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INFINITESIMAL BENDING OF HIGHER ORDER OF ROTATIONAL SURFACES WITH A PLANAR POLE

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ABSTRACT. The present paper is devoted to a study of the connection between the order of flattening of the pole of a rotational surface and the numbers of the regular fundamental fields of infinitesimal bendings of the 1-st order of the surface, which can be extended to regular fields of infinitesimal bendings of the higher order in a neighbourhood of the pole.

1. Introduction. In this paper we investigate the behaviour of the fields of infinitesimal bendings (inf.b.) of higher order in a neighbourhood of a planar pole of a rotational surface. N. V. Efimov was the first to discover [2] the possibility for a rigidity "in the small" of an analytical surface in a neighbourhood of a planar point with respect to relatively analytical inf. b. Complete references for later investigations on inf. b. of surfaces with a planar point can be found in [9,12].

2. Equations of infinitesimal bending of order m. Let in the space $\mathbb{R}^3$ a coordinate system with orths e_1, e_2, e_3 be introduced, and let the surface S be obtained by rotation round the $0z$ axis of the plane curve

$$(1) \qquad L:\ \rho = \rho(z),\ z \in [0,1],\ \rho(z) \in C^q(0,1),\ q \geq 2,\ \rho'(0) = \infty,$$

where ρ is the radius of the corresponding rotational parallel, and $z = 0$ and $z = 1$ are the poles of the surface S. Let us suppose that in a neighbourhood of $z = 0$ the meridian L has a representation

$$(2) \qquad z = \rho^n f_1(z),$$

where $n \geq 2$, $f_1(0) \neq 0$, $f_1(\rho) \in C^A[0,\varepsilon)$, $n - 1 \geq q \geq 2$ when n is an odd integer and $[n] \geq q \geq 2$ for every other n. Then in a neighbourhood of the pole $z = 0$ we have

$$(2) \qquad \rho(z) = z^{n_1}\rho_1(z), \rho_1(0) \neq 0,\ \rho_1(z) \in C^A(0,\varepsilon),\ n_1 = \frac{1}{n} \leq \frac{1}{2}.$$

Let us remark that in the case $n = 2$ the pole $z = 0$ is a nonparabilic point of the surface, i.e. the Gausse curvature $K|_{z=0} > 0$, and for $n > 2$ the pole $z = 0$ is a parabollic point, i.e. $K|_{z=0} = 0$. Moreover when the pole is a parabolic point, then it is a planar point of the surface (the order of flattening of the pole $z = 0$ is $n - 2$).

Let us represent the radius vector of the surface S and its inf. b. S_t of the m-th order in the known way [1].

$$S: \ r(z,\theta) = ze_3 + \rho(z)\mathrm{e}(\theta),$$

$$S_t: \ r(z,\theta,t) = r(z,\theta) + \sum_{j=1}^{m} t^j \overset{j}{z}(z,\theta),$$

$$0 \le z \le 1, \ 0 \le \theta \le 2\pi, \ \mathrm{e}(\theta) = \cos\theta . e_1 + \sin\theta . e_2,$$

where

$$\overset{j}{z}(z,\theta) = \overset{j}{\alpha}(z,\theta).e_3 + \overset{j}{\beta}(z,\theta).\mathrm{e} + \overset{j}{\gamma}(z,\theta).\mathrm{e}', \ j = 1,\dots,m. \tag{3}$$

According to [1] for the fields $\overset{j}{z}$, $j = 1,\dots,m$, the systems of differential equations

$$\overset{1}{\alpha}_z + \rho' \overset{1}{\beta}_z = 0,$$

$$\overset{1}{\beta} + \overset{1}{\gamma}_\theta = 0, \tag{4}$$

$$\overset{1}{\alpha}_\theta + \rho'(\overset{1}{\beta}_\theta - \overset{1}{\gamma}) + \rho \overset{1}{\gamma}_z = 0;$$

$$\overset{j}{\alpha}_z + \rho' \overset{j}{\beta}_z = -\frac{1}{2}\sum_{l=1}^{j-1}(\overset{l}{\alpha}_z \overset{j-l}{\alpha}_z + \overset{l}{\beta}_z \overset{j-l}{\beta}_z + \overset{l}{\gamma}_z \overset{j-l}{\gamma}_z),$$

$$\overset{j}{\gamma}_\theta + \overset{j}{\beta} = -\frac{1}{2\rho}\sum_{l=1}^{j-1}[\overset{l}{\alpha}_\theta \overset{j-l}{\alpha}_\theta + (\overset{l}{\beta}_\theta - \overset{l}{\gamma})(\overset{j-l}{\beta}_\theta - \overset{j-l}{\gamma}) + (\overset{l}{\gamma}_\theta + \overset{l}{\beta})(\overset{j-l}{\gamma}_\theta + \overset{j-l}{\beta})], \tag{5}$$

$$\overset{j}{\alpha}_\theta + \rho'(\overset{j}{\beta}_\theta - \overset{j}{\gamma}) + \rho \overset{j}{\gamma}_z = -\sum_{l=1}^{j-1}[\overset{l}{\alpha}_z \overset{j-l}{\alpha}_\theta + \overset{l}{\beta}_z (\overset{j-l}{\beta}_\theta - \overset{j-l}{\gamma}) + \overset{l}{\gamma}_z (\overset{j-l}{\gamma}_\theta + \overset{j-l}{\beta})],$$

$$j = 2,\dots,m,$$

are satisfied.

Let $\overset{1}{z}_k\,(z,0)$, $k \geq 2$, be a fundamental field [1,5] of inf. b. of the 1-st order of the surface S. Then

$$\overset{1}{\alpha}=\overset{1}{\alpha}_k\,(z)e^{ik\theta}+\overset{1}{\alpha}_{-k}\,(z)e^{-ik\theta},$$

$$\overset{1}{\beta}=\overset{1}{\beta}_k\,(z)e^{ik\theta}+\overset{1}{\beta}_{-k}\,(z)e^{-ik\theta}, \tag{6}$$

$$\overset{1}{\gamma}=\overset{1}{\gamma}_k\,(z)e^{ik\theta}+\overset{1}{\gamma}_{-k}\,(z)e^{-ik\theta}.$$

The fields $\overset{j}{z}\,(z,\theta)$, $j=2,\ldots,m$, are extensions of the field $\overset{1}{z}_k\,(z,\theta)$, and they have [1,11] coordinates

$$\overset{j}{\alpha}\,(z,\theta)=\sum_{h_j=0}^{p_j}[\overset{j}{\alpha}_{(j-2h_j)k}\,(z)e^{(j-2h_j)ki\theta}+\overset{j}{\alpha}_{-(j-2h_j)k}\,(z)e^{-(j-2h_j)ki\theta}],$$

$$\overset{j}{\beta}\,(z,\theta)=\sum_{h_j=0}^{p_j}[\overset{j}{\beta}_{(j-2h_j)k}\,(z)e^{(j-2h_j)ki\theta}+\overset{j}{\beta}_{-(j-2h_j)k}\,(z)e^{-(j-2h_j)ki\theta}], \tag{7}$$

$$\overset{j}{\gamma}\,(z,\theta)=\sum_{h_j=0}^{p_j}[\overset{j}{\gamma}_{(j-2h_j)k}\,(z)e^{(j-2h_j)ki\theta}+\overset{j}{\gamma}_{-(j-2h_j)k}\,(z)e^{-(j-2h_j)ki\theta}],$$

with respect to the moving frame $k, \mathrm{e}, \mathrm{e}'$, where $p_j=\dfrac{j}{2}$ for even j and $p_j=\dfrac{j-1}{2}$ for odd j, $j=2,\ldots,m$.

