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COMPLETE CLASSIFICATION OF CONVEX CONES AND SKELETONS OF DIMENSIONS SEVEN UP TO THIRTEEN

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1. Introduction. Let D be a homogeneous bounded domain in $\mathbb{C}^n$. It is known that a Siegel domain genus I or II corresponds to any such domain. There is also one-to-one bijection between j-algebras (N-algebras and T-algebras) and Siegel domains $D(V, F)$ of genus I or genus II. The classification of Siegel domains genus I is reduced to the study of convex cones V, which are related to skeletons, because there is one-to-one bijection between convex cones and skeletons S_k. A homogeneous Siegel domain $D(V, F)$ of genus I or genus II is irreducible if and only if the associated homogeneous convex cone V is irreducible.

The aim of the present paper is to classify the convex cones of dimension seven up to thirteen.

The paper contains four parts. The first part is the introduction.

The second part deals with the general theory of convex cones and definitions of Siegel domains genus I.

The third part deals with the general theory of the graphic of skeletons.

In the fourth part we give the table of the convex cones and their graphic skeletons of dimension seven up to thirteen.

The classification of cones and skeletons of dimension seven up to thirteen has been studied in [1], [4], [5]. But this classification is not complete. We give a complete classification of these cones and skeletons. This classification contains 29 non-isomorphic cases.

2. Convex cones.

Definition 2.1. *A subset V in n-dimensional Euclidean real space $\mathbb{R}^n$ is called a cone, if together with a point of V it contains the half line connecting it to the origin.*

Definition 2.2. *The set*

$$D(V) = \{Z = X + iY \in \mathbb{C}^n \mid X \in \mathbb{R}^n,\ Y \in V\}$$

is called Siegel domain genus I.

It has been proved that any Siegel domain $S = D(V)$ of genus I is analytically equivalent to a bounded domain. Using the fact that cone V does not comprise a whole line, we can prove the existence of a coordinate system $(y_1,\ y_2,\ \ldots\ y_n)$ in $\mathbf{R}^n$ such that V lies inside the octant $y_1 > 0,\ y_2 > 0,\ \ldots\ y_n > 0$. Then it follows S is included in the product of n-discs.

The subset of V denoted by S_k and defined by $S_k = \{Z = X\}$ is called skeleton of S.

3. Graphic representation of skeletons. We consider a skeleton S_k in $\mathbf{R}^n$ and a regular n-polygon in $\mathbf{R}^2$. On each of its vertices we construct a tine circle. By a regular 1-polygon (resp. 2-polygon) we mean a point (resp. a line segment).

Let us number these circles counterclockwise starting from the vertex at the upper left corner. The i-th circle is called the i-th vertex or simply i-vertex. Some of these circles may be joined by line segments. By notation $i \sim j$ (resp. $i \not\sim j$) we mean that vertices i and j are joined (resp. not joined) by line segment.

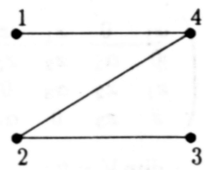
For the connection of two vertices the following statements hold:

To each line segment $\bar{ij}$ that joins two verices i and j we attach a positive integer n_{ij} with the property: if $i < j < k$, $i \sim j$ and $j \sim k$, then $\max(n_{ij}, n_{jk}) \leq n_{ik}$. The above construction by shape is called graphic skeleton of type I.

4. Theorem 4.1. *The complete classification of the convex cones and their graphic skeletons are given in the following table:*

4.1. $$V = \left\{ \begin{pmatrix} \alpha_1 & 0 & 0 & x_1 \\ 0 & \alpha_2 & x_2 & x_3 \\ 0 & x_2 & \alpha_3 & 0 \\ x_1 & x_3 & 0 & \alpha_4 \end{pmatrix} \right\} > 0,$$

