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CALCULUS OF HIGHER ORDER AVERAGED MODULUS OF SMOOTHNESS IN L^p-NORM FOR CONVEX FUNCTIONS OF HIGHER ORDER

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ABSTRACT. Continuing the ideas in [3,4] the averaged moduli of smoothness of order n in L^p norm $(1 \le p < \infty)$ have been calculated for convex functions of order $n-1$ in this paper.

1. Introduction. The so-called averaged modulus of smoothness (or τ-modulus) first introduced by Sendov [14] has become a useful tool for giving estimates in a number of problems, such as quadrature formulae, numerical solutions of differential equations (see e.g. [15], [5]). τ-moduli have been treated in details in [15].

Let

$$M[a,b] = \{f; f \text{ is bounded and measurable on } [a,b]\}.$$

Definition 1.1 (see e.g. [13]). *Let $f \in M[a,b]$ and $\delta \ge 0$. The averaged modulus of smoothness (or τ-modulus) of order n and step δ in L^p-norm* $(1 \le p < \infty)$ *is given by*

$$\tau_n(f;\delta) = \|\omega_n(f,\cdot;\delta)\|_p$$

where $\|\cdot\|_p$ is the classical L^p-norm on $[a,b]$,

$$\omega_n(f,x;\delta) = \sup\{|\Delta_h^n f(t)|; t, t+nh \in I_n(x,\delta)\}, \quad h \in R,$$

$$\Delta_h^n f(t) = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i} f(t+ih) \quad \text{and} \quad I_n(x,\delta) = [x - n\delta/2, x + n\delta/2] \cap [a,b].$$

Remark. In Definition 1.1 h is assumed to be ≥ 0. This obviously follows from the fact that h is ≤ 0, then from $t, t+nh \in I_n(x,\delta)$, by denoting $h' = -h \ge 0$, $t' = t + nh = t - nh' \in I_n(x,\delta)$, we get

$$t' + nh' = t \in I_n(x,\delta) \quad \text{and} \quad |\Delta_h^n f(t)| = |\Delta_{h'}^n f(t')|.$$

Indeed

$$\Delta_{h'}^n f(t') = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i} f(t'+ih') = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i} f(t+(n-i)h) =$$

$$\sum_{j=0}^{n}(-1)^{j}\binom{n}{n-j} f(t+jh) = (-1)^n \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j} f(t+jh) = (-1)^n \Delta_h^n f(t).$$

A first estimate for the approximation error by positive linear operators in terms of τ-moduli was given by V.A.Popov in the following way:

Theorem 1.2 (see [7]). *Let $L : M[a,b] \longrightarrow M[a,b]$ be a positive linear operator with the following properties:*

$$L(1)(x) = 1, \quad L(t)(x) = x + \alpha(x), \quad L(t^2)(x) = x^2 + \beta(x), \quad x \in [a,b].$$

Let

$$\alpha = \sup\{|\beta(x) - 2x\alpha(x)|; x \in [a,b]\} \leq 1.$$

Then for $f \in M[a,b]$ and $1 \leq p < \infty$ the following estimate holds

$$\|f - L(f)\|_p \leq C\tau_1(f; \sqrt{\alpha})_p,$$

where C is an absolute constant ≤ 68.

Remark. Estimates in terms of τ-moduli of higher order can be found in [1,2], [11–13].

We have calculated the uniform moduli of smoothness of higher order for convex functions of higher order in two recent papers [3,4]. Taking into account this idea we calculate the higher order τ-moduli for convex functions of order $n-1$.

2. Calculus of τ-moduli of smoothness. First we need the following.

Definition 2.1 ([9], p.18). *Let n be an integer ≥ -1. A function $F : [a,b] \longrightarrow$ **R** is called convex (concave) of order n on $[a,b]$ if for any system of distinct points $x_1, \ldots, x_{n+2} \in [a,b]$ we have*

$$[x_1, \ldots, x_{n+2}; f] \geq 0 \quad (\leq 0 \text{ respectively}),$$

where by $[x_1, \ldots, x_{n+2}; f]$ the divided difference of f at $x_1, \ldots, x_{n+2}$ is denoted.

The set of all convex functions of order n will be denoted by $K_+^n[a,b]$.

The proofs of our main results require the following two lemmas.