If we substitude (6) and (7) in (4) and (5), then for the functions $\overset{1}{\alpha}_k\,(z)$, $\overset{1}{\beta}_k\,(z)$, $\overset{1}{\gamma}_k\,(z)$ and $\overset{j}{\alpha}_{(j-2h_j)k}\,(z)$, $\overset{j}{\beta}_{(j-2h_j)k}\,(z)$, $\overset{j}{\gamma}_{(j-2h_j)k}\,(z)$, $j=2,\ldots,m$, $h_j=0,\ldots,p_j$, we get the following systems of differential equations

$$\overset{1}{\alpha}{}'_k\,(z)+\rho'\,\overset{1}{\beta}{}'_k\,(z)=0,$$

$$\overset{1}{\beta}_k\,(z)+ik\,\overset{1}{\gamma}_k\,(z)=0, \tag{8}$$

$$ik\,\overset{1}{\alpha}_k\,(z)+\rho'[\overset{1}{\beta}_k\,(z)-\overset{1}{\gamma}_k\,(z)]+\rho\,\overset{1}{\gamma}{}'_k\,(z)=0;$$

$$\overset{j}{\alpha}{}'_{(j-2h_j)k}\,(z)+\rho'\,\overset{j}{\beta}{}'_{(j-2h_j)k}\,(z)=\overset{j}{R}_{1,(j-2h_j)k}\,(z),$$

$$(9) \qquad \overset{j}{\beta}'_{(j-2h_j)k}(z) + i(j-2h_j)k\,\overset{j}{\gamma}_{(j-2h_j)k}(z) = \overset{j}{R}_{2,(j-2h_j)k}(z),$$

$$i(j-2h_j)k\,\overset{j}{\alpha}_{(j-2h_j)k}(z) + \rho'[i(j-2h_j)k\,\overset{j}{\beta}_{(j-2h_j)k}(z) - \overset{j}{\gamma}_{(j-2h_j)k}(z)] +$$

$$\rho\,\overset{j}{\gamma}'_{(j-2h_j)k}(z) = \overset{j}{R}_{3,(j-2h_j)k}(z),$$

where

$$(10) \qquad \overset{j}{R}_{1,(j-2h_j)k} = -\frac{1}{2}\sum_{l=1}^{j-1}\sum_{(r_l)}(\overset{l}{\alpha}'_{r_l k}\overset{j-l}{\alpha}'_{(j-2h_j-r_l)k} +$$

$$\overset{l}{\beta}'_{r_l k}\overset{j-l}{\beta}'_{(j-2h_j-r_l)k} + \overset{l}{\gamma}'_{r_l k}\overset{j-l}{\gamma}'_{(j-2h_j-r_l)k}),$$

$$(11) \qquad \overset{j}{R}_{2,(j-2h_j)k} = -\frac{1}{2\rho}\sum_{l=1}^{j-1}\sum_{(r_l)}\{-k^2 r_l(j-2h_j-r_l)\,\overset{l}{\alpha}_{r_l k}\overset{j-l}{\alpha}_{(j-2h_j-r_l)k}$$

$$+(ir_l k\,\overset{l}{\beta}_{r_l k} - \overset{l}{\gamma}_{r_l k})[i(j-2h_j-r_l)k\,\overset{j-l}{\beta}_{(j-2h_j-r_l)k} - \overset{j-l}{\gamma}_{j-2h_j-r_l)k}]$$

$$+(ir_l k\,\overset{l}{\gamma}_{r_l k} + \overset{l}{\beta}_{r_l k})[i(j-2h_j-r_l)k\,\overset{j-l}{\gamma}_{(j-2h_j-r_l)k} - \overset{j-l}{\beta}_{j-2h_j-r_l)k}]\}$$

$$(12) \qquad \overset{j}{R}_{3,(j-2h_j)k} = -\sum_{l=1}^{j-1}\sum_{(r_l)}\{i(j-2h_j-r_l)k\,\overset{l}{\alpha}'_{r_l k}\overset{j-l}{\alpha}_{(j-2h_j-r_l)k} +$$

$$\overset{l}{\beta}'_{r_l k}[i(j-2h_j-r_l)k\,\overset{j-l}{\beta}_{(j-2h_j-r_l)k} - \overset{j-l}{\gamma}_{(j-2h_j-r_l)k}] +$$

$$\overset{l}{\gamma}'_{r_l k}[i(j-2h_j-r_l)k\,\overset{j-l}{\gamma}_{(j-2h_j-r_l)k} + \overset{j-l}{\beta}_{(j-2h_j-r_l)k}]\}.$$

Remarks:

1) The summation index r_l in (10)-(12) takes values in the set $\{\pm(l-2h_l),\ h_l = 0,1,\dots,p_l\}$ so that the numbers $j-2h_j-r_l$ belong to the set $\{\pm(j-l-2h_{(j-l)}),\ h_{(j-l)} = 0,1,\dots,p_{(j-l)}\}$; 2) To every number j, $2 \le j \le m$, there correspond p_j+1 systems of equations (9); 3) The functions $\overset{j}{\alpha}_{-(j-2h_j)k}(z)$, $\overset{j}{\beta}_{-(j-2h_j)k}(z)$, $\overset{j}{\gamma}_{-(j-2h_j)k}(z)$, $j = 1,\dots,m$, $h_j = 0,1,\dots,p_j$, $p_j = \dfrac{j}{2}$ for j even and $p_j = \dfrac{j-1}{2}$ for j odd, are conjugate to the functions $\overset{j}{\alpha}_{(j-2h_j)k}(z)$, $\overset{j}{\beta}_{(j-2h_j)k}(z)$, $\overset{j}{\gamma}_{(j-2h_j)k}(z)$ and hence they satisfy systems

of equations which are conjugate to (8) and (9), which need not to be considered; 4) For even j the subscipt $(j-2h_j)k$ in (9) takes the values $0, 2k, 4k, \ldots, jk$, and for odd j the values $k, 3k, 5k, \ldots, jk$.

3. Behaviour of the fields of infinitesimal bendings of order m in a neighbourhood of the pole. A field $\overset{j}{z}$ of inf. b. of the j-th order, $j = 1, \ldots, m$, of the surface S is called regular, if it belongs to the class C^q, $q \geq 2$, out of the poles and if it is continuous on the whole surface. If the surface S has regular fields $\overset{2}{z}, \ldots, \overset{j}{z}$ of inf. b. of the 2-nd, ..., j-th order correspondingly, which are extensions of a non-trivial regular field $\overset{1}{z}$ of the 1-st order ($\overset{1}{z}$ is trivial when $\overset{1}{z} = \Omega \wedge r + \omega$ with constant vectors Ω and ω), then it is called nonrigid of order j.

Let $\overset{1}{z}_k$, $k \geq 2$, be a regular fundamental field of inf. b. of the 1-st order in a neighbourhood of the pole $z = 0$ of S. We are looking for its extension into a regular field $\overset{m}{z}$ of inf. b. of m-th order, where m is an arbitrary positive integer. For $m = 2$ and $m = 3$ this problem is studied in [5] and [6] correspondingly. Starting from the results obtained there, we shall solve inductively the problem stated.

Let us remark that: 1) if the field $\overset{1}{z}_k$ belongs to the class C^1 out of the poles, then it belongs there to the class C^q, $q \geq 2$ [7]; 2) here a neighbourhood of the pole $z = 0$ means any part S_0 of the surface S, which contains the pole $z = 0$ and which is restricted by a parallel.

