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APPLICATIONS OF FRACTIONAL CALCULUS TO A CLASS OF INTEGRO-DIFFERENTIAL EQUATIONS

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Dedicated to Academician Ljubomir Iliev on the Occasion of his Eightieth Birthday

1. Introduction. In various articles [9], [10], [12], [13], [14], [19], [20], [22], [23], [30], the author has studied and developed new methods for solving differential equations, integro-differential and integral equations. This can be achieved by representing these equations by their equivalent operator (transform) equations. When the equivalent operator equations are obtained, then their solutions will be those of the corresponding equivalent differential or integro-differential or integral equations. Methods of solving such operator equations are based upon the use of the operational properties of the integro-differential operator ($\overset{x}{\underset{a}{\mathrm{I}}}{}^{s}$) of generalized order [8].

In previous articles, [9], [32], the author has studied the equivalence properties of Gauss' hypergeometric equations and their equivalent operator equations and then he established the existence of Kummer's twenty four solutions ([5], 87-88).

In this paper equivalence properties between a class of fourth order integro-differential equations of Gauss-Papperitz-Riemann types and their operator equations will be presented in the same way as in previous cases. In dealing with the general case it is sufficient to study a particular case as the treatments are similar. In fact, the general case has been dealt with in a separate paper [31]. So, our work will discuss the case of the integro-differential equations of order $(0,4)$ or $(1,3)$ or $(2,2)$ or $(3,1)$ or $(4,0)$, as they represent the same class where (i,j) denotes the order of equations with integral order i and differential order j. The total order may be indicated by $i+j$.

It may be of interest to know that the fourth order equations or its equivalent forms (i,j), $(i=0,1,2,3,4$ and $j=4,3,2,1,0)$ as mentioned above are satisfied by hypergeometric functions of the type $F_D^{(3)}(A,B_1,B_2,B_3;C;x,y,z)$, where $y=y(x)$, $z=z(x)$.

Bailey ([7], p.78) has indicated that F_1 satisfies a certain type of hypergeometric partial differential equations with "at least six solutions" of this type.

Also, it has been mentioned by Appell and Kampé de Fériet that there are sixty solutions of these equations ([6], [7]). On similar subjects some work have appeared in [26]. In our work, in dealing with ordinary differential equations, it is revealed that the number of branch solutions depends upon the order of the equation, the number of its singular points and the number of transformations leaving the principal integrals unaltered. Thus the number of solutions of the general case already presented [31], with total order n and m is the number of singular points of the equation, including $(-\infty)$ or $(+\infty)$, is equal to $2nm(m-1)$, where $2(m-1)$ is the number of transformations. In dealing with equations of fourth order, with five singular points, including $(\pm\infty)$ it will be shown that such equations possess one hundred sixty branch solutions. Also, the extended Riemann P-Function (the M-Function) which is associated with these equations will be discussed.

2. Equivalence properties. As indicated in the above mentioned references, in particular as it has been shown in [9], [31] about the equivalence of the generalized integro- differential equation of Gauss-Papperitz-Riemann type to its operator equation, we find that the operator equation

$$\underset{a}{\overset{x}{\mathrm{I}}}{}^{-w} \prod_{i=1}^{4}(x+a_i)^{s_i} \underset{a}{\overset{x}{\mathrm{I}}}{}^{-1} (x+a_i)^{1-s_i} \underset{a}{\overset{x}{\mathrm{I}}}{}^{w-r}\, y = 0, \tag{2.1}$$

where a_i, s_i are numbers, $r = 0,1,2,3$, $Re(p-w) > 0$, $y \in C^4$ on (a,b), $(p = 1,2,\ldots)$, represents the integro-differential equation of order $(3,1)$ when $r = 0$, of order $(2,2)$ if $r = 1$ or of order $(1,3)$ if $r = 2$ and of order $(0,4)$ when $r = 3$. The case of order $(3,1)$ may be given by the form

$$y' + \sum_{i=1}^{4} \frac{w - s_i + 1}{x + a_i} y + w \sum_{1\le i<j\le 4} \frac{w - s_i - s_j + 1}{(x+a_i)(x+a_j)} \underset{a}{\overset{x}{\mathrm{I}}}\, y +$$

$$w(w-1) \sum_{1\le i<j<k\le 4} \frac{w - s_i - s_j - s_k + 1}{(x+a_i)(x+a_j)(x+a_k)} \underset{a}{\overset{x}{\mathrm{I}}}{}^{2}\, y +$$

$$+w(w-1)(w-2) \frac{w - \sum_{i=1}^{4} s_i + 1}{\prod_{i=1}^{4}(x+a_i)} \underset{a}{\overset{x}{\mathrm{I}}}{}^{3}\, y = 0$$