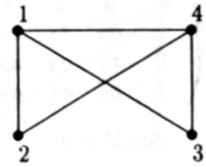
$\dim V = 7.$



$n_{14} = 1$
$n_{24} = 1$
$n_{23} = 1$

4.2. $$V = \left\{ \begin{pmatrix} \alpha_1 & x_1 & x_2 & x_3 \\ x_1 & \alpha_2 & 0 & x_5 \\ x_2 & 0 & \alpha_3 & x_4 \\ x_3 & x_5 & x_4 & \alpha_4 \end{pmatrix} \right\} > 0,$$

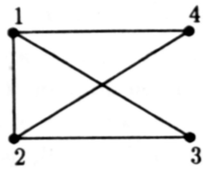
$\dim V = 9.$



$n_{14} = 1$
$n_{12} = 1$
$n_{13} = 1$
$n_{24} = 1$
$n_{34} = 1$

4.3. $$V = \left\{ \begin{pmatrix} \alpha_1 & x_1 & x_2 & x_3 \\ x_1 & \alpha_2 & x_4 & x_5 \\ x_2 & x_4 & \alpha_3 & 0 \\ x_3 & x_5 & 0 & \alpha_4 \end{pmatrix} \right\} > 0,$$

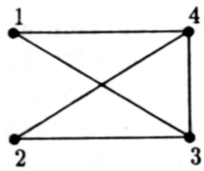
$\dim V = 9.$



$n_{14} = 1$
$n_{12} = 1$
$n_{13} = 1$
$n_{24} = 1$
$n_{23} = 1$

4.4 $$V = \left\{ \begin{pmatrix} \alpha_1 & 0 & x_1 & x_2 \\ 0 & \alpha_2 & x_3 & x_4 \\ x_1 & x_3 & \alpha_3 & x_5 \\ x_2 & x_4 & x_5 & \alpha_4 \end{pmatrix} \right\} > 0,$$

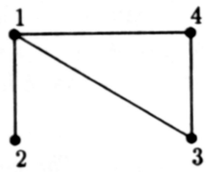
$\dim V = 9.$



$n_{14} = 1$
$n_{13} = 1$
$n_{24} = 1$
$n_{34} = 1$
$n_{23} = 1$

4.5 $$V = \left\{ \begin{pmatrix} \alpha_1 & x_1 & x_2 & z \\ x_1 & \alpha_2 & 0 & 0 \\ x_2 & 0 & \alpha_3 & x_3 \\ \bar{z} & 0 & x_3 & \alpha_4 \end{pmatrix} \right\} > 0,$$

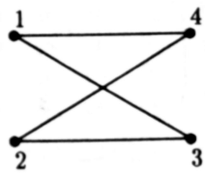
$\dim V = 9.$



$n_{14} = 2$
$n_{12} = 1$
$n_{13} = 1$
$n_{34} = 1$

4.6 $$V = \left\{ \begin{pmatrix} \alpha_1 & 0 & x_1 & z \\ 0 & \alpha_2 & x_2 & x_3 \\ x_1 & x_2 & \alpha_3 & 0 \\ \bar{z} & x_3 & 0 & \alpha_4 \end{pmatrix} \right\} > 0,$$

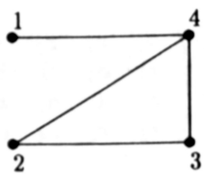
$\dim V = 9.$



$n_{14} = 2$
$n_{13} = 1$
$n_{24} = 1$
$n_{23} = 1$

4.7 $$V = \left\{ \begin{pmatrix} \alpha_1 & 0 & 0 & z \\ 0 & \alpha_2 & x_1 & x_2 \\ 0 & x_1 & \alpha_3 & x_3 \\ \bar{z} & x_2 & x_3 & \alpha_4 \end{pmatrix} \right\} > 0,$$