From the regularity of the field $\overset{1}{z}_k$, $k \geq 2$, it follows that in a neighbourhood of the pole $z = 0$ the equalities

$$
\begin{gathered}
\overset{1}{\alpha}_k(z) = z^{\frac{1}{2}[2n_1 - 1 + \mu_k(n_1)]} \, \overset{1}{\alpha^0}_k(z), \\
\overset{1}{\beta}_k(z) = z^{\frac{1}{2}[1 + \mu_k(n_1)]} \, \overset{1}{\beta^0}_k(z), \qquad (13) \\
\overset{1}{\gamma}_k(z) = z^{\frac{1}{2}[1 + \mu_k(n_1)]} \, \overset{1}{\gamma^0}_k(z), \\
\overset{1}{\alpha^0}_k(0) \neq 0, \quad \overset{1}{\beta^0}_k(0) \neq 0, \quad \overset{1}{\gamma^0}_k(0) \neq 0, \quad n_1 \in (0, \frac{1}{2}],
\end{gathered}
$$

are valid [3,7], where

$$\mu_k(n_1) = \sqrt{1 + 4n_1(1 - n_1)(k^2 - 1)}.legno(14)$$

Let the regular fields $\overset{s}{z}$, $s = 2, \ldots, j-1$, of inf. b. of the order $s = 2, \ldots, j-1$, which are extensions of $\overset{1}{z}_k$, exist. We shall suppose that when s is even, $2 \leq s \leq j-1$, the integration constants $\overset{s}{c}_1$, $\overset{s}{c}_2$, in $\overset{s}{\gamma}_0$ and $\overset{s}{\alpha}_0$ (see [5,6] for $s = 2, 3$) are equal to zero (it always can be achieved by addition of a trivial component of the inf. b. of order s [4,10]).

We shall suppose that in a neighbourhood og the pole $z = 0$ the equalities

$$\overset{s}{\alpha}_{(s-2h_s)k}(z) = z^{\frac{1}{2}[s(2n_1-3+\mu_k(n_1))+2]}\,\overset{s}{\alpha^0}_{(s+2h_s)k}(z),$$

(15) $$\overset{s}{\beta}_{(s-2h_s)k}(z) = z^{\frac{1}{2}[s(2n_1-3+\mu_k(n_1))+4-2n_1]}\,\overset{s}{\beta^0}_{(s-2h_s)k}(z),$$

$$\overset{s}{\gamma}_{(s-2h_s)k}(z) = z^{\frac{1}{2}[s(2n_1-3+\mu_k(n_1))+4-2n_1]}\,\overset{s}{\gamma^0}_{(s-2h_s)k}(z),$$

$$s = 1,\ldots,j-1, \quad h_s = 0,1,\ldots,p_s$$

are valid (for $s = 2$ and $s = 3$ these equalities are proved in [5] and [6] respectively).

Evidently the problem of finding a regular field $\overset{j}{z}_k$ of inf. b. of the j-th order in a neighbourhood of the pole $z = 0$ of the surface S, as an extention of the regular fields $\overset{1}{z}_k, \overset{2}{z}_k, \ldots \overset{j-1}{z}_k$ leads to the problem of finding a regular solution of the system (9) in $[0, z_0]$, $z_0 \in (0,1)$.

3.1 Let $j - 2h_j \neq 0$. Eliminating $\overset{j}{\alpha}_{(j-2h_j)k}$ and $\overset{j}{\gamma}_{(j-2h_j)k}$ from (9) we obtain for $\overset{j}{\beta}_{(j-2h_j)k}$ the equation

(16) $$\rho(z)\,\overset{j}{\beta''}_{(j-2h_j)k}(z) + \rho''(z)[(j-2h_j)^2k^2 - 1]\,\overset{j}{\beta}_{(j-2h_j)k}(z) = \overset{j}{R}_{(j-2h_j)k}(z),$$

where

$$\overset{j}{R}_{(j-2h_j)k} = -\rho''\,\overset{j}{R}_{2,(j-2h_j)k} + \rho\,\overset{j}{R''}_{2,(j-2h_j)k} - (j-2h_j)^2k^2\,\overset{j}{R}_{1,(j-2h_j)k}$$

$$-i(j-2h_j)k\,\overset{j}{R'}_{3,(j-2h_j)k}, \quad h_j = 0,\ldots,p_j.$$

Thus the way to solve the formulated problem is the following: first we look for a solution $\overset{j}{\beta}_{(j-2h_j)k}(z)$ of the equation (16) and after that by its help we determine $\overset{j}{\gamma}_{(j-2h_j)k}(z)$ and $\overset{j}{\alpha}_{(j-2h_j)k}(z)$ from (9_2) and (9_3).

We shall consider the case when $h_j = 0$ (the other cases are investigated similarly).

From the equalities (10)-(12), (17), (13), (15) in a neighbourhood of the pole $z = 0$ we obtain

(18) $$\overset{j}{R}_{1,jk}(z) = z^{\frac{1}{2}[2n_1-3+\mu_k(n_1)]}\,\overset{j}{R^0}_{1,jk}(z),$$

$$\overset{j}{R}_{2,jk}(z) = z^{\frac{1}{2}[j(2n_1-3+\mu_k(n_1))+4-2n_1]} \overset{j}{R^0}_{2,jk}(z),$$

$$\overset{j}{R}_{3,jk}(z) = z^{\frac{1}{2}[j(2n_1-3+\mu_k(n_1))+2]} \overset{j}{R^0}_{3,jk}(z);$$

$$\overset{j}{R}_{jk}(z) = z^{\frac{1}{2}[2n_1-3+\mu_k(n_1)]} \overset{j}{R^0}_{jk}(z). \tag{19}$$

Moreover

$$\overset{j}{R}_{1,jk}(0) = -\frac{1}{8}\sum_{l=1}^{j-1}\Big\{[l(2n_1-3+\mu_k(n_1))+2]$$

$$[(j-l)(2n_1-3+\mu_k(n_1))+2]\, \overset{l}{\alpha^0}_{lk}(0)\, \overset{j-l}{\alpha^0}_{(j-l)k}(0)\Big\},$$

$$\overset{j}{R^0}_{2,jk}(0) = \frac{k^2}{2\rho_1(0)}\sum_{l=1}^{j-1} l(j-l)\, \overset{l}{\alpha^0}_{lk}\overset{j-l}{\alpha^0}_{(j-l)k}(0), \tag{20}$$

$$\overset{j}{R^0}_{3,jk}(0) = -\frac{ik}{2}\sum_{l=1}^{j-1}[l(2n_1-3+\mu_k(n_1))+2](j-l)\, \overset{l}{\alpha^0}_{lk}(0)\, \overset{j-l}{\alpha^0}_{(j-l)k}(0);$$

$$\overset{j}{R^0}_{jk}(0) = \frac{k^2}{4}\sum_{l=1}^{j-1}\Big\{\{j(2n_1-3+\mu_k(n_1))[jl-(j-l)(j-l+2ln_1)] \tag{21}$$

$$+2j^2-4(j-l)(j-l+ln_1)\}\, \overset{l}{\alpha^0}_{lk}(0)\, \overset{j-l}{\alpha^0}_{(j-l)k}(0)\Big\}.$$

