all balls, which contains all points $x_1, x_2, \ldots, x_n$ by $R(x_1, x_2, \ldots, x_n)$. For a Euclidean plane $R(x, y, z)$ is a radius of a circumscribed circle if triangle $\triangle xyz$ is acuteangle and is a half of its maximal side in opposite case.

Definition 1. *Let $0 \leq \alpha < 1$. Non-empty, closed subset P of Banach space is called α–paraconvex iff for any n and for any points $x_1, x_2, \ldots, x_n$ from $P$and for any point q from convex hall* conv$\{x_1, x_2, \ldots, x_n\}$ *of these points the next inequality is true*

$$\rho(q; P) \leq \alpha R(x_1, x_2, \ldots, x_n),$$

where $\rho(q; P) = \inf\{||q - p||:\ p \in P\}$.

The set $P \subset B$ is called paraconvex iff P is α–paraconvex for some $0 \leq \alpha < 1$. Minimum of α for which the set P is α–paraconvex is called an index of paraconvexity of P.

Theorem M1 [2]. *Let $0 \leq \alpha < 1$. Then any l.s.c. map from any paracompact X into Banach space with α–paraconvex values has a singlevalued continuous selection.*

The uniform restriction of the index of paraconvexity of values is essential. But this restriction may be omitted if we strengthen the lower semicontinuity of a map.

Definition 2. *Manyvalued map F from X into Banach space B is called boundly continuous (b.c.) iff for any ball D with $F^{-1}(D) \neq \emptyset$ the map $x \longmapsto F(x) \cap D$ is continuous on $F^{-1}(D)$.*

Lower semicontinuity is a consequence of boundly continuity, which is a consequence of continuity of map. The inverse implications are false.

Theorem 1. *Any b.c. map F from any paracompact X into Banach space B with paraconvex values has a singlevalued continuous selection.*

Proof. Denote the index of paraconvexity of set $F(x)$, $x \in X$, by $\alpha(x)$. It suffice to show that function from X to $[0,1)$ defined by

$$x \longmapsto \alpha(x)$$

is lower semicontinuous. In fact, if it's true, then

$$X = \cup_{n=1}^{\infty} X_n; \; X_1 \subset X_2 \subset X_3 \subset \dots$$

where $X_n = \{x \in X\colon \alpha(x) \leq 1-(\frac{1}{n})\}$ – are the closed subsets of X, nonempty beginning from some n.

The application of theorem M1 to pair's $X_n \subset X_{n+1}$ gives a way to construct a selection $f\colon X \to B$ by extensions f from X_n to X_{n+1}.

Let $x_0 \in X$, $\alpha(x_0) = \alpha$ and $a < \alpha$. We need to find the neighbourhood $V(x_0)$ in which $a < \alpha(x)$, $x \in V(x_0)$.

If $a < 0$, then $V(x_0) = X$. For $a > 0$ choose $a < b < \alpha$. By definition of the index of paraconvexity, the set $F(x_0)$ isn't b–paraconvex, i.e. for some $x_1, \dots, x_n \in F(x_0)$ and for some $q \in \operatorname{conv}\{x_1, \dots, x_n\}$ we have

$$\rho(q; F(x_0)) > bR, \text{ where } R = R(x_1, \dots, x_n).$$

Let V_k be a disjoint δ–neighbourhoods of points x_k and $y_k \in V_k$, $k = 1, 2, \dots, n$. Then for any $\varepsilon > 0$ the ball of radius $R + \varepsilon + \delta$ contains all points y_k and if $\varepsilon \to 0$ then we have

$$R' = R(y_1, \dots, y_n) \leq R + \delta.$$

Further, for any $\varepsilon > 0$ does exist the ball with radius $R' + \varepsilon$, containing all points y_k and the concentric ball with radius $R' + \varepsilon + \delta$ containing all points x_k, i.e. $R \leq R' + \varepsilon + \delta$. If $\varepsilon \to 0$, then we have $|R - R'| < \delta$.

Let $D = D(q; 2R)$ be a ball with centre q and radius $2R$. Then by condition of boundly continuity the map $x \longmapsto F(x) \cap D$ is continuous and x_0 lies at the interior of set $F^{-1}(D)$.

Therefore we may find such a neighbourhood $V(x_0)$ that for any x from $V(x_0)$:

1) $F(x) \cap V_k \neq \emptyset$, $k = 1, 2, \dots, n$;

2) $F(x) \cap D \subset D \setminus D(q; bR)$.

For $x \in V(x_0)$ pick $y_k \in F(x_0) \cap V_k$ and let $q' \in \operatorname{conv}\{y_1, \dots, y_n\}$ and q' have the same coordinates like the coordinates of $q \in \operatorname{conv}\{x_1, \dots, x_n\}$ relatively $x_1, \dots, x_n$. Then $||q - q'|| < \delta$. For any $z \in F(x) \cap D$ we have

$$||z - q'|| \geq ||z - q|| - ||q - q'|| > bR - \delta \geq bR' - \delta(b+1) > bR' - 2\delta.$$

To complete the proof we need to choose δ in such a way, that $bR' - 2\delta > aR'$ or $\delta < R'(b-a)/2$. Since $R - \delta \leq R'$ the last inequality is consequence of

$$\delta < (R - \delta)(b - a)/2 \text{ or } \delta < R(b-a)/(2 + b - a).$$

Finally, the ball with centre $q' \in \text{conv}\{y_1, \ldots, y_n\}$ and radius aR' doesn't intersect the set $F(x) \cap D$ and, consequently, doesn't intersect the set $F(x)$. Therefore the set $F(x)$ is not a–paraconvex, i.e. $\alpha(x) > a$ for all $x \in V(x_0)$.