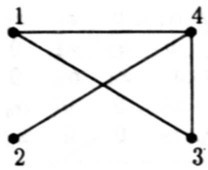
$\dim V = 9.$



$n_{14} = 2$
$n_{24} = 1$
$n_{34} = 1$
$n_{23} = 1$

4.8 $$V = \left\{ \begin{pmatrix} \alpha_1 & 0 & \boldsymbol{x}_1 & z \\ 0 & \alpha_2 & 0 & \boldsymbol{x}_2 \\ \boldsymbol{x}_1 & 0 & \alpha_3 & \boldsymbol{x}_3 \\ \bar{z} & \boldsymbol{x}_2 & \boldsymbol{x}_3 & \alpha_4 \end{pmatrix} \right\} > 0,$$

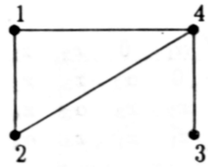
$\dim V = 9.$



$n_{14} = 2$
$n_{13} = 2$
$n_{24} = 1$
$n_{34} = 1$

4.9 $$V = \left\{ \begin{pmatrix} \alpha_1 & \boldsymbol{x}_1 & 0 & z \\ \boldsymbol{x}_1 & \alpha_2 & 0 & \boldsymbol{x}_2 \\ 0 & 0 & \alpha_3 & \boldsymbol{x}_3 \\ \bar{z} & \boldsymbol{x}_2 & \boldsymbol{x}_3 & \alpha_4 \end{pmatrix} \right\} > 0,$$

$\dim V = 9.$

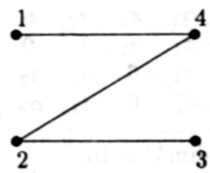


$n_{14} = 2$
$n_{12} = 1$
$n_{24} = 1$
$n_{34} = 1$

4.10 $$V = \left\{ \begin{pmatrix} \alpha_1 & 0 & 0 & C \\ 0 & \alpha_2 & \boldsymbol{x}_1 & \boldsymbol{x}_2 \\ 0 & \boldsymbol{x}_1 & \alpha_3 & 0 \\ C & \boldsymbol{x}_2 & 0 & \alpha_4 \end{pmatrix} \right\} > 0,$$

$\dim V = 9.$

$C = C(X_1, X_2, X_3)$

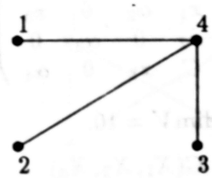


$n_{14} = 3$
$n_{24} = 1$
$n_{23} = 1$

4.11 $$V = \left\{ \begin{pmatrix} \alpha_1 & 0 & 0 & C \\ 0 & \alpha_2 & 0 & \boldsymbol{x}_1 \\ 0 & 0 & \alpha_3 & \boldsymbol{x}_2 \\ C & \boldsymbol{x}_1 & \boldsymbol{x}_2 & \alpha_4 \end{pmatrix} \right\} > 0,$$

$\dim V = 9.$

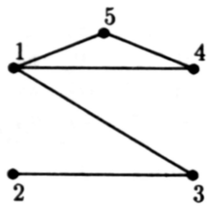
$C = C(X_1, X_2, X_3)$



$n_{14} = 3$
$n_{24} = 1$
$n_{34} = 1$

4.12 $V = \left\{ \begin{pmatrix} \alpha_1 & 0 & \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \\ 0 & \alpha_2 & \boldsymbol{x}_4 & 0 & 0 \\ \boldsymbol{x}_1 & \boldsymbol{x}_4 & \alpha_3 & 0 & 0 \\ \boldsymbol{x}_2 & 0 & 0 & \alpha_4 & \boldsymbol{x}_5 \\ \boldsymbol{x}_3 & 0 & 0 & \boldsymbol{x}_5 & \alpha_5 \end{pmatrix} \right\} > 0,$

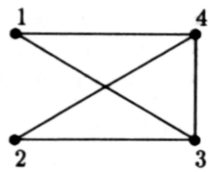
$\dim V = 10.$



$n_{15} = 1$
$n_{45} = 1$
$n_{14} = 1$
$n_{13} = 1$
$n_{23} = 1$

4.13 $V = \left\{ \begin{pmatrix} \alpha_1 & 0 & \boldsymbol{x}_2 & z_1 \\ 0 & \alpha_2 & \boldsymbol{x}_3 & \boldsymbol{x}_1 \\ \boldsymbol{x}_2 & \boldsymbol{x}_3 & \alpha_3 & \boldsymbol{x}_4 \\ \bar{z}_1 & \boldsymbol{x}_1 & \boldsymbol{x}_4 & \alpha_4 \end{pmatrix} \right\} > 0,$