Let $\beta^+_{jk}(z)$ and $\beta^-_{jk}(z)$ be fundamental solutions of the homogeneous equation (16) (for $h_j = 0$) in $(0,1)$, where $\beta^+_{jk}(z)$ is regular and $\beta^-_{jk}(z)$ is nonregular in $z=0$. Then [7] in a neighbourhood of $z=0$

$$\beta^{\pm}_{jk}(z) = z^{\frac{1}{2}[1\pm\mu_{jk}(n_1)]}\beta^{0\pm}_{jk}(z),\quad \beta^{0\pm}_{jk}(0)\neq 0,\quad n_1\in(0,\frac{1}{2}], \tag{22}$$

and

$$W_{jk}(z) = \beta^{+'}_{jk}(z)\beta^-_{jk}(z) - \beta^+_{jk}(z)\beta^{-'}_{jk}(z)\neq 0,\ z\in[0,1). \tag{23}$$

With the Lagrange's method we build the solution

$$\overset{j}{\beta}_{jk}(z) = \beta^+_{jk}(z)(\overset{j}{c^1}_{jk} + \int_{z_0}^{z}\overset{j}{D^-}_{jk}(\tau)d\tau) + \beta^-_{jk}(z)(\overset{j}{c^2}_{jk} - \int_{z_0}^{z}\overset{j}{D^+}_{jk}(\tau)d\tau), \tag{24}$$

$$\overset{j}{c^i}_{jk}= \text{const},\quad i=1,2,\quad z_0\in(0,1),\quad \overset{j}{D^{\pm}}_{jk}(z)=\frac{\overset{j}{R}_{jk}(z)}{\rho(z)W_{jk}(z)}\overset{j}{\beta^{\pm}}_{jk}(z),$$

of the equation (16) (for $h_j=0$) in $(0,1)$.

From (19), (22) and (23) in a neighbourhood of $z=0$ we obtain

$$\overset{j}{D^{\pm}}_{jk}(z)=z^{\frac{1}{2}[j(2n_1-3+\mu_k(n_1))-2n_1+1\pm\mu_{jk}(n_1)]}\,\overset{j}{D^{0\pm}}_{jk}(z), \tag{25}$$

where

$$\overset{j}{D^{\pm}}_{jk}(0)=\frac{\overset{j}{R^0}_{jk}(0)\beta^{0\pm}_{jk}(0)}{\rho_1(0)W^0_{jk}(0)},\quad W^0_{jk}(0)=\beta^{0+}_{jk}(0)\beta^{0-}_{jk}(0)\mu_{jk}(n_1).$$

From (24) and (25) it follows that the solution $\overset{j}{\beta}_{jk}(z)\to 0$ when $z\to 0$ if and only if $\overset{j}{c^2}_{jk}=\int_{z_0}^{0}\overset{j}{D^+}_{jk}(\tau)d\tau$. Let us remark that this integral exists since

$$\frac{1}{2}[j(2n_1-3+\mu_k(n_1))-2n_1+1+\mu_{jk}(n_1)]>\frac{1}{2}[(j-1)(2n_1-3+\mu_k(n_1))+2],$$

and from (15) (because of the regularity of the field $\overset{j-1}{z}$) we have

$$(j-1)(2n_1-3+\mu_k(n_1))+2>0, \tag{26}$$

i.e.

$$k>\frac{1}{j-1}\sqrt{\frac{(j-2)[2(j-1)(1-n_1)-1]}{n_1(1-n_1)}}.$$

Thus the solution

$$\overset{j}{\beta}_{jk}(z)=\beta^+_{jk}(z)(\overset{j}{c^1}_{jk}+\int_{z_0}^{z}\frac{\overset{j}{R}_{jk}(\tau)}{\rho(\tau)W_{jk}(\tau)}\beta^-_{jk}(\tau)d\tau)-$$

$$\beta^-_{jk}(z)\int_0^z\frac{\overset{j}{R}_{jk}(\tau)}{\rho(\tau)W_{jk}(\tau)}\beta^+_{jk}(\tau)d\tau, \tag{27}$$

$\overset{j}{c^1}_{jk}=$ const, $z_0\in(0,1)$, of the equation (16) in $(0,1)$ has the form

$$\overset{j}{\beta}_{jk}(z)=z^{\frac{1}{2}[j(2n_1-3+\mu_k(n_1))+4-2n_1]}\,\overset{j}{\beta^0}_{jk}(z) \tag{28}$$

in a neighbourhood of the pole $z = 0$, where

(29) $$\overset{j}{\beta}{}^0{}_{jk}\,(0) = \frac{\overset{j}{R}{}^0{}_{jk}\,(0)}{\rho_1(0)(\overset{j}{\delta}_1 + 1)(\overset{j}{\delta}_2 + 1)},$$

(29′) $$\overset{j}{\delta}_{1,2} = \frac{1}{2}[j(2n_1 - 3 + \mu_k(n_1)) - 2n_1 + 1 \pm \mu_{jk}(n_1)].$$

From the equation (9_2) and (9_3) for $f_{ij} = 0$ we have

(30) $$\overset{j}{\gamma}_{jk}\,(z) = \frac{1}{jki}[\overset{j}{R}_{2,jk}\,(z) - \overset{j}{\beta}_{jk}\,(z)],$$

$$\overset{j}{\alpha}_{jk}\,(z) = \frac{1}{jki}\{\overset{j}{R}_{3,jk}\,(z) - \rho(z)\,\overset{j}{\gamma}{}'_{jk}\,(z) - \rho'(z)[jki\,\overset{j}{\beta}_{jk}\,(z) - \overset{j}{\gamma}_{jk}\,(z)]\}.$$

Then with the help of (18) and (28) in a neighbourhood of $z = 0$ we obtain

(31) $$\overset{j}{\gamma}_{jk}\,(z) = z^{\frac{1}{2}[j(2n_1-3+\mu_k(n_1))+4-2n_1]}\,\overset{j}{\gamma}{}^0{}_{jk}\,(z),$$

where

(32) $$\overset{j}{\gamma}{}^0{}_{jk}\,(0) = \frac{k}{4i\rho_1(0)(\overset{j}{\delta}_1 + 1)(\overset{j}{\delta}_2 + 1)}\sum_{l=1}^{j-1}\Big\{4(j-l)[lj(n_1-1)+1] - 2j$$

$$+(2n_1 - 3 + \mu_k(n_1))[(j-l)(j + 2jln_1 - 3jl + 2l) - jl]\Big\}\,\overset{l}{\alpha}{}^0{}_{lk}\,(0)\,\overset{j-l}{\alpha}{}^0{}_{(j-l)k}\,(0)$$

and

(33) $$\overset{j}{\alpha}_{jk}\,(z) = z^{\frac{1}{2}[j(2n_1-3+\mu_k(n_1))+2]}\,\overset{j}{\alpha}{}^0{}_{jk}\,(z),$$

where

(34) $$\overset{j}{\alpha}{}^0{}_{jk}\,(0) = \frac{1}{4(\overset{j}{\delta}_1 + 1)(\overset{j}{\delta}_2 + 1)}\sum_{l=1}^{j-1}\overset{j}{\theta}_l\,(n_1)\,\overset{l}{\alpha}{}^0{}_{lk}\,(0)\,\overset{j-l}{\alpha}{}^0{}_{(j-l)k}\,(0),$$

(35) $$\overset{j}{\theta}_l\,(n_1) = (2n_1 - 3 + \mu_k(n_1))\{2(j-l)l(j-1)(3-2n_1)(1-n_1) + j(j-1)(2n_1-3)$$
$$+lj(j-l) + k^2jn_1[j(2l-j) + l(j-l)(1-2n_1)]\}$$

$$+4(j-l)l(j-1)(3-2n_1)(1-n_1)+4l(j-l)(1-n_1)+4(1-n_1)(1-j^2)$$
$$+2k^2n_1[(j-l)l(1-n_1)(2n_1-3)(j-2)+j(2-n_1)(2l-j)].$$

