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THE STRUCTURE OF LOCALLY COMPACT ALGEBRAS

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ABSTRACT. In this paper we study how the topological structure of a topological algebra and the principles of: approximability, openness of congruences and homogeneity in the theory of topological universal algebras, are influenced by the algebraic structure of the algebra.

0. Preliminaries. The present paper is related to the results of A. V. Arhangel'skii [1, 2], A. I. Mal'cev [11], L. S. Pontrjagin [12], I. Prodanoff [13], W. Taylor [16, 17], M. M. Čoban [4, 5, 6, 7, 8, 9], etc. The terminology in [3, 4, 5] is used.

The discrete sum $E = \oplus\{E_n : n \in \mathbf{N} = \{0, 1, 2, \ldots\}\}$ of the spaces $\{E_n\}$ is called a continuous signature. If E is discrete, then the signature E is said to be discrete too. We say that a topological E-algebra A is given if there are continuous mappings $\{e_{nA} : E_n \times A^n \to A : n \in \mathbf{N}\}$ and A is a nonempty space. A class K of topological E-algebras is called a quasivariety if it is closed with respect to subalgebras and Tychonoff products. If all objects of K are T_i-spaces, then K is called T_i-quasivariety. Any topological space is said to be a T_{-1}-space. The T_i-quasivariety K is called a full T_i-quasivariety if the following condition holds: If $(A, \tau_1) \in K$ and if the T_i-topology τ on A is such that (A, τ) is a topological E-algebra, then $(A, \tau) \in K$. A class K of topological E-algebras is called non-trivial if a topological algebra containing at least three different open sets exists in K.

Let K be a quasivariety of topological E-algebras. For every non-empty space X there exists a topological algebra $F(X, K)$ in K and a continuous mapping $k_X : X \to F(X, K)$ such that the set $k_X(X)$ algebraically generates $F(X, K)$ and for each continuous mapping $f : X \to A$, with $A \in K$, there exists a continuous homomorphism $\hat{f} : F(X, K) \to A$ such that $f = \hat{f} \circ k_X$ (see [4, 5]). $((F(X, K), k_X)$ is called a free topological algebra of X in K).

Fix a signature E and $e \in E_n$. Then, for every topological E-algebra A, the mapping $e : A^n \to A$, where $e(x_1, \ldots, x_n) = e_{nA}(e, x_1, \ldots, x_n)$, is continuous and it is called a fundamental operation. A meaningful expression consisting of the variables

$x_1, x_2, \ldots$, parentheses and symbols for fundamental operations is called a term. For any term the type is determined. If A is an E-algebra and in a term $h(x_1, \ldots x_n)$ of type n we fix some variable $x_{i_1} = a_1 \in A, \ldots, x_{i_m} = a_m \in A$, where $n - m \geq 1$, then we obtain a polynomial of type $n - m$. The polynomial of type 1 is called a translation. A translation generated by a fundamental operation is called fundamental. A composition of fundamental translations is called an elementary translation (see [11]). Let P be a set of polynomials with the discrete topology. Then a topological E-algebra A is a topological P-algebra as well. This P-algebra is called a derivative algebra of the E-algebra A.

Theorem 0.1 *For a quasivariety K of topological E-algebras, the following two conditions are equivalent:*

1. *Every algebra $A \in K$ contains a derivative algebra for which the set of all elementary translations is a transitive group of transformations of the space A;*
2. *Every algebra $A \in K$ contains a derivative algebra which is a loop.*

Theorem 0.2 *For a quasivariety K of topological E-algebras, the following two conditions are equivalent:*

1. *For every topological algebra $A \in K$ the group of reversible translations is transitive.*
2. *Each algebra $A \in K$ contains a derivative P-algebra, where $P = \{p, q\}$ and p, q are polynomials of type 3 with identities*

$$p(y, y, x) = q(p(x, y, z), y, z) = p(q(x, y, z), y, z) = x.$$

A.I. Mal'cev proved Theorems 0.1 and 0.2 for any variety [11]. Mal'cev proofs are valid for every quasivariety. Each algebra $A \in K$ with two operations p, q of type 3 with identities $p(y, y, x) = q(p(x, y, z), y, z) = p(q(x, y, z), y, z) = x$ is called a biternary algebra.

1. Approximability of algebras. The topological E-algebra is representable if there exists a topological isomorphism of A into a Tychonoff product of metrizable E-algebras.

