Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

SERDICA — Bulgaricae mathematicae publicationes
19 (1993) 113-120

Mathematical Structures - Informatics - Mathematical Modelling
Papers dedicated to Academician L. Iliev on the Occasion of his Eightieth Birthday

DISTORTION OF THE LEVEL LINES OF THE MODULUS FUNCTIONS UNDER $K - qc$ MAPPINGS

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Dedicated to Academician Ljubomir Iliev
on the Occasion of his Eightieth Birthday

0. Introduction. In the classical type problem of the Riemann surfaces and of the elements of the ideal boundary a role as important as that of the capacity functions is played by the modulus functions. There are many similarities between these classes of functions but also specific aspects which justify this study, whose aim is to extend the results concerning the behaviour of the level lines of the capacity functions under $K - qc$ mappings (K-quasiconformal homeomorphisms) established in [4] to modulus functions.

0.1. Definitions and notations. Let R be an open Riemann surface and Γ its ideal boudary. Throughout the paper we denote by $\{\Pi_n\}$, $n \in \mathbf{N}$, a canonical exhaustion of R, Π_n being a canonical regular region [1, p.26, p.80] and $\Gamma_n = \partial \Pi_n$ consisting of analytic Jordan curves positively oriented with respect to Π_n. Points and parameters on R will be denoted by the same letters.

Let z_0 be a point in R and D a parameter disc containing z_0. The **capacity function** on Γ with respect to z_0 and D [8, p. 179], [9] is a function $p_\Gamma(\ ,z_0) = p_R(\ ,z_0)$ with the properties:

1) p_Γ is harmonic in $R \setminus z_0$,

2) $p_\Gamma(z, z_0) = \log|z - z_0| + h(z, z_0)$, $z \in D$,

where $h(\ ,z_0)$ is a harmonic function in D with $h(z_0, z_0) = 0$, and

3) p_Γ minimazes the integral

$$I(\varphi) = \frac{1}{2\pi}\int_\Gamma \varphi * d\varphi := \lim_{n\to\infty}\int_{\Gamma_n} \varphi * d\varphi$$

in the family of functions $\varphi : R \setminus z_0 \to \mathbf{R}$, which satisfy 1) and 2), $\{\Pi_n\}$ being an exhaustion as above with $z_0 \in \Pi_0$.

One knows that $I(p_\Gamma) := k_\Gamma$ is the **Robin constant** and $c_\Gamma := e^{-k_\Gamma}$ the **capacity** of Γ with respect to z_0 and the parameter in D.

Let Γ_0 be the boundary of a canonical regular region Π_0 on R as above, in particular an analytic Jordan curve, if Π_0 is a Jordan region. The **modulus function** of Γ with respect to Γ_0 [6], [7] is a function $u_\Gamma(\ ,\Gamma_0) = u_R(\ ,\Gamma_0)\ :\ R \setminus \Pi_0 \to \mathbf{R}$ with the properties:

1) u_Γ is harmonic in $R \setminus \overline{\Pi_0}$ and $u_\Gamma = 0$ on Γ_0,

2) $\int_{\Gamma_0} *du_\Gamma = 1$, and

3) u_Γ minimizes the Dirichlet integral

$$D(\varphi) = \int\int_{R\setminus\overline{\Pi_0}} |\operatorname{grad}\varphi|^2 dx dy$$

in the family of functions $\varphi : R \setminus \Pi_0 \to \mathbf{R}$ with the properties 1) – 2). (Sometimes one takes in 2) $\int_{\Gamma_0} *du_\Gamma = 2\pi$, [9, p. 76]).

This minimum $\mu_\Gamma = \mu_R(\Gamma, \Gamma_0) := D(u_\Gamma) \leq \infty$ is the **modulus** of R with respect to Γ_0 and Γ. Together with k_Γ, it is infinite or finite as R is parabolic or hyperbolic.