Lemma 2.2. *For $-\infty < a < b < \infty$, $\delta \geq 0$ and $n \in N$ denote by*

$$I_n(x,\delta) = [x - n\delta/2, x + n\delta/2] \cap [a,b].$$

Then for all $\delta \in [0,(b-a)/n]$ *and all* $x \in [a,b]$ *we have:*

$$I_n(x,\delta) = \begin{cases} [a, x+n\delta/2] & \text{if } x \in [a, a+n\delta/2], \\ [x-n\delta/2, x+n\delta/2] & \text{if } x \in [a+n\delta/2, b-n\delta/2], \\ [x-n\delta/2, b] & \text{if } x \in [b-n\delta/2, b]. \end{cases}$$

Proof. Evidently we have

$$I_n(x,\delta) = [\max\{a, x-n\delta/2\}, \min\{x+n\delta/2, b\}].$$

Let us suppose that $x \in [a, a+n\delta/2]$. Since $x - n\delta/2 \le a + n\delta/2 - n\delta/2 = a$, we obtain $\max\{a, x-n\delta/2\} = a$. Then

$$x + n\delta/2 \le a + n\delta/2 + n\delta/2 = a + n\delta \le a + n(b-a)/n = b,$$

which implies $\min\{x+n\delta/2, b\} = x + n\delta/2$.

Finally we get

$$I_n(x,\delta) = [a, x+n\delta/2].$$

Now let $x \in [a+n\delta/2, b-n\delta/2]$. We have $x - n\delta/2 \ge a + n\delta/2 - n\delta/2 = a$ and $x + n\delta/2 \le b - n\delta/2 + n\delta/2 = b$, which directly implies that

$$I_n(x,\delta) = [x-n\delta/2, x+n\delta/2].$$

Finally, let us suppose that $x \in [b-n\delta/2, b]$. We obtain

$$x - n\delta/2 \ge b - n\delta/2 - n\delta/2 = b - n\delta \ge b - n(b-a)/n = a$$

and

$$x + n\delta/2 \ge b - n\delta/2 + n\delta/2 = b,$$

which immediately implies

$$I_n(x,\delta) = [x-n\delta/2, b]. \quad \square$$

Lemma 2.3. *Let* $n \in N$ *and let* $f \in C^n[A,B]$ *be such that* $f^{(n)}(x) \ge 0$ *for all* $x \in [A,B]$. *We have:*

$$\sup\{|\Delta_h^n f(t)|; t, t+nh \in [A,B]\} = \Delta_{(B-A)/n}^n f(A).$$

Proof. As in the Remark of Definition 1.1 it is easy to see that h can be considered ≥ 0. Let $t, t+nh \in [A,B]$. By the mean value theorem there exists $\xi \in (t, t+nh)$ such that

$$\Delta_h^n f(t) = [t, t+h, \ldots, t+nh; f](h^n n!) = h^n f^{(n)}(\xi) \ge 0,$$

where $|\Delta_h^n f(t)| = \Delta_h^n f(t)$.

Let us fix $t \in [A, B]$. Obviously for all $h \in [0, (B-t)/n]$ we have

$$0 \leq \Delta_h^n f(t) \leq \sup\{\Delta_h^n f(t); h \in [0, (B-t)/n]\}.$$

Let $F(h) = \Delta_h^n f(t)$, where $F : [0, (B-t)/n] \longrightarrow \mathbf{R}$. We have:

$$F'(h) = (\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(t+ih))' = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} i f'(t+ih) =$$

$$n \sum_{i=1}^{n} (-1)^{n-i} \binom{n-1}{i-1} f'(t+ih) = n \sum_{j=0}^{n-1} (-1)^{n-1-j} \binom{n-1}{j} f'(t+h+jh) =$$

$$n\Delta_h^{n-1} f'(t+h) = nh^{n-1} f^{(n)}(\eta),$$

where $\eta \in (t+h, t+nh) \subset [A, B]$.

Hence by hypothesis, $F'(h) \geq 0$ for all $h \in [0, (B-t)/n]$ and therefore we get

$$(1) \quad 0 \leq \Delta_h^n f(t) \leq \Delta_{(B-t)/n)}^n f(t), \quad \text{for all} \quad h \in [0, (B-t)/n] \quad (\text{and all} \quad t \in [A, B]).$$

Now denote $G(t) = \Delta_{(B-t)/n}^n f(t)$ where $G : [A, B] \longrightarrow \mathbf{R}$. As above we get:

$$G'(t) = (\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(t+i(B-t)/n))' = -\sum_{i=0}^{n-1} (-1)^{n-i-1} \binom{n-1}{i} f'(t+i(B-t)/n) =$$

$$-\Delta_{(B-t)/n}^{n-1} f'(t) = -((B-t)/n)^{n-1} f^{(n)}(\gamma),$$

where $\gamma \in [A, B]$.