For $a = 0$ the proof is analogous with the following difference. Pick $\alpha > b_1 > b_2 > a = 0$ and construct the ball with radius $b_2 R'$ which doesn't intersect the set $F(x)$.

3. We begin from a weak version of paraconvexity.

Definition 3. *Closed nonempty subset P of Banach space is called a $1-\text{dim}$–paraconvex iff for some $0 \leq \alpha < 1$ and for any $x, y \in P$ we have*

$$\rho((x+y)/2; P) \leq \alpha R(x; y).$$

If a set is paraconvex, then it is a $1 - dim$–paraconvex. The inverse implication is false. For example, the boundary of equilateral triangle is a $1 - dim$–paraconvex set with coefficient $\alpha = \sin 60^0$, but it isn't a paraconvex set (choose a inscribed circle).

But it was found that for closed connected graphs of functions, i.e. for $P \in \Gamma$, the notion of $1 - dim$–paraconvexity coincides with the notion of paraconvexity.

Lemma 2. *If P is a $1 - dim$-paraconvex subset of Euclidean plane with coefficient α and $P \in \Gamma$, then P is a paraconvex set with coefficient $(1+\alpha)/2$.*

Proof. 1) Let $x, y \in P$, $z = (x+y)/2$ and $q \in [x, y]$, $R = R(x; y) = ||x-y||/2$. If $||q - z|| \leq R(1-\alpha)/2$, then

$$\rho(q, ; P) \leq \rho(z; P) + ||q - z|| \leq R(1+\alpha)/2.$$

If $||\rho(q - z|| > R(1-\alpha)/2$, then

$$||q - x|| < R(1+\alpha)/2 \text{ or } ||q - y|| < R(1+\alpha)/2.$$

2) For $P \in \Gamma$ we fix an orthogonal coordinate system in which P is a graph of some continuous function. Let $x, y, z \in P$. We may assume that x lies into left–hand from y, and y lies into left–hand from z. Through point q lying in the triangle Δxyz draw a line $l(q)$ parallel to line xz. Let q_x, q_y, q_z be the points of intersection of $l(q)$ with the vertical lines, passing through x, y and z. From continuity we receive that

$$P \cap (q_x, q_y) \neq \emptyset \text{ and } P \cap (q_y, q_z) \neq \emptyset.$$

We may assume that q lies in "between" points x and y. If there exists a point $x' \in P \cap (q_x, q_y)$ for which $q \in [x'; q_y)$, then we may draw the line $m(q)$ parallel to the line xy. This line separates points x and x' and points y and x'.

From continuity we may find the segment $[x''; y'']$ with $x'', y'' \in P$ and $q \in [x''; y'']$.

3) Let $x_1, \ldots x_n \in P$ and $q \in \operatorname{conv}\{x_1, \ldots, x_n\}$. Then q lies in some triangle T and from 2) we have

$$\rho(q; P) \leq R(T)(1+\alpha)/2 \leq R(x_1, \ldots, x_n)(1+\alpha)/2.$$

For higher dimensions the next generalization of lemma 2 may be proved.

Theorem 3. *Let V be a convex subset of Euclidean space R^n, $f: V \to R$ be a continuous function, $x_1, \ldots, x_{n+2}$ be any points of the graph Γ_f of function f and q lies in* $\operatorname{conv}\{x_1, \ldots, x_n\}$. *Then there exits a simplex Δ such that:*

- $q \in \Delta$;
- $\dim \Delta = n$;
- *all vertexes of Δ lies on Γ_f;*
- $R(\Delta) \leq R\{x_1, \ldots, x_{n+2}\}$.

We return to the case of subsets of plane.

Lemma 4. *1) If $P \in \Gamma Lip(k)$, then P is α-paraconvex with*

$$2\alpha = 1 + \sin(\arctan k).$$

2) If $P \in \Gamma Mon$, then P is α-paraconvex with

$$2\alpha = 1 + \sqrt{2}/2.$$

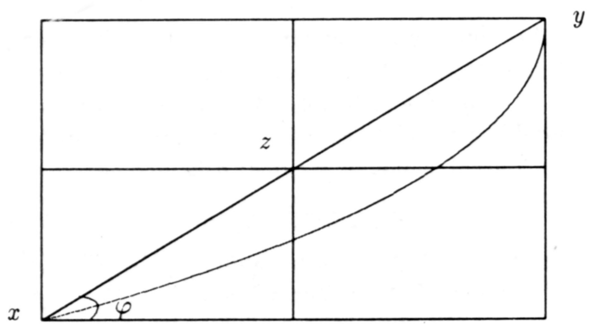
Proof. 1) By lemma 2 it is suffice to see that set P is $1-$dim–paraconvex with coefficient $\sin(\arctan k)$. Through points $x, y \in P$ draw lines with slope k and $-k$ (relatively the coordinate system for which P is graph of some Liptshitz function). "Between" x and y the set P lies in the parallelogram, which we obtained.