3.2. Let $j-2h_j=0$. In this case j is even and $h_j=p_j=\frac{j}{2}$. Now the system (9) gets the form

$$(9') \qquad \begin{aligned} &\overset{j}{\alpha}{}'_0(z)+\rho'\,\overset{j}{\beta}{}'_0(z)=\overset{j}{R}_{1,0}(z),\\ &\overset{j}{\beta}_0(z)=\overset{j}{R}_{2,0}(z),\\ &\rho(z)\,\overset{j}{\gamma}{}'_0(z)-\rho'\,\overset{j}{\gamma}_0(z)=\overset{j}{R}_{3,0}(z). \end{aligned}$$

Formally solving the system, we obtain

$$(36) \qquad \begin{aligned} &\overset{j}{\alpha}_0(z)=\int_0^z(\overset{j}{R}_{1,0}(\tau)-\rho'(\tau)\,\overset{j}{\beta}{}'_0(\tau))d\tau+\overset{j}{c}{}^2_0,\\ &\overset{j}{\beta}_0(z)=\overset{j}{R}_{2,0}(z),\\ &\overset{j}{\gamma}_0(z)=\rho(z)(\overset{j}{c}{}^1_0+\int_0^z\frac{\overset{j}{R}_{3,0}(\tau)}{\rho^2(\tau)}d\tau). \end{aligned}$$

Adding a trivial component of the inf. b. of order j [4,10], we get

$$(36') \qquad \overset{j}{c}{}^1_0=\overset{j}{c}{}^2_0=0.$$

From (10)-(13), (15), (17), (9), (36) and (36′) it is seen that in a neighbourhood of the pole $z=0$ the solution $\overset{j}{\alpha}_{(j-2h_j)k}(z)$, $\overset{j}{\beta}_{(j-2h_j)k}(z)$, $\overset{j}{\gamma}_{(j-2h_j)k}(z)$, $h_j=1,\ldots,p_j$, have a form analogous to (33), (28), (31) correspondingly.

Passing to a variable ρ in a neighbourhood of the pole $z=0$, we obtain

$$(37) \qquad \begin{aligned} &\overset{j}{\alpha}_{(j-2h_j)k}(\rho)=\rho^{j[\nu_k(n)-n]+n}\,\overset{j}{\alpha}{}^0_{(j-2h_j)k}(\rho),\\ &\overset{j}{\beta}_{(j-2h_j)k}(\rho)=\rho^{j[\nu_k(n)-n]+2n-1}\,\overset{j}{\beta}{}^0_{(j-2h_j)k}(\rho),\\ &\overset{j}{\gamma}_{(j-2h_j)k}(\rho)=\rho^{j[\nu_k(n)-n]+2n-1}\,\overset{j}{\gamma}{}^0_{(j-2h_j)k}(\rho), \end{aligned}$$

$$h_j = 0, 1, \dots, p_j,$$

where

$$\nu_k(n) = \sqrt{\left(\frac{n-2}{2}\right)^2 + k^2(n-1)} - \frac{n-2}{2}. \tag{38}$$

4. Investigation of the quantities $\overset{j}{\alpha}{}^0_{jk}(0)$. In this paragraph we shall consider more precisely the quantites $\overset{j}{\alpha}{}^0_{jk}(0)$, $j = 2, \dots, m$, when $2n_1 - 3 + \mu_k(n_1) < 0$, i. e. when $k < \sqrt{\frac{2}{n_1}}$.

Lemma1. *The following inequalities*

$$\overset{2}{\theta}_1(n_1) > 0 \text{ for } n_1 \in (0, \frac{1}{2}) \text{ and } \overset{2}{\theta}_1(0) = \overset{2}{\theta}_1(\overset{1}{\theta}_2) = 0. \tag{39}$$

$$\overset{s}{\theta}_l(n_1) + \overset{s}{\theta}_{s-l}(n_1) < 0 \text{ for} n_1 \in (0, \frac{1}{2}], \; s = 3, 4, \dots, m, \tag{40}$$

are valid.

Proof. From (35) we obtain directly

$$\overset{2}{\theta}_1 = 2(1 - 2n_1)\{\mu_k(n_1)[1 + n_1(k^2 - 1)] + n_1(k^2 - 1)(2n_1 - 3) - 1\}, \tag{41}$$

$$\begin{aligned}\overset{3}{\theta}_1 + \overset{3}{\theta}_2 = {} & 4\{\mu_k(n_1)[(k^2 - 1)(-6n_1^2 + 3n_1) + 2n_1^2 - 11n_1 + 6] \\ & +(k^2 - 1)(-16n_1^3 + 34n_1^2 - 15n_1) - 2n_1^2 + 11n_1 - 6\}.\end{aligned} \tag{42}$$

Evidently $\overset{2}{\theta}_1(0) = \overset{2}{\theta}_1(\frac{1}{2}) = \overset{3}{\theta}_1(0) + \overset{3}{\theta}_2(0) = 0$. Let's denote

$$\overset{2}{\theta}_1(n_1) = 2(1 - 2n_1)\overset{2}{f}(n_1), \quad \overset{2}{f}(n_1) = \overset{2}{A}(n_1) + \mu_k(n_1)\overset{2}{B}(n_1)$$

where $\overset{2}{A}(n_1) = n_1(k^2)(2n_1 - 3) - 1$, $\overset{2}{B}(n_1) = 1 + n_1(k^2 - 1)$. Since for $n_1 \in (0, \frac{1}{2}]$ we have $\overset{2}{A}(n_1) < 0$, $\overset{2}{B}(n_1) > 0$ and $\overset{2}{B^2}(n_1)\mu_k^2(n_1) - \overset{2}{A^2}(n_1) = 4n_1^3(k^2 - 1)^2(1 - n_1)k^2 > 0$, so the inequality (39) is valid.

Analogously we denote

$$\overset{3}{\theta}_1(n_1) + \overset{3}{\theta}_2(n_1) = 4\overset{3}{f}(n_1), \quad \overset{3}{f}(n_1) = \overset{3}{A}(n_1) + \mu_k(n_1)\overset{3}{B}(n_1),$$

where

$$\overset{3}{A}(n_1) = (k^2-1)(-16n_1^3+34n_1^2-15n_1) - 2n_1 + 11n_1 - 6,$$

$$\overset{3}{B}(n_1) = (k^2-1)(-6n_1^2+3n_1) + 2n_1^2 - 11n_1 + 6.$$

Now $\overset{3}{A}(n_1) < 0$, $\overset{3}{B}(n_1) > 0$ for $n_1 \in (0, \frac{1}{2}]$ and $\overset{3}{B^2}(n_1)\mu_k^2(n_1) - \overset{3}{A^2}(n_1) = k^2 n_1^2 (1 - n_1)(k^2-1)\overset{3}{f_1}(n_1)$, where $\overset{3}{f_1}(n_1) = (36k^2-32)n_1^3 + (8-36k^2)n_1^2 + (9k^2+36)n_1 - 18 < 0$ for $n_1 \in (0, \frac{1}{2}]$.

Consequently the inequality (40) is valid for $s = 3$.