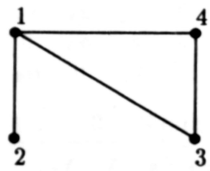
$\dim V = 10.$



$n_{14} = 2$
$n_{34} = 1$
$n_{13} = 1$
$n_{23} = 1$
$n_{24} = 1$

4.14 $V = \left\{ \begin{pmatrix} \alpha_1 & \boldsymbol{x}_1 & \boldsymbol{x}_2 & z_1 \\ \boldsymbol{x}_1 & \alpha_2 & 0 & 0 \\ \boldsymbol{x}_2 & 0 & \alpha_3 & z_2 \\ \bar{z}_1 & 0 & \bar{z}_2 & \alpha_4 \end{pmatrix} \right\} > 0,$

$\dim V = 10.$

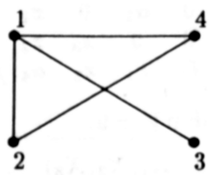


$n_{14} = 2$
$n_{12} = 1$
$n_{13} = 1$
$n_{34} = 1$

4.15 $V = \left\{ \begin{pmatrix} \alpha_1 & \boldsymbol{x}_1 & \boldsymbol{x}_2 & C \\ \boldsymbol{x}_1 & \alpha_2 & 0 & \boldsymbol{x}_3 \\ \boldsymbol{x}_2 & 0 & \alpha_3 & 0 \\ C & \boldsymbol{x}_3 & 0 & \alpha_4 \end{pmatrix} \right\} > 0,$

$\dim V = 10.$

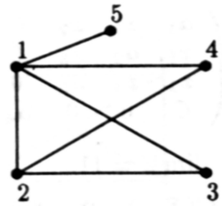
$C = C(X_1, X_2, X_3)$



$n_{14} = 3$
$n_{12} = 1$
$n_{24} = 1$
$n_{13} = 1$

4.16 $$V = \left\{ \begin{pmatrix} \alpha_1 & \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 & \boldsymbol{x}_5 \\ \boldsymbol{x}_1 & \alpha_2 & \boldsymbol{x}_4 & \boldsymbol{x}_6 & 0 \\ \boldsymbol{x}_2 & \boldsymbol{x}_4 & \alpha_3 & 0 & 0 \\ \boldsymbol{x}_3 & \boldsymbol{x}_6 & 0 & \alpha_4 & 0 \\ \boldsymbol{x}_5 & 0 & 0 & 0 & \alpha_5 \end{pmatrix} \right\} > 0,$$

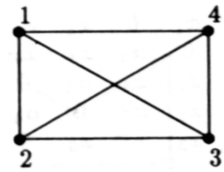
$\dim V = 11.$



$n_{15} = 1$
$n_{14} = 1$
$n_{12} = 1$
$n_{13} = 1$
$n_{24} = 1$
$n_{23} = 1$

4.17 $$V = \left\{ \begin{pmatrix} \alpha_1 & \boldsymbol{x}_1 & \boldsymbol{x}_2 & z_1 \\ \boldsymbol{x}_1 & \alpha_2 & \boldsymbol{x}_3 & \boldsymbol{x}_4 \\ \boldsymbol{x}_2 & \boldsymbol{x}_3 & \alpha_3 & \boldsymbol{x}_5 \\ \bar{z}_1 & \boldsymbol{x}_4 & \boldsymbol{x}_5 & \alpha_4 \end{pmatrix} \right\} > 0,$$