This equation can be easily transformed to equations of other orders and thus they are equivalent. It would be more convenient to deal with one form of the above mentioned equations of order $(0,4)$ given by the fourth order differential equation

$$y^{(\mathrm{iv})} + \sum_{i=1}^{4} \frac{w - s_i + 1}{x + a_i} y''' + w \sum_{1\le i<j\le 4} \frac{w - s_i - s_j + 1}{(x+a_i)(x+a_j)} y'' + \tag{2.2}$$

$$w(w-1)\sum_{1\le i<j<k\le 4}\frac{w-s_i-s_j-s_k+1}{(x+a_i)(x+a_j)(x+a_k)}y'+w(w-1)(w-2)\frac{w-\sum_{i=1}^{4}s_i+1}{\prod_{i=1}^{4}(x+a_i)}y=0,\ldots$$

which is equivalent to the operator equation (2.1) when $r=3$, i.e. the operator equation

$$\text{(2.3)}\qquad \overset{x}{\underset{a}{\mathrm{I}}}{}^{-w}\prod_{i=1}^{4}(x+a_i)^{s_i}\,\overset{x}{\underset{a}{\mathrm{I}}}{}^{-1}\prod_{i=1}^{4}(x+a_i)^{1-s_i}\,\overset{x}{\underset{a}{\mathrm{I}}}{}^{w-3}\,y(x)=0,$$

where the integro-differential operator is given by

$$\overset{x}{\underset{a}{\mathrm{I}}}{}^{s}f=\frac{1}{\Gamma(s+n)}D_x^n\int_a^x(x-t)^{s+n-1}f(t)dt,\quad (n=0,1,2,\ldots),$$

f is a real-valued function of class $C^{(n)}$ on $a\le x\le b, a\le t\le b$, $Re(s+n)>0$, $D_x^n=\frac{d^n}{dx^n}$ and Γ is the Gamma function.

It can be easily shown that any fourth order differential equation of the form (2.2) can be represented by an operator equation if w and $s_i(i=1,2,3,4)$ are determined in the same way as equations of other orders are represented, [26], [28], [30], [31], [32].

3. Solutions of equations. Solutions of the differential equation (2.2) may be obtained by finding solutions of its equivalent operator equations (2.3).

By applying the property that when $\overset{x}{\underset{a}{\mathrm{I}}}{}^{-s}F=0$, then $D_x^2\,\overset{x}{\underset{a}{\mathrm{I}}}{}^{2-s}F=0$, which implies that

$$\overset{x}{\underset{a}{\mathrm{I}}}{}^{2-s}F=\overset{x}{\underset{a}{\mathrm{I}}}{}^{2}\,0=C_1(x-a)+C_2,\text{and}$$

$$F=C_1\frac{(x-a)^{s-1}}{\Gamma(s)}+C_2\frac{(x-a)^{s-2}}{\Gamma(s-1)},$$

C_1 and C_2 are arbitrary constants, also by using the equality $D_x^3\,\overset{x}{\underset{a}{\mathrm{I}}}{}^{3-w}=\overset{x}{\underset{a}{\mathrm{I}}}{}^{-w}$ in (2.3) and by the use of properties of inverse operations of the integro-differential operator of generalized order we may find that

$$y(x;a)=\overset{x}{\underset{a}{\mathrm{I}}}{}^{3-w}\prod_{i=1}^{4}(x+a_i)^{s_i-1}\left[K+\overset{x}{\underset{a}{\mathrm{I}}}\prod_{i=1}^{4}(x+a_i)^{-s_i}\left\{C_1\frac{(x-a)^{w-3}}{\Gamma(w-2)}+\right.\right.$$
$$\left.\left.C_2\frac{(x-a)^{w-2}}{\Gamma(w-1)}+C_3\frac{(x-a)^{w-1}}{\Gamma(w)}\right\}\right],$$