The function u_Γ is constructed as the limit of a sequence or a subsequence of functions $u_n = u_{\Gamma_n}(\ ,\Gamma_0)$ corresponding to Π_n in an exhaustion $\{\Pi_n\}$ as above. One knows that these functions u_n are characterized by the properties 1), 2) and 3′) u_n is equal to a constant μ_n on Γ_n. Here $\mu_n = D_n(u_n)$ is the Dirichlet integral of u_n on $\Pi_n \setminus \overline{\Pi_0}$, $\mu_n \leq \mu_{n+1} \leq \mu_\Gamma$ and $\mu_\Gamma = \lim_{n\to\infty} \mu_n$.

According to μ_Γ being finite or infinite then $\omega_\Gamma = \omega_\Gamma(\ ,\Gamma_0) := u_\Gamma/\mu_\Gamma$ or zero respectively is the **harmonic measure** of Γ with respect to Γ_0. We denote by $\omega_n = u_n/\mu_n$ the harmonic measure of Γ_n with respect to Γ_0 in $\Pi_n \setminus \overline{\Pi_0}$.

There is an obvious connection between p_Γ and u_Γ. Remark that $c_\tau := \{z \in R : p_\Gamma(z, z_0) = \tau\}$ is always compact if τ is sufficiently small. If one takes $\Gamma_0 = c_\tau$ and $\Pi_0 = \{z \in R : p_\Gamma(z, z_0) < \tau\}$ for such a value τ, then

$$u_\Gamma(\ ,\Gamma_0) = \frac{1}{2\pi}[p_\Gamma(\ ,z_0) - \tau]\Big|_{R\setminus\Pi_0}. \tag{0.1}$$

However, starting conversely with an arbitrary Γ_0, the corresponding $u_\Gamma(\ ,\Gamma_0)$ is not derived in this way from a capacity function $p_\Gamma(\ ,z_0)$ for which Γ_0 would be a level line. Indeed, let us consider $R = \{z : |z| < 1\}$. Then $p_\Gamma(z, z_0) = \ln |(z-z_0)(1-\overline{z_0}z_0)/(1-\overline{z_0}z)|$ for every $z_0 \in R$, hence all the level lines of p_Γ are circles, but we can construct $u_\Gamma(\ ,\Gamma_0)$ taking as Γ_0 an arbitrary analytic Jordan curve.

0.2. Setting of the problem. In §§1 and 2 we shall study the distortion of the level lines of the modulus functions and the harmonic measures respectively under

$K - qc$ mappings $f : R \to R'$ between open Riemann surfaces R and R' of class R_p for $\Gamma'_0 = f(\Gamma_0)$. Namely, in §1 we shall compare the images fL_a of the level lines $L_a = \{z \in R : u_\Gamma(z, \Gamma_0) = a\}$ with the level lines $L'_{a'} = \{z' \in R' : u_{\Gamma'}(z', \Gamma'_0) = a'\}$, where Γ' denotes the ideal boundary of R'. In §2 a similar study will be dedicated to the functions $\omega_\Gamma(\ , \Gamma_0)$ and $\omega_{\Gamma'}(\ , \Gamma'_0)$, supposing R and hence R' to be of hyperbolic type. In each paragraph we consider first the action of a single $K - qc$ mapping f by fixing only Γ_0, and then, by fixing both Γ_0 and Γ'_0, the action of the family $\mathcal{F}$ of all $K - qc$ mapping f with $f(\Gamma_0) = \Gamma'_0$ under the hypothesis $\mathcal{F} \neq \emptyset$.

0.3. As in [4] compactness will play an important role.

First it reflects in the condition imposed to R and R' to be of class R_p. By definition an open Riemann surface R belongs to this class R_p if there exists a capacity function on R with compact level lines. However this class may be characterized by the compactness of the level lines of a modulus function $u_\Gamma(\ , \Gamma_0)$ as proved in [3], where it is also shown that the property is independent of the choice of Γ_0 as well of the choice of the logarithmic pole z_0 in the definition based on the capacity function.

