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$ZFI_R + CT$ IS EQUICONSISTENT WITH ZFI_R

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ABSTRACT. The article is the answer to Friedman's question: "... is it known whether ZFI_R is equiconsistent with ZFI_R+ "Every $f \in \omega^\omega$ is recursive"?" The answer is positive and we use the models of realizability type.

The present article is an answer to the question from [1] (see page 2: "... is it known whether ZFI_R is equiconsistent with ZFI_R+ "Every $f \in \omega^\omega$ is recursive"). The answer is positive and was announced by the author first in [2] and then in [3]. This result is possible if we use the models of realizability type and fails in Heyting-valued models (see [4]). In the paper we give the proof for a set theory with two kinds of variables but it can be extended to the usual set theory with only set variables (see also [2]).

1. The language of our theory (we denote it by ZFI_{R2}) consists of variables over natural numbers $a_1^0, a_2^0, \ldots$ and over sets $a_1^1, a_2^1, \ldots$, predicate symbols $\in^0, \in^1$, symbols $0, \perp$, symbols of logical connectives and quantifiers and function symbols for all primitive recursive functions. The notions of a term and a functor are defined in the usual way. The exact wordings of postulates are in [6], but we use the replacement instead the collection and omit Axiom 9 from V, see [6], page 239. Our notations are standard. We use letters x, y, z, u, v for designation of sets and m, n, k, e for designation of natural numbers. Also we identify a partial recursive function with its Gödel's numbers.

2. The main difficulty is the construction of a universum Δ in the spirit of [6], but we must use only replacement in the metatheory because (see [1]) the theory ZFI_{R2} is deductively weaker than ZFI_{R2} with collection. The main thing in our construction is the use of the transfinite induction and the transfinite recursion which is derived by induction. Such constructions are made in [4] and [5]. Here we modify such constructions for realizability models. The definition of an ordinal also is in [4] and [5].

Let On be the class of ordinals and ω be the set of natural numbers. We suppose that for every $\beta < \alpha$ relations are defined:

$$x\ ext_\beta h, \quad x \overset{\vec{h}}{\underset{\beta}{\sim}} y,$$

and the set Δ_β, where $\vec{h}$ and h are fourtuples of natural numbers and a natural number respectively. We define $\Delta_{<\alpha} = \cup_{\beta<\alpha}\Delta_\beta$;

$$x \in \Delta_\alpha \rightleftharpoons x \subseteq [(\omega \times \omega) \cup (\omega \times \Delta_{<\alpha})] \wedge \exists h \forall n \vec{h} y z \beta(\beta < \alpha \wedge y \overset{\vec{h}}{\underset{\beta}{\sim}} z$$

$$\wedge \langle n, y\rangle \in x \rightarrow !h(n, \vec{h}) \wedge \langle h(n, \vec{h}), z\rangle \in x); x\ ext_\alpha \rightleftharpoons x \in \Delta_\alpha;$$

$$x \overset{\vec{h}}{\underset{\alpha}{\sim}} y \rightleftharpoons (x \in \Delta_\alpha) \wedge (y \in \Delta_\alpha) \wedge \forall nkzfg\{((\langle n, k\rangle \in x \rightarrow !h_1(n, k) \wedge$$

$$\langle h_1(n, k), k\rangle \in y) \wedge ((\langle n, k\rangle \in y \rightarrow !h_2(n, k) \wedge \langle h_2(n, k), k\rangle \in x) \wedge$$

$$[xext_\alpha f \wedge yext_\alpha g \rightarrow ((\langle n, z\rangle \in x \rightarrow !h_3(n, f) \wedge \langle h_3(n, f)_1, z\rangle \in y \wedge$$

$$yext_\alpha h_3(n, f)_2) \wedge ((\langle n, z\rangle \in y \rightarrow !h_4(n, g) \wedge \langle h_4(n, g)_1, z\rangle \in x \wedge$$

$$xext_\alpha h_4(n, g)_2]\}; x \sim y \rightleftharpoons \exists \alpha \exists \vec{h}(x \overset{\vec{h}}{\underset{\alpha}{\sim}} y);$$

and we also define $\Delta = \cup_{\alpha \in On}\Delta_\alpha$, $x^* = \{y | \exists k\langle k, y\rangle \in x\}$, $x^{**} = \{n | \exists k\langle k, n\rangle \in x\}$.

We define the range of a set from $\Delta : rng(x) = \cup\{rng(y) + 1 | y \in x^*\}$. Now we prove the following statements by induction:

a) $\forall x \in \Delta.rng(x) \in On$

b) $\forall x \in \Delta.x \in \Delta_{rng(x)+1}$

c) $x \in y^* \Longrightarrow rng(x) < rng(y)$.

