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ON THE GEOMETRY OF ALMOST HERMITIAN MANIFOLDS OF CONSTANT TYPE

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ABSTRACT. In the present paper we give a decomposition of the curvature tensor of any almost Hermitian manifolds of constant type into orthogonal and irreducible components with respect to the unitary group. Using it we prove two theorems analogical with the classical theorem of Schouten and Struik in Riemannian geometry.

1. Preliminary. The notion of constant type of an almost Hermitian manifold was introduced in [1]. The definition there was related to the class of nearly Kähler manifolds. Many results and a number of references have been given in the survey paper [2]. The definition applicable to any class of almost Hermitian manifolds is given in [3]. A great number of recent investigations have been based on the use of this definition, e.g. [4], [5], [6], [7] and [8]. Now our main purpose is to prove the following

Main theorem. *An almost Hermitian manifold of dimension* $2n \geq 8$ *is of constant type iff its curvature tensor has the following decomposition*

$$R = p_1(R) + p_2(R) + p_3(R) + p_4(R) + p_9(R) \tag{1}$$

in the orthogonal components under the induced action of the unitary group $U(n)$ *in the space of all curvature tensors over a real* $2n$*-dimensional vector space with complex structure and a Hermitian product* g.

After proving this theorem we shall give some applications.

2. Decomposition of the curvature tensor. Let us recall the definition of costant type of an almost Hermitian manifold.

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An almost Hermitian manifold (M, g, J) with curvature tensor R is of constant type if for every antiholomorphic plane $x \wedge y$ (i.e. $x \wedge y \perp Jx \wedge Jy$) the equality

$$R(x, y, x, y) - R(x, y, Jx, Jy) = \lambda$$

holds.

Here x, y is an orthonormal pair of vectors at a point $p \in M$ and λ is a function of p.

We consider the tensor R' defined by

$$R'(x, y, z, u) = R(x, y, Jz, Ju).$$

It is easy to see that R' satisfies the identities

$$R'(x, y, z, u) = -R'(y, x, z, u) = -R'(x, y, u, z).$$

Following [9] we can rewrite the decomposition of R' into orthogonal components:

$$R' = R'_1 \oplus R'^{\perp}_1 \tag{2}$$

where R'_1 has zero sectional curvature and $R'^{\perp}_1$ is a curvature tensor so that both tensors R' and $R'^{\perp}_1$ have the same sectional curvatures. From our paper[8] we know that the unique curvature tensor with the last property is the tensor $\gamma(R)$ represented by

$$\begin{aligned} 6\gamma(R) &= 2R(x, y, Jz, Ju) + 2R(Jx, Jy, z, u) + R(x, z, Jy, Ju) + R(Jx, Jz, y, u) \\ &+ R(x, u, Jz, Jy) + R(Jx, Ju, z, y). \end{aligned}$$

Thus we have

$$R' = R'_1 \oplus \gamma(R) \tag{3}$$

From [8] we get also the decomposition

$$\gamma(R) = \sum_{i=1}^{3} p_i(R) - \frac{1}{3}\sum_{j=4}^{6} p_j(R) - p_7(R).$$

Then using the decomposition

$$R = \sum_{i=1}^{10} p_i(R)$$

of the curvature tensor R in [10] we can formulate

Lemma 1. *For any almost Hermitian manifold with curvature tensor R we have the decomposition*

$$R - R' = -R'_1 + \frac{4}{3}(p_4(R) + p_5(R) + p_6(R)) + 2p_7(R) + p_8(R) + p_9(R) + p_{10}(R). \tag{4}$$

Now if the manifold is of constant type for the function λ the following representation holds

(5) $$\lambda = \frac{\tau(R) - \tau^*(R)}{4n(n-1)}$$

(see for example [7]). Then as a consequence of Lemma 1 we have

Lemma 2. *An almost Hermitian manifold (M, g, J) is of constant type iff*

(6) $$\{\frac{4}{3}(p_5(R) + p_6(R)) + 2p_7(R) + \sum_{i=8}^{10} p_i(R)\}(x, y, x, y) = 0$$

for every orthonormal pair of vectors x, y with $x \wedge y = Jx \wedge Jy$.

This result enables us to define the tensor

$$\Gamma(R)(x, y, z, u) = \frac{4}{3}(p_5(R) + p_6(R))(x, y, z, u) + \{2p_7(R) + \sum_{i=8}^{10} p_i(R)\}(x, y, z, u).$$

Evidently $\Gamma(R)$ is a curvature tensor and it has a zero antiholomorphic sectional curvature. By Proposition 3.3 in [7] we get

(7) $$C^*\Gamma(R) = 0$$

where the linear operator C^* can be found also in [8].

Since

$$C^*\Gamma(R) = C_1^*\Gamma(R) + C_2^*\Gamma(R) + p_3\Gamma(R) + p_6\Gamma(R) + p_7\Gamma(R) + p_8\Gamma(R) + p_{10}\Gamma(R)$$

and

$$p_3\Gamma(R) = 0,\ p_6\Gamma(R) = \frac{4}{3}p_6(R),\ p_7\Gamma(R) = 2p_7(R),\ p_8\Gamma(R) = p_8(R),\ p_{10}\Gamma(R) = p_{10}(R)$$

then

$$p_6(R) = p_7(R) = p_8(R) = p_{10}(R), \quad C_1^*\Gamma(R) = 0, \quad C_2^*\Gamma(R) = 0.$$

The condition $C_1^*\Gamma(R) = 0$ is the identity because all components of $\Gamma(R)$ have scalar curvatures $\tau = 0$, $\tau^* = 0$.