Another compactness property concerns the considered family $\mathcal{F}$ of $K - qc$ mappings and is contained in the following.

Lemma [5] 1. *Let R and R' be Riemann surfaces which are not conformally equivalent to $\mathbf{C}$ and $\widehat{\mathbf{C}}$, $M \subset R$ and let $M' \subset R'$ be two compact sets such that there are $K - qc$ mapping $f : R \to R'$ with $f(M) = M'$. Then the family $\mathcal{F}$ of these mapping is normal and closed.*

2. *The same results is valid for $R = R' = \mathbf{C}$ if M contains at least two points.*

In 1.2 and 2.2 we shall take $M = \Gamma_0$ and $M' = \Gamma'_0$.

§1. Level lines of the modulus functions

1.1. Suppose that R and R' are two open Riemann surfaces of class R_p such that there are $K - qc$ homeomorphisms $f : R \to R'$. With the previous notations for the ideal boundaries Γ and Γ', Π_0 and Γ_0, set $\Pi'_0 = f(\Pi_0)$ and $\Gamma'_0 = f(\Gamma_0)$ and consider the modulus functions $u_\Gamma(\ , \Gamma_0)$ on R and $u_{\Gamma'}(\ , \Gamma'_0)$ on R'.

Denote by L_a the level line $L_a = \{z \in R : u_\Gamma(z, \Gamma_0) = a\}$ for every $a \in [0, \mu_\Gamma)$, $L_0 = \Gamma_0$, by Π_a the region $\Pi_a = \{z \in R : 0 < u_\Gamma(z, \Gamma_0) < a\}$, by $L'_{a'} = \{z' \in R' : u_{\Gamma'}(z', \Gamma'_0) = a'\}$ for every $a' \in [0, \mu_{\Gamma'})$ and by $\Pi'_{a'} = \{z' \in R' : 0 < u_{\Gamma'}(z', \Gamma'_0) < a'\}$. Further set $\Pi_{a_1 a_2} = \Pi_{a_2} \setminus \overline{\Pi_{a_1}}$ and $L_{a_1 a_2} = \{L_a : a \in [a_1, a_2]\}$ for $a_1 < a_2$, $a_1, a_2 \in [0, \mu_\Gamma)$ and similarly $\Pi'_{a'_1 a'_2}, L'_{a'_1 a'_2}$ for $a'_1 < a'_2, a'_1, a'_2 \in [0, \mu_{\Gamma'})$.

One knows that the modulus of $\overline{\Pi}_{a_1 a_2}$ which is by definition the modulus of Π_{a_2} with respect to L_{a_1} and L_{a_2} is equal to the modulus of the curve family separating L_{a_2} from L_{a_1} on $\Pi_{a_1 a_2}$ and also equal to the modulus of the extremal family $L_{a_1 a_2}$: Mod $L_{a_2 a_1} = a_2 - a_1$, [2].

In order to study the image $f\ L_a$ of the level line L_a of $u_\Gamma(\ ,\Gamma_0)$ by means of the level lines of $u_{\Gamma'}(\ ,\Gamma'_0)$ we introduce as in [4] the functions

$$a'_0(a,f) = \min\{u_{\Gamma'}(z',\Gamma'_0) : z' \in f\ L_a\}$$

and

$$A'_0(a,f) = \max\{u_{\Gamma'}(z',\Gamma'_0) : z' \in f\ L_a\}.$$

Proposition 1.1. *The functions $a'_0(a,f)$ and $A'_0(a,f)$ are strictly increasing with respect to a and satisfy the inequalities*

$$K^{-1}a'_0(a,f) \le a \le K A'_0(a,f). \tag{1.1}$$

P r o o f. Take $a_1, a_2 \in [0,\mu_\Gamma)$ with $a_1 < a_2$. Since $L_{a_1} \subset \operatorname{int} \Pi_{a_2}$, it follows that $f\ L_{a_1} \subset \operatorname{int} f\ \Pi_{a_2} \subset \Pi'_{A'_0}(a_2,f)$. Hence in any point z' of $f\ L_{a_1}$ the function $u_{\Gamma'}(z',\Gamma'_0) < A'_0(a_2,f)$; in particular $A'_0(a_1,f) < A'_0(a_2,f)$.