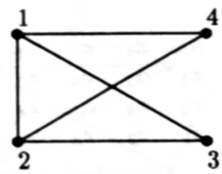
$\dim V = 11.$



$n_{14} = 2$
$n_{12} = 1$
$n_{34} = 1$
$n_{13} = 1$
$n_{23} = 1$
$n_{24} = 1$

4.18 $$V = \left\{ \begin{pmatrix} \alpha_1 & \boldsymbol{x}_1 & \boldsymbol{x}_2 & z_1 \\ \boldsymbol{x}_1 & \alpha_2 & \boldsymbol{x}_3 & z_2 \\ \boldsymbol{x}_2 & \boldsymbol{x}_3 & \alpha_3 & 0 \\ \bar{z}_1 & \bar{z}_2 & 0 & \alpha_4 \end{pmatrix} \right\} > 0,$$

$\dim V = 11.$

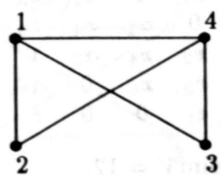


$n_{14} = 2$
$n_{12} = 1$
$n_{13} = 1$
$n_{23} = 1$
$n_{24} = 1$

4.19 $$V = \left\{ \begin{pmatrix} \alpha_1 & \boldsymbol{x}_1 & \boldsymbol{x}_2 & C \\ \boldsymbol{x}_1 & \alpha_2 & 0 & \boldsymbol{x}_3 \\ \boldsymbol{x}_2 & 0 & \alpha_3 & \boldsymbol{x}_4 \\ C & \boldsymbol{x}_3 & \boldsymbol{x}_4 & \alpha_4 \end{pmatrix} \right\} > 0,$$

$\dim V = 11.$

$C = C(X_1, X_2, X_3)$

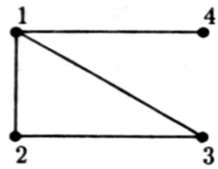


$n_{14} = 3$
$n_{12} = 1$
$n_{13} = 1$
$n_{34} = 1$
$n_{24} = 1$

4.20 $V = \left\{ \begin{pmatrix} \alpha_1 & \boldsymbol{x}_1 & \boldsymbol{x}_2 & C \\ \boldsymbol{x}_1 & \alpha_2 & \boldsymbol{x}_3 & 0 \\ \boldsymbol{x}_2 & \boldsymbol{x}_3 & \alpha_3 & 0 \\ C & 0 & 0 & \alpha_4 \end{pmatrix} \right\} > 0,$

$\dim V = 11.$

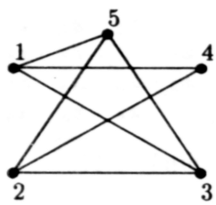
$C = C(X_1, X_2, X_3, X_4)$



$n_{14} = 4$
$n_{12} = 1$
$n_{13} = 1$
$n_{23} = 1$

4.21 $V = \left\{ \begin{pmatrix} \alpha_1 & 0 & \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \\ 0 & \alpha_2 & \boldsymbol{x}_4 & \boldsymbol{x}_7 & \boldsymbol{x}_5 \\ \boldsymbol{x}_1 & \boldsymbol{x}_4 & \alpha_3 & 0 & \boldsymbol{x}_6 \\ \boldsymbol{x}_2 & \boldsymbol{x}_7 & 0 & \alpha_4 & 0 \\ \boldsymbol{x}_3 & \boldsymbol{x}_5 & \boldsymbol{x}_6 & 0 & \alpha_5 \end{pmatrix} \right\} > 0,$

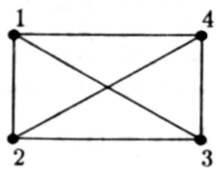
$\dim V = 12.$



$n_{15} = 1$
$n_{14} = 1$
$n_{13} = 1$
$n_{25} = 1$
$n_{24} = 1$
$n_{23} = 1$
$n_{35} = 1$