Theorem 1.1 *Let $E' = \cup\{E_n : n \geq 1\}$ be a σ-compact space and A be a regular σ-compact E-algebra. Then for every open cover ω of the space A, there exists a metrizable E-algebra B and a continuous homomorphism $f : A \to B$ such that* $\dim B \leq \dim A$ *and f is an ω-mapping. If in addition A is a p-space, then the mapping f is perfect.*

Proof. There exists a continuous pseudometric d such that $\{B(x, 1) = \{y \in A : d(x, y) < 1\} : x \in A\}$ refines ω. Let $E' = \cup\{H_m : m \in \mathbf{N}\}$ and $A = \cup\{A_m : m \in \mathbf{N}\}$ be such that $H_m \subset H_{m+1}$, $A_m \subset A_{m+1}$ and the sets $H_0, A_0, H_1, A_1, \ldots$

are compact. Every continuous mapping is uniformly continuous in any compact set. Moreover, there exists a sequence $\{d_m : m \in \mathbf{N}\}$ of continuous pseudometrics such that: $d(x,y) \leq d_0(x,y) \leq d_m(x,y) \leq d_{m+1}(x,y)$; $\dim(A,d_m) \leq \dim A$; the mapping e_{nA} is continuous at any point of the sets $(E_n \cap H_m) \times A^n_m$ for every $n, m \in \mathbf{N}$ with respect to the topology induced by the pseudometric d_m. Let τ be the topology induced by the pseudometrics $\{d_m : m \in \mathbf{N}\}$. Then (A,τ) is a topological E-algebra. There exists a mapping $f : A \to B$ such that $f(x) = f(y)$ iff $\Sigma\{d_m(x,y) : m \in \mathbf{N}\} = 0$. Then the pseudometrics $\{d_m\}$ induce on B the metrizable topology τ', the space (B,τ') is a topological E-algebra, $\dim B \leq \dim A$ and f is a continuous ω-homomorphism. □

Corollary 1.1 *If $E' = \cup\{E_n : n \in \mathbf{N}\}$ is a σ-compact space and A is a regular σ-compact E-algebra, then the E-algebra A is a representable one.*

A continuous mapping $f : X \to Y$ has a metrizable kernel (see [10]) if there exist a compact metrizable space Z and an embedding $g : X \to Y \times Z$ such that $f = h \circ g$ and $h(y,z) = y$ for all $(y,z) \in Y \times Z$.

Corollary 1.2 *Let E be a σ-compact space and A be a regular σ-compact Čech complete E-algebra. Then:*

1. *The topological E-algebra A can be represented as a limit of an inverse system $\{A_\alpha, p^\alpha_\beta : \alpha,\beta \in M\}$ of metrizable E-algebras where $p^\alpha_\beta : A_\alpha \to A_\beta$ and $p_\alpha : A \to A_\alpha$, which is onto, are perfect homomorphisms for all $\alpha,\beta \in M$;*

2. *The topological algebra A can be represented as a limit of well-ordered inverse system $\{B_\alpha, q^\alpha_\beta : \alpha,\beta \in M\}$ of topological E-algebras , indexed by the ordinals less than some m and satisfying the conditions: B_0 is a metrizable space; for all limit ordinals $\gamma \in M$, the natural mapping from the algebra B_γ to $\lim\{B_\alpha, q^\alpha_\beta : \beta \leq \alpha < \gamma\}$ is a topological isomorphism; for all $\beta \in M$, $q_\beta : A \to B_\beta$, which is onto, is a perfect homomorphism and $q^{\beta+1}_\beta$ is a perfect homomorphism with a metrizable kernel.*

Remark 1.1 For compact algebras of discrete signature Theorem 1.1 is proved in [13].

2. Openness of congruences. The homomorphism $h : A \to B$ of the E-algebra A into B induces the congruence $\alpha = c(h)$ in A such that $x\alpha y$ iff $h(x) = h(y)$. If θ is an equivalence relation of A and $L \subset A$, by θ-saturation of L we mean $\{x \in A : x\theta y$ for some $y \in L\}$. The equivalence relation θ of the space A is open iff the θ-saturation of any open set is also open. An algebra A has permutable congruences iff $\alpha\beta = \beta\alpha$ for any congruences α and β, where $\alpha\beta = \{(x,y) : x\alpha z$ and $z\beta y$ for some $z \in A\}$.