With the help of (35) we obtain

$$\overset{s}{\theta_l}(n_1) + \overset{s}{\theta}_{s-l}(n_1) = 2(2n_1 - 3 + \mu_k(n_1))\{(3-2n_1)(s-1)$$
$$[2l(s-l)(1-n_1) - s] + sl(s-l) + k^2 s n_1 (s-l)(1-2n_1)\}$$
$$+8(s-l)l(s-1)(3-2n_1)(1-n_1) + 8(1-n_1)[1 - s^2 + l(s-l)]$$
$$+4k^2 n_1 (s-l) l (1-n_1)(2n_1-3)(s-2),$$
$$s = 2, 3, \dots, m, \quad l = 1, \dots, \left[\frac{s}{2}\right]. \tag{43}$$

Then

$$\overset{s}{\theta_l}(n_1) + \overset{s}{\theta}_{s-l}(n_1) = P(n_1)s^2 + Q(n_1)s + R(n_1), \tag{44}$$

where

$$P(n_1) = 2(2n_1 - 3 + \mu_k(n_1))\{(3-2n_1)[2l(1-n_1) - 1] + l + k^2 n_1 l (1-2n_1)\}$$
$$+4(1-n_1)\{(3-2n_1)l(2-k^2n_1) - 2\},$$
$$Q(n_1) = 2(2n_1 - 3 + \mu_k(n_1))\{(3-2n_1)[1 - 2l(1-n_1)(1+l)]$$
$$-l^2 - k^2 n_1 l^2 (1-2n_1)\} + 4l(1-n_1)$$
$$\{2 - 2(3-2n_1)(l+1) - k^2 n_1 (l+2)(2n_1-3)\},$$
$$R(n_1) = 4(1-n_1)[(2n_1 - 3 + \mu_k(n_1))(3-2n_1)l^2 +$$
$$2l^2(3-2n_1) + 2(1-l^2) + 2k^2 l^2 n_1 (2n_1-3)].$$

From here we have

$$P(n_1) = 2[C(n_1) + \mu(n_1)D(n_1)],$$

where

$$C(n_1) = -8l(k^2-1)n_1^3 + [18l(k^2-1) - 6l + 4]n_1^2 + [15l - 9l(k^2-1) - 8]n_1 + 5 - 9l,$$

$$D(n_1) = (3-2n_1)[2l(1-n_1)-1] + l + k^2 n_1 l(1-2n_1) > 0 \text{ for } n_1 \in [0, \frac{1}{2}].$$

Since $C(n_1) \to +\infty$ when $n_1 \to -\infty$, $C(0) < 0$, $C(\frac{1}{2}) < 0$, $C(1) > 0$, $C(n_1) \to -\infty$ when $n_1 \to +\infty$, so $C(n_1) < 0$ for $n_1 \in [0, \frac{1}{2}]$. We shall show that

$$P(n_1) < 0 \text{ for } n_1 \in (0, \frac{1}{2}]. \tag{45}$$

We have

$$\begin{aligned} D^2(n_1)\mu_k^2(n_1) - C^2(n_1) = {} & [-16l^2(k^2-1) - 32l^2(k^2-1)^2 - 16l^2(k^2-1)^3]n_1^6 \\ & +[32l(2l+1)(k^2-1) + 32l(1+3l)(k^2-1)^2 + 32l^2(k^2-1)^3]n_1^5 \\ & +[-32l^2 + 48l - 16 - 4(33l^2+12l+4)(k^2-1) \\ & -4l(30l+24)(k^2-1)^2 - 20l^2(k^2-1)^3]n_1^4 \\ & +[144l^2 - 208l + 64 + 4(47l^2 - 22l + 16)(k^2-1) \\ & +4l(22+20l)(k^2-1)^2 + 4l^2(k^2-1)^3]n_1^3 \\ & +[-224l^2 + 324l - 100 + 2(-76l^2 + 94l - 42)(k^2-1) - (24l^2+24l)(k^2-1)^2]n_1^2 \\ & +[144l^2 - 212l + 68 + 2(24l^2 - 42l + 18)(k^2-1)]n_1 - 32l^2 + 48l - 16. \end{aligned}$$

Then

$$D^2(n_1)\mu_k^2(n_1) - C^2(n_1) = 4(n_1 - 1)g(n_1), \tag{46}$$

where

$$\begin{aligned} g(n_1) = {} & [-4l^2(k^2-1) - 8l^2(k^2-1)^2 - 4l^2(k^2-1)^3]n_1^5 \\ & +[(12l^2+8l)(k^2-1) + (16l^2+8l)(k^2-1)^2 + 4l^2(k^2-1)^3]n_1^4 \\ & +[-8l^2 + 12l - 4 - (21l^2 + 4l^2 + 4)(k^2-1) - (14l^2+16l)(k^2-1)^2 - l^2(k^2-1)^3]n_1^3 \\ & +[28l^2 - 40l + 12 + (26l^2 - 26l + 12)(k^2-1) + (6l^2+6l)(k^2-1)^2]n_1^2 \\ & +[-28l^2 + 41l^2 - 13 + (-12l^2 + 21l - 9)(k^2-1)]n_1 + 8l^2 - 12l + 4. \end{aligned} \tag{47}$$

We have

$$\begin{aligned} g'''(n_1) = {} & -240l^2(k^2-1)k^4 n_1^2 \\ & +8l(k^2-1)[-6l + 5 + (-36l+19)k^2 + 6lk^4]n_1 - 48l^2 + 72l - 24 \\ & -(126l^2 + 24l + 24)(k^2-1) - (84l^2 + 96l)(k^2-1)^2 - 6l^2(k^2-1)^3. \end{aligned}$$

Since the discriminant of the quadratic equation $g'''(n_1) = 0$ is negative, so $g'''(n_1) < 0$ for all $n_1 \in (-\infty, +\infty)$. Then $g''(n_1)$ have not more than one real zero. From $g''(n_1) \to -\infty$ when $n_1 \to +\infty$ and $g''(1) > 0$ it follows that $g''(n_1)$ has only the zero $n_1^* > 1$. Consequently $g'(n_1)$ has not more than two real zeros and $g(n_1)$ – not more than three. From (47) it is seen that $g(n_1)$ has the zero $n_1' \in (\frac{1}{2}, 1)$, because $g(n_1) > 0$ for $n_1 \leq 0$, $g(\frac{1}{2}) = 4(k^2-1)[-3l^2+8l-4+l^2(k^2-1)] > 0$, and $g(1) < 0$. Then in the interval $(0, \frac{1}{2})$ either $g(n_1)$ has not zeros, or it has exactly two zeros. We will show that the last is not valid. In fact, if $g(n_1)$ has two zeros in $(0, \frac{1}{2})$, so $g'(n_1)$ should have not less than two zeros in $(0, n_1')$ and $g''(n_1)$ – not less than one zero in $(0, n_1')$, $n_1' < 1$. But we proved that $g''(n_1)$ has only the zero $n_1^* > 1$.

Thus we have $g(n_1) > 0$ for $n_1 \in (0, \frac{1}{2}]$. Then from (46) we obtain $D^2(n_1)\mu_k^2(n_1) - C^2(n_1) < 0$ for $n_1 \in (0, \frac{1}{2}]$. From here and from $C(n_1) < 0$, $D(n_1) > 0$ for $n_1 \in [0, \frac{1}{2}]$ it follows (45).