4.22 $V = \left\{ \begin{pmatrix} \alpha_1 & \boldsymbol{x}_1 & \boldsymbol{x}_2 & z_1 \\ \boldsymbol{x}_1 & \alpha_2 & \boldsymbol{x}_4 & z_2 \\ \boldsymbol{x}_2 & \boldsymbol{x}_4 & \alpha_3 & \boldsymbol{x}_3 \\ \bar{z}_1 & \bar{z}_2 & \boldsymbol{x}_3 & \alpha_4 \end{pmatrix} \right\} > 0,$

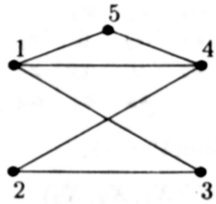
$\dim V = 12.$



$n_{14} = 2$
$n_{12} = 1$
$n_{34} = 1$
$n_{13} = 1$
$n_{23} = 1$
$n_{24} = 2$

4.23 $V = \left\{ \begin{pmatrix} \alpha_1 & 0 & \boldsymbol{x}_2 & \boldsymbol{x}_3 & z_1 \\ 0 & \alpha_2 & \boldsymbol{x}_4 & \boldsymbol{x}_6 & 0 \\ \boldsymbol{x}_2 & \boldsymbol{x}_4 & \alpha_3 & 0 & 0 \\ \boldsymbol{x}_3 & \boldsymbol{x}_6 & 0 & \alpha_4 & \boldsymbol{x}_1 \\ \bar{z}_1 & 0 & 0 & \boldsymbol{x}_1 & \alpha_5 \end{pmatrix} \right\} > 0,$

$\dim V = 12.$

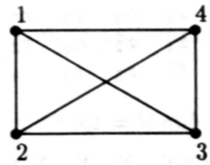


$n_{15} = 2$
$n_{45} = 1$
$n_{14} = 1$
$n_{13} = 1$
$n_{24} = 1$
$n_{23} = 1$

4.24 $$V = \left\{ \begin{pmatrix} \alpha_1 & x_5 & x_1 & C \\ x_5 & \alpha_2 & x_2 & x_3 \\ x_1 & x_2 & \alpha_3 & x_4 \\ C & x_3 & x_4 & \alpha_4 \end{pmatrix} \right\} > 0,$$

$\dim V = 12.$

$C = C(X_1, X_2, X_3)$

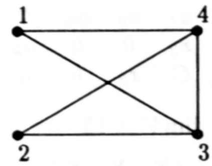


$n_{14} = 3$
$n_{12} = 1$
$n_{34} = 1$
$n_{13} = 1$
$n_{23} = 1$
$n_{24} = 1$

4.25 $$V = \left\{ \begin{pmatrix} \alpha_1 & 0 & x_1 & C \\ 0 & \alpha_2 & x_2 & x_3 \\ x_1 & x_2 & \alpha_3 & x_4 \\ C & x_3 & x_4 & \alpha_4 \end{pmatrix} \right\} > 0,$$

$\dim V = 12.$

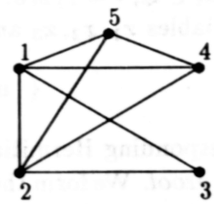
$C = C(X_1, X_2, X_3, X_4)$



$n_{14} = 4$
$n_{34} = 1$
$n_{13} = 1$
$n_{23} = 1$
$n_{24} = 1$

4.26 $$V = \left\{ \begin{pmatrix} \alpha_1 & x_6 & x_1 & x_2 & x_3 \\ x_6 & \alpha_2 & x_4 & x_7 & x_8 \\ x_1 & x_4 & \alpha_3 & 0 & 0 \\ x_2 & x_7 & 0 & \alpha_4 & x_5 \\ x_3 & x_8 & 0 & x_5 & \alpha_5 \end{pmatrix} \right\} > 0,$$

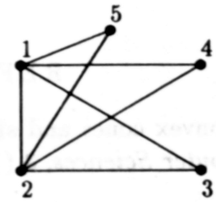
$\dim V = 13.$



$n_{15} = 1$
$n_{45} = 1$
$n_{25} = 1$
$n_{14} = 1$
$n_{13} = 1$
$n_{24} = 1$
$n_{12} = 1$
$n_{23} = 1$