Denote by $D = \{0,1\}$ a two-point discrete space.

Remark 2.1 A.I. Mal'cev [11] has proved the following statements. If A is a topological E-algebra with a polynomial $p(x,y,z)$ fulfilling the equations $p(y,y,x) = p(x,y,y) = x$ then :

1. Every quotient homomorphism of the algebra A is an open mapping;

2. Any congruence in A is open;

3. If α is a congruence on A, then the quotient space A/α is a topological E-algebra;

4. The algebra A has permutable congruences;

5. If A is a T_0-space, then A is a T_2-space.

If for an E-algebra A there exists a polynomial $p(x, y, z)$ with equations $p(y, y, x) = p(x, y, y) = x$, then A is called an algebra with Mal'cev equations.

Theorem 2.1 *Let K be a quasivariety of topological E-algebras and let the free algebra $F(D, K)$ be discrete. Then the following conditions are equivalent:*

1. *Any algebra $A \in K$ has permutable congruences;*

2. *There exists a term $p(x, y, z)$ such that the equations $p(x, y, y) = p(y, y, x) = x$ hold identically in K;*

3. *Any congruence in an algebra of K is open.*

Proof. The equivalence $1 \iff 2$ for variety K is proved by A.I. Mal'cev [11]. Malc'ev's proof remains valid for every quasivariety. The implication $2 \Rightarrow 3$ is an immediate consequence of Remark 2.1. Let α and β be the non-permutable congruences of the algebra $A \in K$. There are elements $a, b, c \in A$ for which $a\alpha b$, $b\beta c$ and $(c, a) \notin \beta\alpha$. Consider the infinite zero-dimensional compact space X with an isolated point x_3. Let x_2 be a non-isolated point of the space X. Then the algebra $F(X, K)$ is algebraically free in K and k_X is a topological embedding (see [5]). Suppose that $X = k_X(X) \subset F(X, K)$. There exists a homomorphism $h : F(X, K) \to A$ such that $h(x_2) = b$, $h(x_3) = c$, $h(X \setminus \{x_2, x_3\}) = a$. Then $\alpha' = h^{-1}(\alpha)$ and $\beta' = h^{-1}(\beta)$ are non-permutable congruences of the algebra $F(X, K)$. The mapping $f : X \to D$, where $f^{-1}(1) = x_3$, generates a continuous homomorphism $\hat{f} : F(X, K) \to F(D, K)$ and $c(\hat{f}) \subset \alpha'$. Consider the homomorphism $\hat{g} : F(X, K) \to F(D, K)$ generated by the mapping $g : X \to D$, where $g^{-1}(1) = \{x_2, x_3\}$. Thus, by construction $c(\hat{g}) \subset \beta'$, the set $W = \hat{f}^{-1}(1)$ is open in $F(X, K)$, $W \cap X = \{x_3\}$ and $\hat{g}^{-1}\hat{g}W \cap X = \{x_2, x_3\}$. Therefore, the set $\hat{g}^{-1}\hat{g}W$ is not open in $F(X, K)$ and the congruence $c(\hat{g})$ is not open. □

Corollary 2.1 *Let $i \in \{-1; 0; 1; 2; 3; 3.5\}$ and K be a full T_i-quasivariety of topological algebras of discrete signature E. The quasivariety K has permutable congruences iff any congruence in any algebra of K is open.*

Remark 2.2 Theorem 2.1 does not hold for an arbitrary topological E algebra. For full variety of topological algebras of discrete signature the assertion of Corollary 2.1 is announced in [16].

The mapping $f : X \to Y$ is paracompactly factorizable if there exist a paracompact T_2-space Z and continuous mappings $g : X \to Z$ and $h : Z \to Y$ such that $f = h \circ g$. A compact space X is called a Dugundji space (see [14]) if every embedding $h : X \to Y$ into a Tychonoff space Y admits a linear positive extension operator $u : C(X) \to C(Y)$, where $u(1_X) = 1_Y$ and $u(f) \circ h = f$ for every $f \in C(X)$.

The next assertion can be proved as in [6, 7, 10] with the help of Theorems 1.1 and 2.1.