We denote

$$R(n_1) = 4(1-n_1)h(n_1)$$

where

$$h(n_1) = (2n_1 - 3 + \mu_k(n_1))(3-2n_1)l^2 +$$
$$2l^2(3-2n_1) + 2(1-l^2) + 2k^2l^2n_1(2n_1-3).$$

We shall represent $h(n_1)$ in the form

$$h(n_1) = C_1(n_1) + \mu_k(n_1)D_1(n_1),$$

where

$$C_1(n_1) = 4l^2(k^2-1)n_1^2 - 6l^2(k^2-1)n_1 + 2l^2n_1 - 5l^2 + 2 < 0,$$
$$D_1(n_1) = l^3(3-2n_1) > 0$$

for $n_1 \in (0, \frac{1}{2}]$. Then

$$C_1^2(n_1) - D_1^2(n_1)\mu_k^2(n_1) = 4\{4l^4(k^2-1)k^2n_1^4 - 12l^4(k^2-1)k^2n_1^3 +$$
$$l^2(k^2-1)(9l^2k^2 - 4l^2 + 4)n_1^2 + 2l^2(l^2-1)(3k^2-4)n_1 + 4l^4 - 5l^2 + 1\} \geq$$
$$4\{4l^4(k^2-1)k^2n_1^4 + l^2(k^2-1)(3l^3k^2 - 4l^2 + 4)n_1^2 +$$
$$2l^2(l^2-1)(3k^2-4)n_1 + 4l^4 - 5l^2 + 1\} > 0.$$

Consequently $h(n_1) < 0$ for $n_1 \in (0, \frac{1}{2}]$ and

(48) $$R(n_1) = 4(1 - n_1)h(n_1) < 0 \text{ for } n_1 \in (0, \frac{1}{2}].$$

We set $y = \overset{s}{\theta}_l + \overset{s}{\theta}_{s-l}$. Then from (44) we have

(49) $$y = P(n_1)s^2 + Q(n_1)s + R(n_1).$$

Since for fixed $n_1 \in (0, \frac{1}{2}]$ and l, $1 \le l \le [\frac{s}{2}]$, (49) is an equation of a parabola and $P(n_1) < 0$, $y(0) = R(n_1) < 0$, $y(2) = 2\overset{2}{\theta}_1(n_1) > 0$, $y(3) = \overset{3}{\theta}_1(n_1) + \overset{3}{\theta}_2(n_1) < 0$, then $y(s) = \overset{s}{\theta}_l(n_1) + \overset{s}{\theta}_{s-l}(n_1) < 0$ for $s \ge 3$. Thus the lemma 1 is proved.

Directly we obtain:

(50) $$\overset{s}{\theta}_l(0) = 0 \text{ only for } l = 1 \text{ and } l = s - 1;$$

(51) $$\overset{s}{\theta}_l(0) < 0 \text{ for } 1 < l < s - 1;$$

(52) $$\overset{s}{\theta}_l(\frac{1}{2}) = 0 \text{ only for } s = 2;$$

(53) $$\overset{s}{\theta}_l(1) = 0 \text{ only for } l = \frac{s}{2};$$

(54) $$\overset{s}{\theta}_l(1) + \overset{s}{\theta}_{s-l}(1) = 0;$$

(55) $$(\overset{s}{\delta}_1 + 1)(\overset{s}{\delta}_2 + 1) = \frac{1}{4}\{[s(2n_1 - 3 + \mu_k(n_1)) - 2n_1 + 3]^2 - \mu_{sk}^2(n_1)\} < 0$$

for $n_1 \in (0, \frac{1}{2}]$, $2n_1 - 3 + \mu_k(n_1) < 0$ and $(s-1)(2n_1 - 3 + \mu_k(n_1)) + 2 > 0$;

(56) $$(\overset{s}{\delta}_1 + 1)\Big|_{n_1=1} = \frac{1}{2}[s(2n_1 - 3 + \mu_k(n_1) - 2n_1 + 3 - \mu_{sk}(n_1)]|_{n_1=1} = 0.$$

We shall consider the quantity $\overset{j}{\alpha}{}^0_{jk}(0)$, $j = 2, \dots, m$. From (34) we obtain

$$\overset{j}{\alpha}{}^0_{jk}(0) = \frac{-1}{4(\overset{j}{\delta}_1 + 1)(\overset{j}{\delta}_2 + 1)}[(\overset{j}{\theta}_1 + \overset{j}{\theta}_{j-1})\overset{1}{\alpha}{}^0_{1k}(0)\overset{j-1}{\alpha}{}^0_{(j-1)k}(0) +$$

$$(\overset{j}{\theta}_2 + \overset{j}{\theta}_{j-2})\, \overset{2}{\alpha^0}_{2k}(0)\, \overset{j-2}{\alpha^0}_{(j-2)k}(0) + \ldots + \overset{j}{\theta}_{\frac{j}{2}} \overset{\frac{j}{2}}{\alpha^0}_{\frac{jk}{2}}(0)\, \overset{\frac{j}{2}}{\alpha^0}_{\frac{jk}{2}}(0)\Big],$$

when j is even, and

$$\overset{j}{\alpha^0}_{jk}(0) = \frac{-1}{4(\overset{j}{\delta}_1 + 1)(\overset{j}{\delta}_2 + 1)}$$

$$\Big[(\overset{j}{\theta}_1 + \overset{j}{\theta}_{j-1})\, \overset{1}{\alpha^0}_{1k}(0)\, \overset{j-1}{\alpha^0}_{(j-1)k}(0) + (\overset{j}{\theta}_2 + \overset{j}{\theta}_{j-2})\, \overset{2}{\alpha^0}_{2k}(0)\, \overset{j-2}{\alpha^0}_{(j-2)k}(0)$$

$$+ \ldots + (\overset{j}{\theta}_{\frac{j-1}{2}} + \overset{j}{\theta}_{\frac{j+1}{2}})\, \overset{\frac{j-1}{2}}{\alpha^0}_{\frac{(j-1)k}{2}}(0)\, \overset{\frac{j+1}{2}}{\alpha^0}_{\frac{(j+1)k}{2}}(0)\Big],$$

when j is odd.

Applying the equality (34), we obtain

$$\overset{2}{\alpha^0}_{2k}(0) = \frac{-\overset{2}{\theta}_1}{4(\overset{2}{\delta}_1 + 1)(\overset{2}{\delta}_2 + 1)} \left(\overset{1}{\alpha^0}_{1k}(0)\right)^2; \tag{57}$$

$$\overset{3}{\alpha^0}_{2k}(0) = \frac{\overset{2}{\theta}_1 \left(\overset{1}{\alpha^0}_{1k}(0)\right)^3}{4^2(\overset{3}{\delta}_1 + 1)(\overset{3}{\delta}_2 + 1)} \; \frac{\overset{3}{\theta}_1 + \overset{3}{\theta}_2}{(\overset{2}{\delta}_1 + 1)(\overset{2}{\delta}_2 + 1)}; \tag{58}$$

$$\overset{j}{\alpha^0}_{jk}(0) = \frac{(-1)^{j-1}\, \overset{2}{\theta}_1 \left(\overset{1}{\alpha^0}_{1k}(0)\right)^j}{4^{j-1}(\overset{j}{\delta}_1 + 1)(\overset{j}{\delta}_2 + 1)} \tag{59}$$

$$\left[\frac{(\overset{j}{\theta}_1 + \overset{j}{\theta}_{j-1})(\overset{j-1}{\theta}_1 + \overset{j-1}{\theta}_{j-2}) \ldots (\overset{3}{\theta}_1 + \overset{3}{\theta}_2)}{(\overset{j-1}{\delta}_1 + 1)(\overset{j-1}{\delta}_2 + 1)(\overset{j-2}{\delta}_1 + 1)(\overset{j-2}{\delta}_2 + 1) \ldots (\overset{2}{\delta}_1 + 1)(\overset{2}{\delta}_2 + 1)} + \ldots\right],$$

$$j = 4, \ldots, m$$

In the square brackets of the equality (59) it is written the first term only. The rest terms (they are finite number) have a form analogous to that one of the first – the numerator of each term contains $j-2$ factors of the form $\overset{s}{\theta}_l + \overset{s}{\theta}_{s-l}$ (for $l = \frac{s}{2}$ this factor is division by two) and the denominator has $j-2$ factors of the form $(\overset{s}{\delta}_1 + 1)(\overset{s}{\delta}_2 + 1)$, $s = 2, \ldots, j$, $l = 1, \ldots, \left[\frac{s}{2}\right]$. Moreover the numerator of any term, except the first, contains as a factor $\overset{2}{\theta}_1$ in some positive degree. Thus each term in the square brackets of the equality (59) has the form

$$\frac{\overset{j}{M}(n_1) + \mu_k(n_1)\overset{j}{N}(n_1)}{\overset{j}{M}_1(n_1) + \mu_k(n_1)\overset{j}{N}_1(n_1)},$$

where $\overset{j}{M}(n_1)$ and $\overset{j}{N}(n_1)$ ($\overset{j}{M}_1(n_1)$ and $\overset{j}{N}_1(n_1)$) are polynomials correspondingly of degree $3(j-1)$ and $3(j-2)-1$ ($2(j-2)$ and $2(j-2)-1$).

