Serdica
Bulgariacae mathematicae publicationes

Сердика
Българско математическо списание

SERDICA — Bulgaricae
mathematicae publicationes
18 (1992) 286-292

ON SENDOV'S BOUND FOR ZEROS OF A POLYNOMIAL

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ABSTRACT. In this paper we obtain upper and lower bounds for the positive roots of algebraic equations, then the results are applied to determine the R-order of convergence of iterative numerical processes.

1. Introduction. A method, originally due to Bl.Sendov [2] for simultaneous approximate calculation of all positive roots of the equation

$$f(x) = a_0 + a_1x + \ldots + a_mx^m = 0 \tag{1}$$

is based on the following theorem given by H.Poincare [1] : Let f be a polynomial with real coefficients. If k is a large natural number, then the number of positive roots of the equation (1) is equal to the number of variations in sign in the sequence of nonnegative coefficients of the polynomial

$$g(x) = (1+x)^k f(x).$$

Let $0 < x_1 \leq x_2 \leq \ldots \leq x_p$, $p \leq m$ be positive roots of equation (1), $(a_0 > 0,\ a_m \neq 0)$ and

$$(1+x)^k f(x) = \sum_{\nu=0}^{m+k} b_k(\nu)x^\nu. \tag{2}$$

Let us denote by $\nu_k(1)$ the smallest integer number, for which $b_k(\nu_k(1)) \geq 0$ and $b_k(\nu_k(1)+1) < 0$, $b_k(0) = a_0 > 0$. In general, $\nu_k(s)$ is the smallest integer number, for which $(-1)^{s-1}b_k(\nu_k(s)) \geq 0$, $(-1)^{s-1}b_k(\nu_k(s)+1) < 0$, $\nu_k(s) > \nu_k(s-1)$. Then we obtain the numbers

$$\nu_k(1), \nu_k(2), \ldots, \nu_k(s). \tag{3}$$

According to Poincare, there exists such number $k_0 = k_0(f)$, that for every $k \geq k_0$ we have $s_k = p$, where p is the number of positive roots of the equation (1).

Theorem A (Bl.Sendov [2]). *The numbers* (3) *satisfy*

$$\frac{\nu_k(s)}{k-\nu_k(s)+1} \le \xi(k,\nu,s) \le \frac{\nu_k(s)+1}{k-\nu_k(s)}, \tag{4}$$

$$\lim_{k\to\infty} \frac{\nu_k(s)}{k-\nu_k(s)+1} = \lim_{k\to\infty} \xi(k,\nu,s) = x_s, \tag{5}$$

$$f(x_s) = 0, \quad s = 1,2,\ldots,p.$$

Another approach is given in [3].

We consider the polynomial equation

$$P_{n,p,q}(s) = s^n - (p+1)\sum_{i=0}^{n-1} q^i s^{n-i-1}, \quad p \ge 0, q > 0, n \ge 2. \tag{6}$$

By Descartes' rule, $P_{n,p,q}(s)$ has exactly one positive root $\sigma_{p,q}^{(n)}$.

The case $p \ge 0$, $q = 1$ has been treated by Traub in [4] (see,also [5]) and the case $p = 1$, $q = 2$ was considered by Scharlach in [6]. The following theorems are used:

Theorem B (J.Herzberger [7]). *For* $n > q/(p+1)$ *it holds*

$$\frac{n}{n+1}(p+q+1) < \sigma_{p,q}^{(n)} < p+q+1. \tag{7}$$

Theorem C (N.Kjurkchiev [8]).

$$p+q+1-\frac{(p+1)q^n}{(p+q+1)^n}(1+1/n)^n < \sigma_{p,q}^{(n)} < p+q+1-\frac{(p+1)q^n}{(p+q+1)^n}. \tag{8}$$

The problem of determining the bounds for $\sigma_{p,q}^{(n)}$ is considered in [9-14].

2. Main results.

Theorem 1. *Using the same notation as in Sendov's theorem the next inequalities hold*

$$\frac{k}{k+1}\frac{\nu}{k-\nu+1} \le \xi \le \frac{\nu+1}{k-\nu}. \tag{9}$$

Proof. According to Bl.Sendov, there exist such numbers $\delta = \delta(f)$, $\delta \in (0,1)$ and $N = N(f)$, that for $k > N$ we have $\operatorname{sgn} b_k(\nu) = \operatorname{sgn} a_0$, $0 \le \nu \le \delta k$,$\operatorname{sgn} b_k(\nu) = \operatorname{sgn} a_m$, $(1-\delta)k \le \nu \le m+k$. If $0 < \delta < 1$ and $k\delta \le \nu \le (1-\delta)k$, then