Corollary 2.2 *Let X be a compact G_δ-set in some regular topological E-algebra A with Mal'cev equations. Then X is a Dugundji space.*

Corollary 2.3 *Let A be a Hausdorff locally compact E-algebra with Mal'cev equations. Then:*

1. *If the space A is paracompact, then* $\mathrm{Ind}A = \mathrm{ind}A$ *and there exist a discrete space Y, a cardinal number m and a perfect inductively open mapping $h : Y \times D^m \to A$ which is onto;*
2. *If $h : A \to B$ is a quotient homomorphism and $f : X \to B$ is a paracompactly factorizable mapping of the zero-dimensional space X, then there exists a continuous mapping $g : X \to A$ such that $f = h \circ g$;*
3. *If $h : A \to B$ is a quotient homomorphism onto σ-compact algebra B and $f : X \to B$ is a Baire measurable mapping, then there exists a Baire measurable mapping $g : X \to A$ such that $f = h \circ g$.*

Remark 2.3 For compact algebra with Mal'cev equations Corollary 2.2 was proved by V.V. Uspenskii as well.

Remark 2.4 Let E be a σ-compact space. Then Corollary 2.2 holds if A is a regular σ-compact E-algebra with open congruences and Corollary 2.3 holds provided A is a Hausdorff σ-compact locally compact E-algebra with open congruences.

3. Homogeneous algebras. If for the E-algebra A there exist two polynomials $p(x,y)$ and $q(x,y)$ with equations $p(x,x) = p(y,y)$, $p(x,q(x,y)) = y$, then A is called a homogeneous algebra. We have $q(x,p(x,y)) = y$. Let $c = p(x,x)$, $P_a(x) = p(a,x)$, $Q_a(x) = q(a,x)$, $h(x,y,z) = q(x,p(y,z))$. Then, $h(x,y,y) = h(y,y,x) = x$, P_a and Q_a are homeomorphisms, $P_a(a) = c$ and $Q_a^{-1} = P_a$. An algebra A is homogenous iff A is a biternary algebra. Any homogeneous algebra is an algebra with Mal'cev equations.

The mapping $g : X \times X \to X \times X$ is a centralizing mapping with center $a \in X$ if g is a homeomorphism, $g(x,x) = (x,a)$ and $\{x\} \times X = g(\{x\} \times X)$ for every $x \in X$. The space X is called a centralizable space if there exists a centralizable mapping for X (see [15]).

Theorem 3.1 *The space X is a centralizable space iff for some two continuous binary operations X is a homogeneous algebra.*

P r o o f. Let for the continuous binary operations p and q the space X be a homogeneous algebra. Then $h(x,y) = (x, p(x,y))$ is a centralizing mapping with center $p(x,x)$. Let h be a centralizing mapping with center $a \in X$. We put $g(x,y) = y$, $p(x,y) = g(h(x,y))$ and $q(x,y) = g(h^{-1}(x,y))$. Then $p(x,x) = a$ and $q(x,p(x,y)) = y$. □

Lemma 3.1 *Let $f : A \to B$ be a homomorphism of the homogeneous topological E-algebra A onto the E-algebra B. Then for every $x, y \in B$ the spaces $f^{-1}(x)$ and $f^{-1}(y)$ are homeomorphic.*

In the following statements all the spaces are assumed to be regular.

Theorem 3.2 *Let X be a compact G_δ-set of a homogeneous topological E-algebra A, $m = w(X)$ and X have no isolated points. Then:*

1. *X is a Dugundji space;*
2. *X is an inductively open image of D^m;*
3. *X is an irreducible image of D^m;*
4. *X contains a homeomorphic image of D^m and the spaces $C(X)$ and $C(D^m)$ are linearly homeomorphic;*
5. *If $\dim X = 0$, then the spaces X and D^m are homeomorphic;*
6. *The spaces X and D^m are Baire homeomorphic;*
7. *If the dimension $\operatorname{ind} X$ is defined, then $\dim X = \operatorname{Ind} X = \operatorname{ind} X$;*
8. *If $\operatorname{ind} X = n$, then the space X is perfectly n-dimensional and $X = \cup\{X_i : i \leq n\}$, where $X_0, \ldots, X_n$ are Lindelöf zero-dimensional G_δ-subspaces.*

Proof. The key of proving all these facts is provided by Theorems 1.1 and 2.1, Lemma 3.1 and the methods used in [6, 7, 8]. □

Corollary 3.1 *Let A be a non-discrete locally compact paracompact homogeneous E-algebra and $m = \chi(X)$. Then, there exists a discrete space Z for which:*