Next let us examine condition $C_2^*\Gamma(R) = 0$. Following [11] we have

$$C_2^*\Gamma(R) = \frac{(n+1)\varphi - \psi}{n+1}(S_1 + 3S_2)\Gamma(R).$$

For

$$S_1\Gamma(R) = \frac{1}{16(n+2)}\{(\rho + 3\rho^*)(\Gamma + L_3\Gamma) - \frac{(\tau + 3\tau^*)\Gamma}{n}g\}$$

since

$$\Gamma(R) = \frac{4}{3}p_5(R) + p_9(R), \quad L_3\Gamma(R) = \frac{4}{3}p_5(R) - p_9(R),$$

it follows that

$$S_1\Gamma(R) = 0$$

For $S_2\Gamma(R)$ we have

$$\begin{aligned} S_2\Gamma(R) &= \frac{1}{16(n+2)}\{(\rho - \rho^*)(\Gamma + L_3\Gamma) - \frac{(\tau - \tau^*)\Gamma}{n}g\} \\ &= \frac{1}{4(n-2)}\{(\rho - \rho^*)(R + L_3\Gamma) - \frac{(\tau - \tau^*)R}{n}g\} \end{aligned}$$

Using the fact $p_5(R) = 0$ iff

$$(\rho - \rho^*)(R + L_3) - \frac{(\tau - \tau^*)R}{n}g = 0$$

we can formulate

Lemma 3. *The tensor $S_2\Gamma(R) = 0$ is zero iff $p_5(R) = 0$.*

The main theorem follows from Lemmas 1,2 and 3.

3. Two applications. The following classical theorem of Schouten and Struik [12] is well known in Riemannian geometry.

A Riemannian manifold of dimension $n > 3$ is of constant sectional curvature iff it is Einsteinian and conformal by flat manifold.

Now we give two theorems which are entirely analogic to this theorem.

Theorem 1. *An almost Hermitian manifold of dimension $2n > 6$ is of constant sectional curvature iff it is of constant type and conformal flat.*

Proof. Weyl's conformal curvature tensor can be written in the form [10]

$$C(R) = C_1(R) + C_2(R) + (p_3 + p_6 + p_7 + p_9 + p_{10})(R).$$

When the manifold is conformal flat, then all components of $C(R)$ are zero. $C_1(R) = 0$ is equivalent to $(\tau - (2n-1)\tau^*)(R) = 0$ and $C_2(R) = 0$

$$(3\varphi - (n-1)\psi)(S_2 - S_1) = 0.$$

From here using the operator ρ^* we get

$$S_2 - S_1 = 0.$$

Condition $p_5(R) = 0$ gives $S_1 = 0$. Then $p_2(R) = 0$. Since $p_3(R) = p_6(R) = p_7(R) = p_9(R) = p_{10}(R) = 0$ the decomposition (1) is reduced to the equality

$$R = p_1(R) + p_4(R)$$

which is equivalent to

$$R = \frac{\tau(R)}{2n(2n+1)}\pi_1.$$

The converse is trivial.

Now we shall replace $C(R) = 0$ by condition $C^*(R) = 0$ where $C^*(R)$ is a tensor analogical to Weyl's conformal curvature tensor $C(R)$ [11]. Namely $C^*(R)$ is a part of R with the property: the image of this part under the action of the operator ρ^* is zero. For this tensor we have found the following decomposition into orthogonal components:

$$C^*(R) = C_1^*(R) + C_2^*(R) + (p_3 + p_6 + p_7 + p_8 + p_{10})(R),$$

where

$$C_1^*(R) = \frac{((2n+1)\tau - 3\tau^*)R}{8n(n^2-1)(2n+1)}((2n+1)\pi_1 - \pi_2),$$

$$C_2^*(R) = \frac{1}{n+1}\{(n+1)\varphi - \psi\}(S_1 + 3S_2).$$

Condition $C_1^*(R) = 0$ is equivalent to the relation $(2n+1)\tau(R) - 3\tau^*(R) = 0$ and condition $C_2^*(R) = 0 - S_1 + 3S_2 = 0$. Condition $p_5(R) = 0$ gives $S_1 = S_2$ and then $p_2(R) = 0$.

We have

$$R = p_1(R) + p_4(R) + p_9(R)$$

or equivalently

$$R = \frac{\tau^*(R)}{2n(2n+1)}\pi_2 + p_9(R).$$

Thus we can formulate the following

Theorem 2. *An almost Hermitian manifold of dimension $2n > 6$ is of constant type with $C^*(R) = 0$ iff its curvature tensor can be represented in the form (7).*

Remark. Such manifolds do not exist in the class AH_3 (see Theorem 12.7 in [7]). Then we state the following problem: Do there exist almost Hermitian manifolds with curvature tensor of form (7)?

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