The inequalities (1.1) result as a special case from

Proposition 1.2. *Let $a_1 < a_2$ as before. Then*

$$K^{-1}[a'_0(a_2,f) - A'_0(a_1,f)] \le a_2 - a_1 \le K[A'_0(a_2,f) - a'_0(a_1,f)]. \tag{1.2}$$

P r o o f. i) By means of Grötzsch's inequalities we can write $K^{-1}(a_2-a_1) = K^{-1}$ Mod $L_{a_1a_2} \le$ Mod $f\ L_{a_1a_2} \le$ Modulus of the whole family of curves which separates $L'_{a'_0(a_1,f)}$ from $L'_{A'_0(a_2,f)} =$ Mod $L'_{a'_0(a_1,f),A'_0(a_2,f)} = A'_0(a_2,f) - a'_0(a_1,f)$.

ii) Suppose that $A'_0(a_1,f) < a'_0(a_2,f)$. Then $a'_0(a_2,f) - A'_0(a_1,f) =$ Mod $L'_{a'_0(a_2,f),A'_0(a_1,f)} \le$ Modulus of the whole family of curves separating $f\ L_{a_1}$ and $f\ L_{a_2} \le$ K Modulus of the family separating L_{a_1} and $L_{a_2} = K$ Mod $L_{a_1a_2} = K(a_2 - a_1)$.

If $A'_0(a_1,f) \ge a'_0(a_2,f)$ the left inequality in (1.2) is trivial.

Taking $a_1 = 0$ and $a_2 = a$, (1.2) reduces to (1.1).

Corollary. *If R and R' are of hyperbolic type, then*

$$K^{-1}[\mu_{\Gamma'} - A'_0(a,f)] \le \mu_\Gamma - a \le K[\mu_{\Gamma'} - a'_0(a,f)]. \tag{1.3}$$

P r o o f. The inequalities (1.3) follow from (1.2) by taking $a_1 = a$ and letting $a_2 \to \mu_\Gamma$.

Further for $a = 0$, (1.3) implies the Grötzsch inequalities

$$K^{-1}\mu_{\Gamma'} \le \mu_\Gamma \le K\mu_{\Gamma'}. \tag{1.4}$$

Remarks. Since $f\ L_a \subset \Pi'_{a'_0(a,f)A'_0(a,f)}$ a measure for its distortion with respect to the level lines of $u_{\Gamma'}(\ ,\Gamma'_0)$ is given by Mod $L_{a'_0(a,f)A'_0(a,f)} = A'_0(a,f) - a'_0(a,f)$.

An interesting case of $K-qc$ mapping in $\mathcal{F}$ is represented by mappings f with $f\ L_a = L'_{a'}$, $a' = a'(a,f)$. In this case (1.2) becomes

$$K^{-1}(a'_2 - a'_1) \le a_2 - a_1 \le K(a'_2 - a'_1),\ a_1 < a_2. \tag{1.5}$$

The equality in the right or the left side of (1.2) is realized exactly in this case, when f satisfies the supplementary conditions: the dilatation quotient of f is the constant K and the major axes of the characteristics ellipses of f are orthogonal or tangent respectively to the level lines L_a a.e. in R. We have then $a_2 - a_1 = K(a'_2 - a'_1)$ or $K^{-1}(a'_2 - a'_1) = a_2 - a_1$. Evidently, the equality in both sides corresponds only to $K = 1$.