1. *A is a perfect inductively open image of $Z \times D^m$ and each canonically closed subset of A is a G_δ-set;*
2. *A is a perfect irreducible image of $Z \times D^m$;*
3. *If $\operatorname{ind} A = n - 1$, then A is a perfect irreducible n-fold image of $Z \times D^m$;*
4. *If $\dim A = 0$, then the spaces A and $Z \times D^m$ are homeomorphic;*
5. *The spaces $Z \times D^m$ and A are coabsolute;*
6. *If $\operatorname{ind} A$ is defined, then $\dim A = \operatorname{Ind} A = \operatorname{ind} A$;*

Corollary 3.2 *Suppose that a paracompact space X satisfies one of the following conditions: X is a p-space and a subspace of locally compact homogeneous algebra A for which $\operatorname{ind} A$ is defined; X is a locally compact G_δ-subspace of homogeneous E-algebra A and $\operatorname{ind} X$ is defined. Then:*

1. *X admits a perfect zero-dimensional mapping onto a metric space;*
2. *$\dim X = \operatorname{Ind} X$ and if $\dim X = n$, then X is perfectly n-dimensional and $X = \cup\{X_i : i \leq n\}$, where $X_0, \ldots, X_n$ are paracompact zero-dimensional G_δ-sets;*
3. *If X is locally connected, then X is metrizable.*

Corollary 3.3 *Any Tychonoff product of centralizable spaces is a centralizable space.*

4. Some remarks and questions. Let $E = E_3 = \{p\}$. We denote by V the class of all topological E-algebras with Mal'cev equations $p(x,y,y) = p(y,y,x) = x$. By W_1 we denote the class of all algebras $A \in V$ with the equation $p(y,x,y) = x$. Further, let W_2 designate the class of all algebras $A \in V$ with the equation $p(x,y,x) = x$ and $W = W_1 \cup W_2$.

Proposition 4.1 *If $B \subset A \in V$ and $|B| = 1$, then B is a subalgebra of the algebra A. If $H \subset C \in W$ and $|H| = 2$, then H is a subalgebra of the algebra C.*

Proposition 4.2 *If $K \subset W$ is a quasivariety, then $K \subset W_1$ or $K \subset W_2$.*

The class K of topological E-algebras is said to be closed under m-discrete sum, where m is a cardinal number, if for each family $\{A_\mu \in K : \mu \in M\}$, where $|M| \leq m$, of pairwise disjoint algebras in the discrete sum $A = \oplus\{A_\mu : \mu \in M\}$ there exists a structure of E-algebra such that $A \in K$ and any A_μ is a subalgebra of A.

This definition yields:

Proposition 4.3 *If the full T_i-variety K of topological E-algebras is closed under 2-discrete sum, then K is closed under m-discrete sum for any cardinal number m.*

Proposition 4.4 *The classes V, W_1, W_2 are closed under m-discrete sum for any cardinal number m.*

Remark 4.1 Let $A = [a,b]$. Then $A \in V$ for $e_{3A}(x,y,z) = \min\{b, max\{a, x - y - z\}\}$.

Remark 4.2 Let A be a topological group. Then $A \in V$ for $e_{3A}(x,y,z) = xy^{-1}z$. If B is a topological quasigroup, then $B \in V$ for some e_{3B}. The topological quasigroups are homogeneous algebras.

Remark 4.3 If X is a zero-dimensional metrizable space, then:

1. $X \in W_1$ for some e_{3X};
2. $X \in W_2$ for some e_{3X}.

Remark 4.4 V.V. Uspenskii has proved the following statement: If $A \in V$, $B \subset A$ and $r : A \to B$ is a retraction, then $B \in V$ for $e_{3B}(x,y,z) = r(e_{3A}(x,y,z))$.

Question 4.1 Let $A \in V$ be a pseudocompact space. Is A^2 a pseudocompact space?

Question 4.2 Let A be a pseudocompact homogeneous algebra. Is A^2 a pseudocompact space?

Question 4.3 Let $A \in V$ be a compact space. Is it true that $\dim A = \operatorname{ind} A$?

Question 4.4 Let A be a compact homogeneous algebra or a compact loop. Is it true that $\dim A = \operatorname{ind} A$?

Question 4.5 Let A be a locally compact homogeneous algebra or a locally compact loop. Is A a paracompact space?

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