4.27 $$V = \left\{ \begin{pmatrix} \alpha_1 & x_1 & x_2 & x_3 & z_1 \\ x_1 & \alpha_2 & x_4 & x_5 & x_6 \\ x_2 & x_4 & \alpha_3 & 0 & 0 \\ x_3 & x_5 & 0 & \alpha_4 & 0 \\ \bar{z}_1 & x_6 & 0 & 0 & \alpha_5 \end{pmatrix} \right\} > 0,$$

$\dim V = 13.$

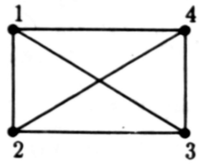


$n_{15} = 2$
$n_{25} = 1$
$n_{14} = 1$
$n_{13} = 1$
$n_{24} = 1$
$n_{12} = 1$
$n_{23} = 1$

$$4.28 \quad V = \left\{ \begin{pmatrix} \alpha_1 & x_4 & x_1 & C \\ x_4 & \alpha_2 & x_2 & x_5 \\ x_1 & x_2 & \alpha_3 & x_3 \\ C & x_5 & x_3 & \alpha_4 \end{pmatrix} \right\} > 0,$$

$$\dim V = 13.$$

$$C = C(X_1, X_2, X_3, X_4)$$

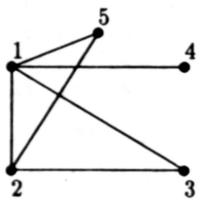


$n_{14} = 4$
$n_{34} = 1$
$n_{13} = 1$
$n_{23} = 1$
$n_{24} = 1$
$n_{12} = 1$

$$4.29 \quad V = \left\{ \begin{pmatrix} \alpha_1 & x_4 & x_1 & x_2 & C \\ x_4 & \alpha_2 & x_5 & 0 & x_3 \\ x_1 & x_5 & \alpha_3 & 0 & 0 \\ x_2 & 0 & 0 & \alpha_4 & 0 \\ C & x_3 & 0 & 0 & \alpha_5 \end{pmatrix} \right\} > 0,$$

$$\dim V = 13.$$

$$C = (X_1, X_2, X_3)$$



$n_{15} = 3$
$n_{25} = 1$
$n_{14} = 1$
$n_{13} = 1$
$n_{12} = 1$
$n_{23} = 1$

In the table $a_i \in \mathbf{R}$, $i = 1,2,3,4$, $x_j \in \mathbf{R}$, $j = 1,2,3,4,5$; $z \in \mathbf{C}$, $C(x_1, x_2, x_3)$ is a function of the variables x_1, x_2, x_3 and the symbol

$$\{ \text{ matrix } \} > 0$$

means that the corresponding Hermitian matrix has positive characteristic values.

Sketch of the proof. We form the set of all Hermitian and the set of all symmetric matrices of three parameters which cover the dimensions from seven up to nine. Such a matrix represents a convex cone if and only if their submatrices have determinant of the form cone. After that we form the graphic skeletons. This procedure gives a method for the complete classification.

REFERENCES

[1] DIMOU, G.E. Convex cones and skeletons up to dimension 8. *Institute of Mathematics and Computer Sciences*, **3** (1990) 239-243, Calcutta 7.

[2] DIMOU, G.E. Some properties of non-symmetric homogeneous domain dimension 7. Proceedings of Diff. Geometry and Topology Meeting. Bucharest, 1991.

[3] PYATETSKII-SHAPIRO, I.I. Automorphic Functions and the Geometry of Classical Domains. M., 1969.

[4] KANEYUKI, S., T. TSUJI. Classification of homogeneous bounded domains of lower dimension. *Nagoya Math.J.*, **53** (1974) 1-46.

[5] VINDERG, E.B. The theory of convex homogeneous cones. *Trudy Mosk. Math. Obsc.*, **12** (1963) 303-308.

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