3. The definition of realizability. We consider only points which contain the formulas with variables over sets. So let $f : \omega \longrightarrow \omega$, $g : \omega \longrightarrow \Delta$ be valuations for numerical and set theoretic variables.

1. $R(e, f, g, s = t) \rightleftharpoons |s(f)| = |t(f)|$, where $|s(f)|$ is the meaning of term s with valuation f (it is very easy to define by induction on term's construction).
2. $R(e, f, g, s \in a_n^1) \rightleftharpoons \langle e, |s(f)|\rangle \in g(n)$
3. $R(e, f, g, a_m^1 \in a_n^1) \rightleftharpoons \langle e_1, g(m)\rangle \in g(n) \wedge g(n)exte_2$
4. $R(e, f, g, \forall x^1\varphi(x^1)) \rightleftharpoons \forall x \in \Delta.R(e, f, g_x^n, \varphi(a_n^1))$
5. $R(e, f, g, \exists x^1\varphi(x^1)) \rightleftharpoons \exists x \in \Delta.R(e, f, g_x^n, \varphi(a_n^1))$.

4. The proof of axiom realizability. The verification of the postulates of predicate logic and the axioms of arithmetic proceeds in the same way as for Kleene's original realizability (see also [7]).

In the group of set-theoretic axioms we consider realizabilities of replacement and the transfinite induction on sets. The realizabilities of other axioms are to be found in [7].

a) realizability of the $\in$ — induction on sets

$$\forall x^1[\forall y^1(y^1 \in x^1 \to \varphi(y^1)) \to \varphi(x^1)] \to \forall x^1 \varphi(x^1).$$

Let p realize the premiss. We have (by the recursion theorem) a number I such that $I(p) \cong p(u)$, where $u(k) \cong I(p)$. If $R(k, f, g^{nm}_{xy}, a^1_m \in a^1_n)$, then $rng(y) < rng(x)$ and by induction hypothesis $R(I(p), f, g^{nm}_{xy}, \varphi(a^1_m))$, i.e. $R(u, f, g^n_x, \forall y^1(y^1 \in a^1_n \to \varphi(y^1)))$ and hence $R(p(u) \cong I(p), f, g^n_x, \varphi(a^1_n))$. We employ the external transfinite induction and obtain that

$$\forall x \in \Delta . R(I(p), f, g^n_x, \varphi(a^1_n)) \rightleftharpoons R(I(p), f, g, \forall x^1 \varphi(x^1)).$$

b) realizability of the replacement axiom:

$$\forall x^0 \in a^1_k \exists! y^1 \varphi(x^0, y^1) \land \forall x^1 \in a^1_k \exists! y^1 \psi(x^1, y^1) \to$$

$$\exists z^1[\forall x^0 \in a^1_k \exists y^1 \in z^1 \varphi(x^0, y^1) \land \forall x^1 \in a^1_k \exists y^1 \in z^1 \psi(x^1, y^1)].$$

We note that if $R(k, f, g^{nm}_{xy}, a^1_m = a^1_n)$, then $x \sim y$. Here we consider a simplified variant of the replacement which consists of only a "set" part. Let

$$R(k, f, g^k_a, \forall x^1 \in a^1_k \exists y^1[\psi(x^1, y^1) \land \forall u^1(\psi(x^1, u^1) \to u^1 = y^1)]);$$

if $x \in a^*$, then $\exists e \exists y \in \Delta . R(e, f, g^{nm}_{xy}, \psi(a^1_n, a^1_m))$. Now let us consider $B_x = \{u \in \Delta \mid \exists e . R(e, f, g^{nm}_{xy}, \psi(a^1_n, a^1_m))\}$. Since $u \sim y$, then that class is a set. Employing the axioms of separation and extensionality we have:

$$\forall x \in a^* \exists! B_x \forall y \in \Delta[y \in B_x \leftrightarrow \exists e . R(e, f, g^{nm}_{xy}, \psi(a^1_n, a^1_m))].$$

Then we use external replacement and obtain that

$$\exists v \forall x \in a^* \exists B_x \in v \forall y \in \Delta(y \in B_x \leftrightarrow \exists e . R(e, f, g^{nm}_{xy}, \psi(a^1_n, a^1_m))).$$

Let $z = \{0\} \times U_v$. This set is used for the proof of the realizability of replacement. Then we must name a number which realizes the conclusion of the replacement but it is constructed just as for the collection (see [7], page 204, point 3.6).

5. The fundamental result. So we have proved the following

Theorem. *If $ZFI_{R2} + CT \vdash \varphi$, then there exists a number e such that $\forall f g R(e, f, g, \varphi)$.*

Corollary. *As our proof was embedded in ZFI_{R2}, we have that $ZFI_{R2} + CT$ is consistent relative to ZFI_{R2}.*

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