Hence $G'(t) \leq 0$ for all $t \in [A, B]$ and consequently $G(t) \leq G(A)$, for every $t \in [A, B]$, which can be rewritten as follows:

$$(2) \qquad \Delta_{(B-t)/n}^n f(t) \leq \Delta_{(B-A)/n}^n f(A), \quad \text{for all} \quad t \in [A, B].$$

From (1) and (2) we derive

$$0 \leq \Delta_h^n f(t) \leq \Delta_{(B-A)/n}^n f(A) \quad \text{for all} \ t \ \text{and} \ h \ \text{satisfying} \ A \leq t \leq t+nh \leq B,$$

which proves the lemma. □

The first main result is

Corollary 2.4. *Let us suppose that* $n \in N$, $1 \leq p < \infty$ *and that* $f \in C^n[A, B]$ *satisfies* $f^{(n)}(x) \geq 0$, *for all* $x \in [a, b]$. *Then for all* $\delta \in [0, (b-a)/n]$ *we have:*

$$(3) \quad \tau_n(f;\delta)_p = (\int_a^{a+n\delta/2} (\Delta_{(x-a)/n+\delta/2}^n f(a))^p dx + \int_{a+n\delta/2}^{b-n\delta/2} (\Delta_\delta^n f(x-n\delta/2))^p dx +$$

$$\int_{b-n\delta/2}^{b} (\Delta^n_{(b-x)/n+\delta/2} f(x - n\delta/2))^p dx)^{1/p}.$$

Proof. Let $I_n(x,\delta) = [A_n(x), B_n(x)]$ where $I_n(x,\delta)$ is defined from Lemma 2.2. From Definition 1.1 and from Lemma 2.3 we obtain

$$\omega_n(f,x;\delta) = \Delta^n_{(B_n(x)-A_n(x))/n} f(A_n(x)), \quad x \in [a,b].$$

Hence from Definition 1.1 and from Lemma 2.2, and taking into account that

$$\int_a^b = \int_a^{a+n\delta/2} + \int_{a+n\delta/2}^{b-n\delta/2} + \int_{b-n\delta/2}^{b},$$

by simple calculus we immediately obtain (3). □

Now we can formulate the second main result.

Theorem 2.5. *Let us suppose that $n \in N$, $1 \le p < \infty$ and $f \in C[a,b] \cap K_+^{n-1}[a,b]$. Then for all $\delta \in [0,(b-a)/n]$ (3) holds.*

Proof. For $m \in N$ and $x \in [a,b]$ let us denote by $B_m(f)(x)$ the Bernstein polynomials defined by

$$B_m(f)(x) = (1/(b-a)^m) \sum_{k=0}^{m} \binom{m}{k} (x-a)^k (b-x)^{m-k} f(x_k),$$

where $x_k = a + k(b-a)/m$. It is well known that $f \in K_+^{n-1}[a,b]$ implies $B_m(f) \in K_+^{n-1}[a,b]$, i.e. $[B_m(f)]^{(n)}(x) \ge 0$, $x \in [a,b]$, $m \in N$ (see [10] or e.g. [8], p.125-126). Also, from $f \in [a,b]$ we obtain that $\lim_{m\to\infty} B_m(f) = f$ uniformly on $[a,b]$ (see e.g. [6]). Now applying Corollary 2.4 to $B_m(f)$, relation (3) holds by replacing f by $B_m(f)$. Then passing to limit with $m \to \infty$ and taking into account that $\lim_{m\to\infty} B_m(f) = f$ uniformly on $[a,b]$, we get

$$\lim_{m\to\infty} \tau_n(B_m(f);\delta)_p = E, \tag{4}$$

where E is the term on the right side in (3) (written for f).