where K and C_i $(i=1,2,3)$ are arbitrary constants. Let $C_1=\Gamma(w-2), C_2=\Gamma(w-1)$ and $C_3=\Gamma(w)$. Then the fundamental solutions of (2.3) may be given by

$$(S_1)\qquad y_1(x;a)=K\,\overset{x}{\underset{a}{\mathrm{I}}}{}^{3-w}\prod_{i=1}^{4}(x+a_i)^{s_i-1}$$

$$(S_2)\qquad y_2(x;a) = \overset{x}{\underset{a}{\mathrm{I}}}{}^{3-w} \prod_{i=1}^{4}(x+a_i)^{s_i-1} \overset{x}{\underset{a}{\mathrm{I}}} \prod_{i=1}^{4}(x+a_i)^{-s_i}(x-a)^{w-3}$$

$$(S_3)\qquad y_3(x;a) = \overset{x}{\underset{a}{\mathrm{I}}}{}^{3-w} \prod_{i=1}^{4}(x+a_i)^{s_i-1} \overset{x}{\underset{a}{\mathrm{I}}} \prod_{i=1}^{4}(x+a_i)^{-s_i}(x-a)^{w-2}$$

$$(S_4)\qquad y_4(x;a) = \overset{x}{\underset{a}{\mathrm{I}}}{}^{3-w} \prod_{i=1}^{4}(x+a_i)^{s_i-1} \overset{x}{\underset{a}{\mathrm{I}}} \prod_{i=1}^{4}(x+a_i)^{-s_i}(x-a)^{w-1}.$$

Solutions $(S_i), (i = 1,2,3,4)$ are determined according to the singular points of the differential equation (2.2). These singular points are: $-a_i, \pm\infty, i = 1,2,3,4$. Thus the lower limit (a) of integrals may be replaced by these singular points in performing integration. For each lower limit we obtain four solutions. Consequently twenty principal solutions are obtained by using the singular points.

Applying the following transformations

$$u = v,\ u = 1-v,\ u = \frac{v}{1-x_i+vx_i},\ u = \frac{1-v}{1-vx_i},\quad i = 1,2,3,$$

on principal solutions, then any one of them can be expressed in eight forms, which may be called branch solutions, and thus the total number of branch solutions would be one hundred and sixty solutions. (For further information about such transformation, leaving the integrals unaltered, the reader may see other papers of the author, mentioned earlier). In general, the number of solutions of an equation of n-th order with $(m+1)$ singular points including $(\pm\infty)$ may be given by the following: The differential equation has n fundamental solutions, $n(m+1)$ principal solutions and $2nm(m+1)$ branch solutions, since there are $2m$ transformations which may leave the integrals unaltered as indicated.

4. The M-function. In a previous work [9] the author has shown that Gauss' hypergeometric second order differential equation can be derived from the equation of Riemann type

$$(4.1)\qquad y'' + \sum_{i=1}^{3}\frac{w-s_i+1}{x+a_i}y' + w\sum_{1\le i<j\le 3}\frac{w-s_i-s_j+1}{(x+a_i)(x+a_j)}y +$$

$$w(w-1)\frac{w-\sum_{i=1}^{3}s_i+1}{\prod_{i=1}^{3}(x+a_i)}\overset{x}{\underset{r}{\mathrm{I}}}\, y = 0,$$

(r is chosen to be the lower limit of the operator).

Let

(i) $w - \sum_{i=1}^{3} s_i + 1 = 0$,
(ii) $a_2 \to \pm\infty,\ a_1,\ a_3 = 0$,
(iii) $w = a,\ s_1 = c - b,\ s_3 = a - c + 1$, or $w = b,\ s_1 = c - a,\ s_3 = b - c + 1$,

then (4.1) would represent the Gauss' equation

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0. \tag{4.2}$$

Condition (i) is satisfied by the values of w and s_i given by the matrix equation

$$\begin{bmatrix} w \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = U \begin{bmatrix} a+b+c \\ a+b+c \\ a'+b'+c' \\ a+b+c \end{bmatrix} \tag{4.3}$$

where U is 4×4 unit matrix and a, b, c, a', b', c' are the indices of Riemann P-Function associated with (4.2):

$$P \left\{ \begin{matrix} a_1 & a_2 & a_3 & \\ a & b & c & x \\ a' & b' & c' & \end{matrix} \right\} \tag{4.4}$$

where $a + b + c + a' + b' + c' = 1$.