1.2. We shall now fix Γ_0 and Γ'_0 and establish inequalities valid for the family $\mathcal{F}$ defined in 0.2., i.e. the family of all $K-qc$ homeomorphisms $f: R \to R'$ with $f(\Gamma_0) = \Gamma'_0$ or equivalently $f(\Pi_0) = \Pi'_0$, under the hypothesis $\mathcal{F} \neq \emptyset$. To this purpose we introduce for $a \in [0,\mu_\Gamma)$ the functions

$$a'_0(a) = \inf\{a'_0(a,f),\ f \in \mathcal{F}\}$$

and

$$A'_0(a) = \sup\{A'_0(a,f),\ f \in \mathcal{F}\}.$$

Proposition 1.3. *There exist mappings $f_{0,a}$ and $F_{0,a}$ in $\mathcal{F}$ with $a'_0(a) = a'_0(a,f_{0,a})$ and $A'_0(a) = A'_0(a,F_{0,a})$.*

Proof. Let $\{f_n\}$ be a sequence in $\mathcal{F}$ such that $a'_0(a,f_n) \to a'_0(a)$. Since Lemma in 0.3 implies that $\mathcal{F}$ is normal and closed, the sequence $\{f_n\}$ contains a subsequence, denoted again by $\{f_n\}$, which locally uniformly converges on R to a mapping $f_{0,a} \in \mathcal{F}$. For every n we choose a point $z_n \in L_a$ such that $u_{\Gamma'}(f_n(z_n),\Gamma'_0) = a'_0(a,f_n)$. By passing if necessary to a subsequence and renumbering, we may suppose that $z_n \to z^* \in L_a$, therefore $f_n(z_n) \to f_{0,a}(z^*)$. Hence $a'_0(a) = \lim_{n\to\infty} u_{\Gamma'}(f_n(z_n),\Gamma'_0) = u_{\Gamma'}(f_{0,a}(z^*),\Gamma'_0) \ge a'_0(a,f_{0,a})$ and by definition $a'_0(a) = a'_0(a,f_{0,a})$. The proof for $A'_0(a)$ is similar.

Proposition 1.3 shows that the functions $a'_0(a)$ and $A'_0(a)$ are finite.

Proposition 1.4. *The functions $a'_0(a)$ and $A'_0(a)$ are strictly increasing and satisfy the inequalities*

$$K^{-1}[a'_0(a_2) - A'_0(a_1)] \le a_2 - a_1 \le K[A'_0(a_2) - a'_0(a_1)] \tag{1.6}$$

for $a_1 < a_2$, a_1 and $a_2 \in [0,\mu_\Gamma)$.

Proof. i) $A'_0(a_1) = A'_0(a_1,F_{0,a_1}) < A'_0(a_2,F_{0,a_1}) \le A'_0(a_2)$.

ii) In order to obtain (1.6) it is sufficient to replace in (1.2) the functions $a_0'(a_j, f)$ and $A_0'(a_j, f)$ by $a_0'(a_j)$ and $A_0'(a_j)$ respectively, $j = 1, 2$.

From (1.6) one deduces immediately

$$K^{-1}a_0'(a) \leq a \leq K A_0'(a)$$

(1.7) and

$$K^{-1}[\mu_{\Gamma'} - A_0'(a)] \leq \mu_\Gamma - a \leq K[\mu_{\Gamma'} - a_0'(a)].$$

The compact Riemann surfaces can be studied also by means of modulus functions. Let S and S' be two such surfaces and choose besides $\Pi_0 \subset S$ and $\Pi_0' \subset S'$ the points $z_\infty \in S \setminus \overline{\Pi_0}$ and $z'_\infty \in S' \setminus \overline{\Pi_0'}$. The family $\mathcal{F}$ will consists of all the $K-qc$ mappings $f : S \to S'$ with $f(\Pi_0) = \Pi_0'$ and $f(z_\infty) = z'_\infty$.

§2. **Level lines of the harmonic measures.** We shall now consider hyperbolic Riemann surfaces of class R_p and reformulate the results for the harmonic measure.

Since the harmonic measure

$$\omega_\Gamma(\ ,\Gamma_0) = u_\Gamma(\ ,\Gamma_0)/\mu_\Gamma,$$

the level lines of ω_Γ coincide with the level lines of u_Γ. If we denote by $l_b = \{z \in R : \omega_\Gamma(z, \Gamma_0) = b\}$ for $b \in [0, 1)$, then $l_b = L_a$ for $b = a/\mu_\Gamma$.