From the lemma 1, the equality (52) and the inequality (55) it follows that the quantity in the square brackets of the equality (59) for $n_1 = \frac{1}{2}$ has positive value and it is equal to the value of the first term. For $n_1 \in (0, \frac{1}{2})$ the first term and the terms which contain a factor $\overset{2}{\theta}_1$ in even degree are positive, and all rest terms are negative. Then there exists a number $\overset{j}{n^*}_1 \in [0, \frac{1}{2})$ such that $\overset{j}{\overset{0}{\alpha}}_{jk}(0) \neq 0$ for $n_1 \in (\overset{j}{n^*}_1, \frac{1}{2})$. From (57) and (58) it can be seen that $\overset{2}{n^*}_1 = \overset{3}{n^*}_1 = 0$.

Let us remark that a reduction to a common denominator in the square brackets of the equality (59) in each term there appear $q_j = \left[\frac{j}{2}\right] - 1$ factors of the form $(\overset{s}{\delta}_1 + 1)(\overset{s}{\delta}_2 + 1)$. Thus the quantity in the square brackets (59) gets the form

$$\frac{\overset{j}{\tilde{M}}(n_1) + \mu_k(n_1)\overset{j}{\tilde{N}}(n_1)}{\overset{j}{\tilde{M}}_1(n_1) + \mu_k(n_1)\overset{j}{\tilde{N}}_1(n_1)},$$

where $\overset{j}{\tilde{M}}(n_1)$ and $\overset{j}{\tilde{N}}(n_1)$ are polynomials respectively of degree $3(j-2)+2q_j$ and $3(j-2)+2q_j-1$. Since the function $\overset{j}{\tilde{M}}(n_1) + \mu_k(n_1)\overset{j}{\tilde{N}}(n_1)$ may have no more than $6(j-2)+4q_j$ zeroes and $n_1 = 0$ is at least its simple zero, and $n_1 = 1$ is at least its $(j-2+q_j)$-tuple zero (see (54) and (56)), so if it has zeroes in the interval $(0, \frac{1}{2})$, they are not more than $5(j-2)+3q_j$.

5. Main results for the fields of infinitesimal bendings in a neighbourhood of the pole. Now we shall consider the question of the regularity of the found field $\overset{j}{z}$. From (33), (28), (31) it can be seen that the field $\overset{j}{z}$ is regular if $2n_1 - 3 + \mu_k(n_1) \geq 0$, i. e. $k \geq \sqrt{\frac{2}{n_1}}$.

Let $2n_1 - 3 + \mu_k(n_1) < 0$, i. e. $k < \sqrt{\frac{2}{n_1}}$. In view of the inequality (26) we have

$$j(2n_1 - 3 + \mu_k(n_1)) + 4 - 2n_1 > 0 \text{ for } n_1 \in (0, \frac{1}{2}].$$

Then from (28), (31) and (33) it can be seen that $\overset{j}{\beta}_{(j-2h_j)k}(0) = \overset{j}{\gamma}_{(j-2h_j)k}(0) = 0$, and $\overset{j}{\alpha}_{(j-2h_j)k}(0) = 0$ if the inequality

$$(60) \qquad j(2n_1 - 3 + \mu_k(n_1)) + 2 > 0 \text{ i.e. } k > \frac{1}{j}\sqrt{\frac{n(j-1)[n(2j-1)-2j]}{n-1}},$$

is true.

Moreover if $n_1 \in (0, \frac{1}{2})$ is such that the inequality $\overset{j}{\alpha^0}_{(j-2h_j)k}(0) \neq 0$ is valid at least for one $h_j = 0, 1, \ldots, p_j$, then the condition (60) is also necessary for the field $\overset{j}{z}$ to be regular. In the paragraph 4 we have shown: 1) $\overset{2}{\alpha^0}_{2k}(0) \neq 0$, $\overset{3}{\alpha^0}_{3k}(0) \neq 0$ for any $n_1 \in (0, \frac{1}{2})$ (see (39), (40), (57), (58)); 2) if $\overset{j}{\alpha^0}_{jk}(0)$ for $j > 3$ has zeroes $n_1 \in (0, \frac{1}{2})$, so they are not more than $5(j-2) + 3q_j$.

We denote

$$(61) \qquad A(j,n) = \frac{1}{j}\sqrt{\frac{n(j-1)[n(2j-1)-2j]}{n-1}}, \; j \geq 2, \; n \geq 2.$$

Immediately it is seen that the quantity $A(j,n)$ is an increasing function of j as well as of n, $2 \leq j, n < +\infty$, $A(j,n) < \sqrt{2n}$, $\lim_{j \to +\infty} A(j,n) = \sqrt{2n}$ and $\lim_{n \to \infty} A(j,n) = \infty$.

Thus the following statements are valid:

Theorem 1. *The condition*

$$(61) \qquad k > A(m,n), \quad m \geq 2, \quad n \geq 2,$$

is necessary (except may be for not more than $5(m-2) + 3(\left[\frac{m}{2}\right] - 1)$ *values of* $n > 2$ *for* $m > 3$*) and sufficient (for each* $n \geq 2$ *for* $m \geq 2$*) in a neighbourhood of the pole* $z = 0$ *of the surface* S*, so that the regular fundamental field* $\overset{1}{z}_k$ *of inf. b. of the 1-st order can be extended to a regular field* $\overset{m}{z}$ *of inf. b. of the m-th order.*

Corollary 1. *In a neighbourhood of the pole* $z = 0$ *any regular fundamental field* $\overset{1}{z}_k$, $k \geq \sqrt{2n}$, *of inf. b. of the 1-st order can be extended to a regular field of inf. b. of every order.*

Corollary 2. *If from the closed surface S it is moved away an arbitrary small neighbourhood of the pole $z = 1$, which neighbourhood is bounded by a parallel, then the remaining part S_0 of the surface S is nonrigid of any order.*

Corollary 3. *If the pole $z = 0$ of the surface S is a nonparabolic point, i. e. $n = 2$, then in its neighbourhood any regular fundamental field $\overset{1}{z}_k$, $k \geq 2$, of inf. b. of the 1-st order can be extended to a regular field of inf. b. of every order.*

Let $n > 2$ be such that the condition (62) of theorem 1 is necessary. Then the following statement is valid.

Corollary 4. *In a neighbourhood of $z = 0$ any regular fundamental field $\overset{1}{z}$, $k < \sqrt{2n}$, of inf. b. of the 1-st order canot be extended to a regular field $\overset{m}{z}$ of inf. b. of order $m \geq \dfrac{n}{n - \nu_k(n)}$.*

A summary of the results in this paper is contained in the note [8].

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