The twenty four Kummer's solutions are associated with the P-Function, end represented by the sets of branch solutions $(P^{(a)}, P^{(a')})$, $(P^{(b)}, P^{(b')})$, $(P^{(c)}, P^{(c')})$, as listed in ([5], p. 88). Each set contains four branch solutions. In a similar way we can develope and extend Riemann P-Function to M-Function associated with differential equations of higher orders. In fact, the author has already extended the M-Function to n-th order differential equations [31]. As to the differential equations of fourth order we notice that the operator equation:

$$\overset{x}{\underset{r}{\mathrm{I}}}{}^{w} \prod_{i=1}^{5} (x + a_i)^{s_i} \overset{x}{\underset{r}{\mathrm{I}}}{}^{-1} \prod_{i=1}^{5} (x + a_i)^{1-s_i} \overset{x}{\underset{r}{\mathrm{I}}}{}^{w-3}\, y(x) = 0, \tag{4.5}$$

is equivalent to the integro-differential equation:

$$y^{(\mathrm{iv})} + \sum_{i=1}^{5} \frac{w - s_i + 1}{x + a_i} y''' + w \sum_{1 \le i < j \le 5} \frac{w - s_i - s_j + 1}{(x + a_i)(x + a_j)} y'' +$$

$$+ w(w-1) \sum_{1 \le i < j < k \le 5} \frac{w - s_i - s_j - s_k + 1}{(x + a_i)(x + a_j)(x + a_k)} y' +$$

$$+\prod_{p=0}^{2}(w-p)\sum_{1\le i<j<k<m\le 5}\frac{w-s_i-s_j-s_k-s_m+1}{(x+a_i)(x+a_j)(x+a_k)(x+a_m)}y+$$

$$\prod_{p=0}^{3}(w-p)\frac{w-\sum_{i=1}^{5}s_i+1}{\prod_{i=1}^{5}(x+a_i)}\overset{x}{\underset{r}{\mathrm{I}}}\,y=0. \tag{4.6}$$

This equation is reduced to the differential equation (2.2), and (4.5) is reduced to the operator equation (2.3) if:

(i) $w-\sum_{i=1}^{5}s_i+1=0$
(ii) $a_5\to\pm\infty$.

w and s_i $(i=1,2,3,4,5)$ are given by the matrix equation

$$\begin{bmatrix} w\\ s_1\\ s_2\\ s_3\\ s_4\\ s_5\end{bmatrix}=U\begin{bmatrix} a+b+c+d+e\\ a+b+c+d+e\\ a'+b'+c'+d'+e'\\ a''+b''+c''+d''+e''\\ a'''+b'''+c'''+d'''+e'''\\ a+b+c+d+e\end{bmatrix} \tag{4.7}$$

where U is 6×6 unit matrix and $\sum(a+a'+a''+a''')=1$, $\sum a=a+b+c+d+e$, etc.

The scheme of the M-Function may be given by

$$M\begin{Bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & \\ a & b & c & d & e & \\ a' & b' & c' & d' & e' & x\\ a'' & b'' & c'' & d'' & e'' & \\ a''' & b''' & c''' & d''' & e''' & \end{Bmatrix},$$

where $A_i=-a_i$ are the singular points of the equation. In this case for the equation of fourth order the principal sets of branch solutions which correspond to the singular points are:

$$(M^{(a)},M^{(a')},M^{(a'')},M^{(a''')}),\ (M^{(b)},M^{(b')},M^{(b'')},M^{(b''')}),$$
$$(M^{(c)},M^{(c')},M^{(c'')},M^{(c''')}),\ (M^{(d)},M^{(d')},M^{(d'')},M^{(d''')}),$$
$$(M^{(e)},M^{(e')},M^{(e'')},M^{(e''')}).$$

Each set represents eight branch solutions and so the total of branch solutions of the fourth order differential equations of this type is one hundred sixty expressed in terms of hypergeometric functions $F_D^{(3)}$. It can be easily shown that the principal solutions are linearly independent but not the branch solutions. Also, it can be shown that these solutions satisfy the equation.

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