$$\frac{b_k(\nu)}{\binom{k}{\nu}} = f\Big(\frac{k}{k+1}\frac{\nu}{k-\nu+1}\Big) + C_1\theta_{k,\nu}\frac{1}{k+1}, \tag{10}$$

where $C_1 = C_1(f)$ does not depend on k and ν and $|\theta_{k,\nu}| \leq 1$. From (2) we obtain

$$\frac{b_k(\nu)}{\binom{k}{\nu}} = f\Big(\frac{k}{k+1}\frac{\nu}{k-\nu+1}\Big) + \sum_{i=1}^{m} B(k,\nu,i)a_i \tag{11}$$

where

$$B(k,\nu,i) = \frac{\nu}{k-\nu+1}\cdot\frac{\nu-1}{k-\nu+2}\cdots\frac{\nu-(i-1)}{k-\nu+i} - \Big(\frac{k}{k+1}\frac{\nu}{k-\nu+1}\Big)^i.$$

Letting $a+1 = \nu/(k-\nu+1) \leq (1-\delta)/\delta$ we get

$$|B(k,\nu,i)| \leq \Big|(a+1)^i - \Big(\frac{k}{k+1}(a+1)\Big)^i\Big| \leq i\Big(\frac{1-\delta}{\delta}\Big)^i\frac{1}{k+1}. \tag{12}$$

The equality (10) follows from (11) and (12) by setting

$$C_1 = C_1(f) = \sum_{i=1}^{m} i\Big(\frac{1-\delta}{\delta}\Big)^i |a_i|.$$

From (10) and $(-1)^{s-1}b_k(\nu_k(s)) \geq 0$, $(-1)^{s-1}b_k(\nu_k(s)+1) < 0$, we obtain

$$(-1)^{s-1}\Big(f\Big(\frac{k}{k+1}\frac{\nu}{k-\nu+1}\Big) + C_1\theta_1\frac{1}{k+1}\Big) \geq 0,$$

$$(-1)^{s-1}\Big(f\Big(\frac{k}{k+1}\frac{\nu+1}{k-\nu}\Big) + C_1\theta_2\frac{1}{k+1}\Big) < 0,$$

where $\theta_1, \theta_2 \in [-1,1]$.

The theorem is proved. □

The estimates (4) and (9) are in some sense analogous to the Herzberger's estimates (7) (see, also (8)) for localization of the unique positive root of the equation (6).

In the following theorem, we shall derive some upper and lower bounds for the root $\sigma_{p,q}^{(n)}$ of the equation (6).

Theorem 2. *Using the same notations as in Herzberger's theorem the next inequalities hold*

$$p+q+1 - A_{p,q}^{(n)}\frac{(p+1)q^n}{(p+q+1)^n} < \sigma_{p,q}^{(n)} < p+q+1 - B_{p,q}^{(n)}\frac{(p+1)q^n}{(p+q+1)^n}, \tag{13}$$

where

$$A_{p,q}^{(n)} = \frac{2}{1+\sqrt{1-4n(p+1)q^n/(p+q+1)^{n+1}}}, \quad B_{p,q}^{(n)} = \Big(1+\varepsilon_{p,q}^{(n)}\Big)^n,$$

$$\varepsilon_{p,q}^{(n)} = \frac{2(p+1)q^n((p+q+1)^{n+1} - (p+1)q^n(n+1))^{-1}}{1 + \sqrt{1 - 4n(p+1)^2 q^{2n}/((p+q+1)^{n+1} - (p+1)q^n(n+1))^2}}.$$

P r o o f. Let $m_{p,q}^{(n)}$, $M_{p,q}^{(n)}$ be positive numbers and

$$P_{n,p,q}(m_{p,q}^{(n)}) < 0, \quad P_{n,p,q}(M_{p,q}^{(n)}) > 0.$$

Then

$$m_{p,q}^{(n)} < \sigma_{p,q}^{(n)} < M_{p,q}^{(n)}.$$

Let us suppose that

$$m_{p,q}^{(n)} = p + q + 1 - A_{p,q}^{(n)} \frac{(p+1)q^n}{(p+q+1)^n}, \quad P_{n,p,q}(m_{p,q}^{(n)}) < 0,$$

i.e.

$$\left(1 - A_{p,q}^{(n)} \frac{(p+1)q^n}{(p+q+1)^{n+1}}\right)^n A_{p,q}^{(n)} > 1. \tag{14}$$