2.1. Suppose that $f : R \to R'$ is a $K - qc$ homeomorphism, choose Π_0 in R and take $\Pi_0' = f(\Pi_0)$, hence $\Gamma_0' = f(\Gamma_0)$. As in 1.1 define

$$b_0'(b, f) = \min\{\omega_{\Gamma'}(z', \Gamma_0') : z' \in fl_b\}$$

and

$$B_0'(b, f) = \max\{\omega_{\Gamma'}(z', \Gamma_0') : z' \in fl_b\}.$$

Then $b_0'(b, f) = a_0'(\mu_\Gamma b, f)/\mu_{\Gamma'}$ and $B_0'(b, f) = A_0'(\mu_\Gamma b, f)/\mu_{\Gamma'}$. Thus the main inequalities (1.2) become for $b_1 < b_2$, b_1 and $b_2 \in [0, 1)$

$$K^{-1}[b_0'(b_2, f) - B_0'(b_1, f)] \leq \mu_\Gamma \mu_{\Gamma'}^{-1}(b_2 - b_1) \leq K[B_0'(b_2, f) - b_0'(b_1, f)], \tag{2.1}$$

whence, taking into account that $K^{-1} \leq \mu_\Gamma \mu_{\Gamma'}^{-1} \leq K$,

$$K^{-2}[b_0'(b_2, f) - B_0'(b_1, f)] \leq (b_2 - b_1) \leq K^2[B_0'(b_2, f) - b_0'(b_1, f)].$$

Consequently, for $b_1 = 0$ and $b_2 = b$ one gets

$$K^{-1}b_0'(b, f) \leq \mu_\Gamma \mu_{\Gamma'}^{-1} b \leq K B_0'(b, f) \tag{2.2}$$

and for $b_1 = b$, $b_2 \to 1$

$$K^{-1}[1 - B'_0(b, f)] \le \mu_\Gamma \mu_{\Gamma'}^{-1}(1 - b) \le K[1 - b'_0(b, f)]. \tag{2.3}$$

The equality in (2.1) – (2.2) arises under the same conditions as in (1.2).

2.2. In the same manner as in 1.2, we can introduce the family $\mathcal{F}$ and the functions

$$b'_0(a) = \inf\{b'_0(b, f),\ f \in \mathcal{F}\}$$

and

$$B'_0(b) = \sup\{B'_0(b, f),\ f \in \mathcal{F}\}.$$

Again we have $b'_0(b) = a'_0(\mu_\Gamma b)/\mu_{\Gamma'}$ and $B'_0(b) = A'_0(\mu_\Gamma b)/\mu_{\Gamma'}$, hence by Proposition 1.3 we get

$$b'_0(b) = b'_0(b, f_{0,\mu_\Gamma b}) \text{ and } B'_0(b) = B'_0(b, F_{0,\mu_\Gamma b}).$$

In a similar way Proposition 1.4 implies that the functions $b'_0(b)$ and $B'_0(b)$ are strictly increasing and satisfy the inequalities

$$K^{-1}[b'_0(b_2) - B'_0(b_1)] \le \mu_\Gamma \mu_{\Gamma'}^{-1}(b_2 - b_1) \le K[B'_0(b_2) - b'_0(b_1)] \tag{2.4}$$

for $b_1 < b_2$, b_1 and $b_2 \in [0, 1)$ as well as the inequalities

$$K^{-1} b'_0(b) \le \mu_\Gamma \mu_{\Gamma'}^{-1} b \le K B'_0(b) \tag{2.5}$$

and

$$K^{-1}[1 - B'_0(b)] \le \mu_\Gamma \mu_{\Gamma'}^{-1}(1 - b) \le K[1 - b'_0(b, f)] \tag{2.6}$$

for $b \in [0, 1)$.

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