Through the point $z = (x+y)/2$ drop perpendiculars on sides of this parallelogram with slope k ($-k$) if the point y lies above and to the right of point x (y lies to the right and below of x). The lengths of this perpendiculars are equal to $R(x; y)\sin\beta$, where $0 \leq \beta \leq \sin(\arctan k)$ and the set P intersects one of these perpendiculars.

2) By lemma 2 it suffice to see that the set P is $1-$dim–paraconvex with coefficient $\sqrt{2}/2$.

But it's practically obvious.

The length of one of the sides of a right–angled triangle with a hypotenuse $R = R(x; y)$ is less or equal than $(\sqrt{2}/2)R$ and from continuity and monotony of a function it's easy to see that set P intersects both such sides.



In fact it may be proved that the set P in this case is exactly $(\sqrt{2}/2)$–paraconvex. The example of the functions x^n shows that $\sqrt{2}/2$ is an exact constant of paraconvexity.

From lemma 4 and the above mentioned theorem M1 we obtain the following theorem.

Theorem 5. *Let $k \geq 0$. Then any l.s.c. map from any paracompact to Euclidean space with values in $\Gamma Mon \cap \Gamma Lip(k)$ has a singlevalued continuous selection.*

Theorem 5 is false for graphs of all Lipchitz functions with all possible constants. Indeed, even for a fixed coordinate system the set of that graph isn't a equi-LC^0. But in this situation we may use a notion of *b.c.* map and theorem 1.

Theorem 6. *Let $F: X \to R^2$ be a boundly continuous map from paracompact X and for any $x \in X$*

$$F(X) \in \Gamma Mon \cup \Gamma Lip, \;\; \Gamma Lip = \cup_{n=0}^{\infty} \Gamma Lip(n).$$

Then F has a singlevalued continuous selection.

4. For a fixed coordinate system the situation is easier. We may identify a function with its graph and in this case the the paraconvexity may be omitted.

For Banach spaces B, E and constant $k \geq 0$ we define $\Gamma Lip(B, E; k)$ – the class of all graphs of all Lipchitz functions from B into E with constant k and with closed convex domains of definition.

Theorem 7. *Let $k \geq 0$ and $F: X \to B \oplus E$ be a l.s.c. map from paracompact X into a Decart's sum of Banach spaces B and E. Let $F(X) \in \Gamma Lip(B, E; k)$ for all $x \in X$. Then F has a singlevalued continuous selection.*

Proof. Let $p: B \oplus E \to B$ be a projection on the first item and $G = p \circ F$. Then $G: X \to B$ is a *l.s.c.* map with closed convex values which are domains of definition of elements of $\Gamma Lip(B, E; k)$. By the well known theorem M of E. Michael the map G

has a continuous selection $g: X \to B$, $g(x) \in G(x)$. Define a singlevalued selection f for map F:

$$f(x) = F(x) \cap p^{-1}(g(x)),\ x \in X.$$

If we identify the set $F(x)$ with the function whose graph is $F(x)$, then we have

$$f(x) = (g(x); F(x)[g(x)]).$$

Briefly, we lift an element $g(x) \in B$. The continuity of f can be checked in a standard manner.

Let $D_1 = D(g(x); \varepsilon/4k)$ be an open ball with centre $g(x)$ and radius $\varepsilon/4k$, $D_2 = D(F(x)[g(x)];\ \varepsilon/2)$ and $D = D_1 \oplus D_2$ be a neighbourhood of point $f(x)$. Choose a neighborhood U of point $x: U = F^{-1}(D) \cap g^{-1}(D_1)$.

If $y \in U$, then

$$\text{(1)} \qquad ||f(x) - f(y)|| = \max\{||g(x) - g(y)||, ||F(x)[g(x)] - F(y)[g(y)]||\}$$

and $||g(x) - g(y)|| < \varepsilon/4k$. Let estimate the second part.

Choose an element $b \in B$ so that $(b; F(y)[b]) \in D$. It's possible – the set $F(y)$ intersects the ball D. Then the second part in (1) is less or equal than

$$||F(y)[b] - F(y)[g(y)]|| + ||F(x)[g(x)] - F(y)[b]|| \leq k||b - g(y)|| + \frac{\varepsilon}{2} < \varepsilon.$$

In the last inequalities we used that b and $g(y)$ lie in ball D_1 and that $F(x)[g(x)]$ and $F(y)[b]$ lies in ball D_2.

The Lipshitz condition at this theorem isn't a necessary condition. The theorem remains true for equicontinuous set of functions $\{F(x)\}$, $x \in X$. For $\dim B < \infty$ and $\dim E < \infty$ and for perfectly normal X the domains of definition of functions may be convex only.

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