Therefore

$$\left(1 - A_{p,q}^{(n)} \frac{(p+1)q^n}{(p+q+1)^{n+1}}\right)^n A_{p,q}^{(n)} > \left(1 - nA_{p,q}^{(n)} \frac{(p+1)q^n}{(p+q+1)^{n+1}}\right) A_{p,q}^{(n)}$$

the inequality (14) is true, when the inequality

$$n(p+1)q^n \left(A_{p,q}^{(n)}\right)^2 - (p+q+1)^{n+1} A_{p,q}^{(n)} + (p+q+1)^{n+1} \leq 0 \tag{15}$$

is true. Define

$$M_{p,q}^{(n)} = p + q + 1 - B_{p,q}^{(n)} \frac{(p+1)q^n}{(p+q+1)^n}$$

so that

$$P_{n,p,q}(M_{p,q}^{(n)}) > 0,$$

i.e.

$$\left(1 - B_{p,q}^{(n)} \frac{(p+1)q^n}{(p+q+1)^{n+1}}\right)^n B_{p,q}^{(n)} < 1, \quad 1 \leq B_{p,q}^{(n)} < 2.$$

We take

$$B_{p,q}^{(n)} = (1 + \varepsilon_{p,q}^{(n)})^n.$$

Then

$$\left(1 - (1 + \varepsilon_{p,q}^{(n)})^n \frac{(p+1)q^n}{(p+q+1)^{n+1}}\right)(1 + \varepsilon_{p,q}^{(n)}) < 1.$$

From

$$\Big(1-(1+n\varepsilon_{p,q}^{(n)})\frac{(p+1)q^n}{(p+q+1)^{n+1}}\Big)(1+\varepsilon_{p,q}^{(n)})<1$$

we have

$$(16)\ n(p+1)q^n\left(\varepsilon_{p,q}^{(n)}\right)^2-\varepsilon_{p,q}^{(n)}((p+1)q^n(n+1)-(p+q+1)^{n+1})+(p+1)q^n>0.$$

From (15) and (16) we obtain $A_{p,q}^{(n)}$, $\varepsilon_{p,q}^{(n)}$ (resp.$B_{p,q}^{(n)}$) in (13). This completes the proof of the theorem. □

3. Numerical results. The bounds (8) and (13) are tested numerically.

Table of $\sigma_{p,q}^{(n)}$ ($p=q=1$, $n=3,4,6,7$)

Bound(8):			Bound(13):		
2.824417010	$<\sigma_{1,1}^{(3)}<$	2.925925926	2.919435246	$<\sigma_{1,1}^{(3)}<$	2.919654783
2.939718364	$<\sigma_{1,1}^{(4)}<$	2.975308642	2.974437378	$<\sigma_{1,1}^{(4)}<$	2.974449635
2.993081958	$<\sigma_{1,1}^{(6)}<$	2.997256516	2.997241295	$<\sigma_{1,1}^{(6)}<$	2.997241344
2.997671239	$<\sigma_{1,1}^{(7)}<$	2.999085505	2.999083546	$<\sigma_{1,1}^{(7)}<$	2.999083547

Remark. Then the results are applied so as to determine the R-order of convergence of iterative numerical processes. Let IP denote a general iterative process that produces a sequence of approximations $\{t^{(k)}\}$ with the limit point t^*. For the errors

$$\eta^{(k)}=\|t^*-t^{(k)}\|\geq 0$$

it is often possible to derive a difference inequality like

$$(17)\qquad \eta^{(k+1)}\leq\gamma\prod_{i=0}^{n}\left(\eta^{(k-i)}\right)^{q^i(p+1)},\quad p,q,\gamma>0.$$

According to J.Schmidt in [15] the recurrence (17) has R-order of convergence – $O_R(IP,t^*)$ (see J.Ortega and W.Rheinboldt [16]) of at least $\sigma_{p,q}^{(n+1)}$, where $\sigma_{p,q}^{(n+1)}$ is the unique positive root of the polynomial $P_{n+1,p,q}(s)$.

Using the estimate (13) we may establish

$$O_R(0,\{\eta^{(k)}\})\geq\sigma_{p,q}^{(n+1)}>p+q+1-A_{p,q}^{(n+1)}\frac{(p+1)q^{n+1}}{(p+q+1)^{n+1}}.$$

The R-order is one of the most important measures to characterize the speed of convergence of sequences obtained by iterative processes in normed spaces (see W.Burmeister and J.Schmidt [17], J.Herzberger [18], F.Potra [19], N.Kjurkchiev [20]).

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Received 11.06.1992