But it is known (see e.g. [15]) that as function of f τ_n-modulus is a semi-norm, i.e.

$$\tau_n(f+g;\delta)_p \le \tau_n(f;\delta)_p + \tau_n(g;\delta)_p \quad \text{and} \quad \tau_n(\lambda f;\delta)_p = |\lambda|\tau_n(f;\delta), \quad \lambda \in \mathbf{R}.$$

This immediately implies

$$|\tau_n(f;\delta)_p - \tau_n(g;\delta)_p| \le \tau_n(f-g;\delta)_p, \quad f,g \in C[a,b].$$

Since

$$\tau_n(f-g;\delta)_p \le 2^n \|f-g\|(b-a)^{1/p},$$

where by $\|\cdot\|$ the uniform norm is denoted and taking into account $g = B_m(f)$ we get

$$|\tau_n(f;\delta)_p - \tau_n(B_m(f);\delta)_p| \le 2^n \|f - B_m(f)\| (b-a)^{1/p}. \tag{5}$$

Passing to limit with $m \to \infty$ in (5) and taking into account (4) we get $\tau_n(f;\delta)_p = E$, which proves (3). □

Remark. For $n = 1$, in Theorem 2.5 f is supposed to be continuous and monotonous on $[a,b]$. However it can be proved that continuity of f on $[a,b]$ is not necessary. For example we obtain:

Corollary 2.6. *Let $f : [a,b] \longrightarrow \mathbf{R}$ increasing on $[a,b]$ and let $1 \le p < \infty$. For all $\delta \in [0, b-a]$ we have*

$$\tau_1(f;\delta)_p = (\int_a^{a+\delta/2} (f(x+\delta/2) - f(a))^p dx + \tag{6}$$

$$\int_{a+\delta/2}^{b-\delta/2} (f(x+\delta/2) - f(x-\delta/2))^p dx + \int_{b-\delta/2}^{b} (f(b) - f(x-\delta/2))^p dx)^{1/p}.$$

Proof. Since f is increasing on $[a,b]$, from Lemma 2.2 and from Definition 1.1 for $n = 1$ we get (replacing ω_1 by ω):

$$\omega(f,x;\delta) = \begin{cases} f(x+\delta/2) - f(a), & x \in [a, a+\delta/2], \\ f(x+\delta/2) - f(x-\delta/2), & x \in [a+\delta/2, b-\delta/2], \\ f(b) - f(x-\delta/2), & x \in [b-\delta/2, b]. \end{cases} \tag{7}$$

But

$$\int_a^b = \int_a^{a+\delta/2} + \int_{a+\delta/2}^{b-\delta/2} + \int_{b-\delta/2}^{b},$$

therefore from Definition 1.1 and from (7) we get (6) immediately. □

Remarks. 1). Since for $n > 1$, $f \in K_+^{n-1}[a,b]$ implies that f is continuous on the open interval (a,b) (see [9], p.27), condition $f \in C[a,b]$ in Theorem 2.5 one reduces to the continuity of f at a and b.

2). Let us consider, for example, $f : [0,\pi/2] \longrightarrow \mathbf{R}$ defined by $f(x) = \sin x$. Since f is increasing on $[0,\pi/2]$, by Corollary 2.6 (for $p = 1$), we get:

$$\tau_1(\sin;\delta)_1 = \int_0^{\delta/2} \sin(x+\delta/2)dx + \int_{\delta/2}^{(\pi-\delta)/2} (\sin(x+\delta/2) - \sin(x-\delta/2))dx +$$

$$\int_{(\pi-\delta)/2}^{\pi/2} (1 - \sin(x-\delta/2))dx = -\cos(x+\delta/2)|_0^{(\pi-\delta)/2} + \cos(x-\delta/2)|_{\delta/2}^{\pi/2} + \delta/2 =$$

$$-\cos(\pi/2) + \cos(\delta/2) + \cos(\pi/2 - \delta/2) - \cos 0 + \delta/2 = \cos(\delta/2) + \sin(\delta/2) + \delta/2 - 1.$$

Clearly, Corollary 2.6 can be used to calculate the τ-moduli of smoothness for many other elementary functions.

3). By Remark 2 and Theorem 1.2 we get the estimate:

$$\| \sin - L(\sin)\|_1 \leq C(\sin(\sqrt{\alpha}/2) + \cos(\sqrt{\alpha}/2) + \sqrt{\alpha}/2 - 1).$$

4). We think that Corollary 2.6 can be used, for example, to improve (for monotonous functions) the absolute constant C which appears in the estimate of Theorem 1.2.

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